Asymptotics of dissipative nonlinear evolution equations with ellipticity: different end states

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Abstract

In this paper, we consider the global existence and asymptotic behaviors of solutions to the Cauchy problem for the following nonlinear evolution equations with ellipticity and dissipative effects:

\[
\begin{aligned}
\psi_t &=- (1-\alpha)\psi - \theta_x + \alpha \psi_{xx}, \\
\theta_t &= - (1-\alpha)\theta + \nu \psi_x + 2\psi \theta_x + \alpha \theta_{xx},
\end{aligned}
\]

with initial data

\[
(\psi, \theta)(0) = (\psi_0(x), \theta_0(x)) \rightarrow (\psi_\pm, \theta_\pm) \quad \text{as} \quad x \rightarrow \pm \infty,
\]

where \(\alpha\) and \(\nu\) are positive constants such that \(\alpha < 1\), \(\nu < 4\alpha(1-\alpha)\). Under the assumption that \(|\psi_+ - \psi_-| + |\theta_+ - \theta_-|\) is sufficiently small, we show that if the initial data is a small perturbation of the diffusion waves defined by (2.5) which are obtained by the diffusion equations (2.1), solutions to Cauchy problem (E) and (I) tend asymptotically to those diffusion waves with exponential rates. The analysis is based on the energy method. The same problem was studied by Tang and Zhao [J. Math. Anal. Appl. 233 (1999) 336–358] for the case of \((\psi_\pm, \theta_\pm) = (0, 0)\).

Keywords: Evolution equations; Diffusion waves; Decay rate; Energy method; A priori estimates

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1. Introduction

Many systems, describing the essential mechanism of nonlinear interaction between ellipticity and dissipation, arise broadly from physical and mechanical fields, cf. [3,6,7,9]. These systems are far from being completely investigated in mathematics so far due to their complexity. However, the well understanding of these mathematical models, reversely, will give us much help to yield insight into the physical phenomenon and mechanical law. A set of simplified equations was thus proposed by Hsieh [5],

\[
\begin{align*}
\psi_t &= -(\sigma - \alpha)\psi - \sigma \theta_x + \alpha \psi_{xx}, \\
\theta_t &= -(1 - \beta)\theta + \nu \psi_x + 2 \psi \theta_x + \beta \theta_{xx},
\end{align*}
\]

(1.1)

where \(\alpha, \beta, \sigma\) and \(\nu\) are positive constants such that \(\alpha < \sigma\) and \(\beta < 1\).

Ignoring the damping and diffusion terms temporarily, system (1.1) is simplified as

\[
\begin{pmatrix}
\psi \\
\theta
\end{pmatrix}_t = \begin{pmatrix}
0 & -\sigma \\
\nu & 2 \psi
\end{pmatrix} \begin{pmatrix}
\psi \\
\theta
\end{pmatrix}_x.
\]

(1.2)

It is easy to see that system (1.2) is elliptic for \(|\psi| < \sqrt{\sigma \nu}\), and hyperbolic for \(|\psi| > \sqrt{\sigma \nu}\). Around the zero equilibrium, system (1.2) subject to initial small disturbance is unstable owing to the ellipticity and \(|\psi|\) will grow because of the inherent instability of system (1.2). When the growth of \(|\psi|\) leads to \(|\psi| > \sqrt{\sigma \nu}\), system (1.2) becomes hyperbolic at once and \(\psi\) ceases to grow. Thus, a “switching back and forth” phenomenon is expected due to the interplaying among ellipticity, hyperbolicity and dissipation for suitable coefficients, which makes system (1.2) quite complicated. These have been numerically verified in [4,5]. But we may predict that the damping and diffusion terms joining to system (1.2) will prevent \(\psi\) from growing and make the solutions stable.

As to the study of system (1.1), there are only a few rigorous results available so far due to the complexity of system (1.1) as we have mentioned above.

To make the analysis easier for reading, Tang and Zhao in [10] discussed the Cauchy problem of the system (1.1) with \(\alpha = \beta\) and \(\sigma = 1\). That is,

\[
\begin{align*}
\psi_t &= -(1 - \alpha)\psi - \theta_x + \alpha \psi_{xx}, \\
\theta_t &= -(1 - \alpha)\theta + \nu \psi_x + 2 \psi \theta_x + \alpha \theta_{xx},
\end{align*}
\]

(1.3)

with initial data

\[
\begin{pmatrix}
\psi(x, 0) \\
\theta(x, 0)
\end{pmatrix} = \begin{pmatrix}
\psi_0(x) \\
\theta_0(x)
\end{pmatrix}.
\]

(1.4)

Under the assumptions that \(\nu < 4\alpha(1 - \alpha)\) and the initial data

\[
\begin{pmatrix}
\psi_0(x) \\
\theta_0(x)
\end{pmatrix} \in L^2 \cap W^{1,\infty}(\mathbb{R}, \mathbb{R}^2),
\]

(1.5)

they proved the global existence of the solutions to Cauchy problem (1.3), (1.4) and obtained the decay rates of the solutions by the Fourier analysis and the energy method. However, the assumption (1.5) implies

\[
\begin{pmatrix}
\psi_0(x) \\
\theta_0(x)
\end{pmatrix} \to (0, 0) \quad \text{as } x \to \pm \infty,
\]

(1.6)

which is a rigorous restriction on the initial data \(\begin{pmatrix}\psi_0(x) \\
\theta_0(x)\end{pmatrix}\).
To consider more general problems in application, one always assumes that the initial data \((\psi(x, 0), \theta(x, 0)) = (\psi_0(x), \theta_0(x))\) satisfies
\[
(\psi(x, 0), \theta(x, 0)) = (\psi_0(x), \theta_0(x)) \rightarrow (\psi_{\pm}, \theta_{\pm}) \quad \text{as} \quad x \rightarrow \pm \infty,
\] (1.7)
where \(\psi_{\pm}, \theta_{\pm}\) are constant states and \((\psi_+ - \psi_-, \theta_+ - \theta_-) \neq (0, 0)\). This case was considered in [11].

In this paper, we will further study the global existence and decay rates of solutions to the Cauchy problem (1.3) and (1.7) by applying the energy method. For this, we discuss in Section 2 the linear diffusion equations obtained by approximating the system (1.3) and give the decay estimates of the corresponding diffusion waves. In Sections 3, the global existence results are obtained from the local existence and a priori estimates. Finally, we get in Section 4 decay rates to the diffusion waves for solutions to (1.3) and (1.7). Furthermore, we also obtain the optimal decay rates.

**Notations.** Throughout this paper, we denote positive constants by \(C\). Moreover, the character “\(C\)” may differ in different places. \(L_p = L_p(\mathbb{R}) (1 \leq p \leq \infty)\) denotes the usual Lebesgue space on \(\mathbb{R} = (-\infty, \infty)\) with its norm \(\|f\|_{L_p} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p}, 1 \leq p < \infty, \|f\|_{L_\infty} = \sup_{\mathbb{R}} |f(x)|\), and when \(p = 2\), we write \(\|\cdot\|_{L_2(\mathbb{R})} = \|\cdot\|\). \(H^l(\mathbb{R})\) denotes the usual \(l\)th order Sobolev space with its norm \(\|f\|_{H^l(\mathbb{R})} = \left(\sum_{i=0}^{l} \|\partial^i f\|^2\right)^{1/2}\). For simplicity, \(\|f(\cdot, t)\|_{L_p}\) and \(\|f(\cdot, t)\|_l\) are denoted by \(\|f(t)\|_{L_p}\) and \(\|f(t)\|_l\), respectively.

### 2. Analysis of the linear diffusion waves

As in [2,12], we expect that the solutions of (1.3) time-asymptotically behave as those of the following linear system:
\[
\begin{aligned}
\bar{\psi}_t &= -(1 - \alpha) \bar{\psi}_x + \alpha \bar{\psi}_{xx}, \\
\bar{\theta}_t &= -(1 - \alpha) \bar{\theta}_x + \alpha \bar{\theta}_{xx},
\end{aligned}
\] (2.1)

Since both equations in (2.1) are independent, it suffices to solve (2.1)_1. By setting the transformation \(\tilde{\psi}(x, t) = \phi(x, t)e^{-(1-\alpha)t}\), we can derive a heat equation from (2.1)_1, that is
\[
\tilde{\phi}_t = \alpha \tilde{\phi}_{xx}.
\] (2.2)

We hope to find the solution \(\tilde{\phi}(x, t)\) of the following form:
\[
p(\xi) = p\left(\frac{x}{\sqrt{1 + t}}\right), \quad -\infty < \xi < \infty,
\] (2.3)
satisfying boundary conditions \(p(\pm \infty) = \psi_{\pm}\), where \(\xi = x/\sqrt{1 + t}\).

It follows from (2.2) and (2.3),
\[
\begin{aligned}
-\frac{1}{2} \xi p'(\xi) &= \alpha p''(\xi), \\
n(\pm \infty) &= \psi_{\pm}.
\end{aligned}
\] (2.4)
We get by the direct calculation
\[ \bar{\psi}(x,t) = p(\xi) = \frac{\psi_+ - \psi_-}{\sqrt{4\pi\alpha(1+t)}} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{4\alpha(1+t)}\right) dy + \psi_- , \]
which gives the solutions of (2.1)
\[
\begin{align*}
\bar{\psi}(x,t) &= e^{-(1-\alpha)t}\left((\psi_+ - \psi_-) \int_{-\infty}^{x} G(y, t+1) dy + \psi_-\right), \\
\bar{\theta}(x,t) &= e^{-(1-\alpha)t}\left((\theta_+ - \theta_-) \int_{-\infty}^{x} G(y, t+1) dy + \theta_-\right),
\end{align*}
\]
where
\[ G(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right) \]
is the heat kernel function. It is easy to show
\[
\begin{align*}
\bar{\psi}(x,t) &\to \psi_\pm e^{-(1-\alpha)t}, \quad x \to \pm \infty, \\
\bar{\theta}(x,t) &\to \theta_\pm e^{-(1-\alpha)t}, \quad x \to \pm \infty.
\end{align*}
\]
Now we will consider the asymptotic behavior of \( \bar{\psi}(x,t) \), \( \bar{\theta}(x,t) \) and their derivatives in \( L^p(\mathbb{R}) \). First for the heat kernel function, it has the following properties.

**Lemma 2.1.** When \( 1 \leq p \leq +\infty \), \( 0 \leq l, k < +\infty \), we have
\[ \| \partial^l_t \partial^k_x G(t) \|_{L^p} \leq C t^{-\left(\frac{1}{2} - \frac{l}{2p} - \frac{k}{2}\right)}. \]

From the above lemma, we can get the following results by simple calculations.

**Lemma 2.2.** The solutions \( \bar{\psi}(x,t) \) and \( \bar{\theta}(x,t) \) to (2.1) satisfy the following properties:

(i) \[ \| \partial^l \bar{\psi}(t) \|_{L^\infty} \leq C e^{-(1-\alpha)t}, \quad \| \partial^l \bar{\theta}(t) \|_{L^\infty} \leq C e^{-(1-\alpha)t}, \quad l = 0, 1, 2, \ldots; \]

(ii) for any \( p \) with \( 1 \leq p \leq +\infty \), it holds that
\[
\begin{align*}
\| \partial^l_k \partial^k_x \bar{\psi}(t) \|_{L^p} &\leq C |\psi_+ - \psi_-| e^{-(1-\alpha)t}(1+t)^{1/(2p) - k/2}, \\
k &= 1, 2, \ldots, l = 0, 1, 2, \ldots, \]
\[
\| \partial^l_k \partial^k_x \bar{\theta}(t) \|_{L^p} &\leq C |\theta_+ - \theta_-| e^{-(1-\alpha)t}(1+t)^{1/(2p) - k/2}, \\
k &= 1, 2, \ldots, l = 0, 1, 2, \ldots.
\end{align*}
\]

3. Global existence of solutions

3.1. Reformulation of the problem

Let
\[
\begin{align*}
u(x,t) &= \psi(x,t) - \bar{\psi}(x,t), \\
u(x,t) &= \theta(x,t) - \bar{\theta}(x,t).
\end{align*}
\]

\[ (3.1) \]
Then from (2.1), we can rewrite problem (1.3) and (1.7) as follows:

\[
\begin{align*}
  u_t &= -(1-\alpha)u - v_x + \alpha u_{xx} - \tilde{\theta}_x, \\
  v_t &= -(1-\alpha)v + vv_x + 2u v_x + \alpha v_{xx} + 2\tilde{\psi} v_x + 2\tilde{\theta}_x u + F(x, t),
\end{align*}
\]  

with initial data

\[
\begin{align*}
  u(x, 0) &= \psi_0(x) - \tilde{\psi}(x, 0) \rightarrow 0, \quad x \rightarrow \pm \infty, \\
  v(x, 0) &= \theta_0(x) - \tilde{\theta}(x, 0) \rightarrow 0, \quad x \rightarrow \pm \infty,
\end{align*}
\]  

where

\[
F(x, t) = \nu \tilde{\psi}_x + 2\tilde{\psi} \tilde{\theta}_x.
\]  

We seek the solutions of (3.2), (3.3) in the set of functions \(X(0, T)\) defined by

\[
X(0, T) = \{(u, v) \mid u, v \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3)\}.
\]  

Now we state our first main results as follows.

**Theorem 3.1.** Let \((u_0(x), v_0(x)) \in H^2(\mathbb{R}, \mathbb{R}^2)\). Furthermore, Suppose that both \(\delta = |\psi_+ - \psi_-| + |\theta_+ - \theta_-|\) and \(\delta_0 = \|u_0\|_2^2 + \|v_0\|_2^2\) are sufficiently small. Then for any \(0 < \alpha < 1, v < 4\alpha(1-\alpha)\), the Cauchy problem (3.2), (3.3) admits a unique global solution \((u(x, t), v(x, t)) \in X(0, T)\) satisfying

\[
\|u(t)\|_2^2 + \|v(t)\|_2^2 + \int_0^t (\|u(\tau)\|_3^2 + \|v(\tau)\|_3^2) \, d\tau \leq C(\delta + \delta_0)
\]  

and

\[
\sup_{x \in \mathbb{R}} \left(\|u(x, t)\| + \|v(x, t)\|\right) \rightarrow 0 \quad \text{as} \ t \rightarrow \infty.
\]  

**3.2. Local existence of solutions**

In this subsection, we will study the local existence of Cauchy problem (3.2) and (3.3). To this end, we rewrite the Cauchy problem (3.2) and (3.3) in the following integral forms:

\[
\begin{align*}
  u(x, t) &= G(x, t) * u_0(x) - (1-\alpha) \int_0^t G(x, t-s) * u(x, s) \, ds \\
  &\quad + \int_0^t G_x(x, t-s) * v(x, s) \, ds - \int_0^t G(x, t-s) * \tilde{\theta}_x(x, s) \, ds, \\
  v(x, t) &= G(x, t) * v_0(x) - (1-\alpha) \int_0^t G(x, t-s) * v(x, s) \, ds \\
  &\quad - v \int_0^t G_x(x, t-s) * u(x, s) \, ds + 2 \int_0^t G(x, t-s) * (uv)_x(x, s) \, ds \\
  &\quad + 2 \int_0^t G(x, t-s) * (\tilde{\psi} v_x)(x, s) \, ds + 2 \int_0^t G(x, t-s) * (\tilde{\theta} u)(x, s) \, ds \\
  &\quad + \int_0^t G(x, t-s) * F(x, s) \, ds,
\end{align*}
\]  

where the convolutions are taken with respect to the space variable \(x\). We can construct the approximate solution sequences and obtain the local existence by implementing the standard arguments with Brower fixed point principle [1].
Lemma 3.2 (Local existence). If \((u_0(x), v_0(x)) \in H^2(\mathbb{R}, \mathbb{R}^2)\), then there exists \(t_0\) depending only on \(\|(u_0(x), v_0(x))\|_{H^2(\mathbb{R}, \mathbb{R}^2)}\), such that the Cauchy problem (3.2), (3.3) admits a unique smooth solution \((u(x, t), v(x, t)) \in X(0, t_0)\) satisfying
\[
\| (u(x, t), v(x, t)) \|_{H^2(\mathbb{R}, \mathbb{R}^2)} \leq 2 \| (u_0(x), v_0(x)) \|_{H^2(\mathbb{R}, \mathbb{R}^2)}.
\]

3.3. Global existence of solutions

By the local existence result, in order to get the global existence of Cauchy problem (3.2) and (3.3), it is sufficient to get a priori estimates. Precisely, we need to prove that there exists a constant \(C\) depending only on \(\|(u_0(x), v_0(x))\|_{H^2(\mathbb{R}, \mathbb{R}^2)}\), such that any solution \((u, v)(x, t)\) in \(X(0, T)\) satisfy
\[
\| u_0 \|_2 + \| v_0 \|_2 \leq C \text{ for any } t \in [0, T].
\]
Next, we devote ourselves to the estimate of the solution \((u(x, t), v(x, t))\) of (3.2), (3.3) under the a priori assumption
\[
N(T) = \sup_{0 < t < T} \left\{ \sum_{k=0}^{2} \| \partial_x^k u(t) \|^2 + \sum_{k=0}^{2} \| \partial_x^k v(t) \|^2 \right\} \leq \delta_1^2,
\]
where \(0 < \delta_1 \ll 1\).

By Sobolev inequality \(\| f \|_{L^\infty} \leq \| f \|_{L^1/2} \| f_x \|_{L^{1/2}}\), we have
\[
\| (u, u_x, v, v_x) \|_{L^\infty} \leq \delta_1,
\]
which will be used later.

Moreover, if \(\nu < 4\alpha(1 - \alpha)\), we can find \(\epsilon \in (0, 2)\), \(c_0 > 0\) such that
\[
\begin{align*}
2c_0\alpha - \frac{1}{(1 - \alpha)} &> 0, \\
2(1 - \alpha) - \frac{c_0\alpha^2}{\alpha} &> 0.
\end{align*}
\]
In fact, from \(\nu < 4\alpha(1 - \alpha)\), we know that there exists a constant \(k \in (0, 1)\), such that \(\nu = 4k\alpha(1 - \alpha)\). Choosing
\[
\epsilon = k + 1 \quad \text{and} \quad c_0 = \frac{1}{2\alpha(1 - \alpha)} \left( \frac{1}{2(k + 1)} + \frac{k + 1}{8k^2} \right),
\]
one can easily verify that \(\epsilon\) and \(c_0\) satisfy (3.10).

What follows will be a series of lemmas contributing to our desired estimates.

Lemma 3.3. Suppose that the assumptions in Theorem 3.1 hold and \((u(x, t), v(x, t))\) is a solution to (3.2), (3.3) obtained in Lemma 3.2, then it holds that for any \(\nu < 4\alpha(1 - \alpha)\),
\[
\int_{\mathbb{R}} (u^2 + v^2) \, dx + \int_0^t \int_{\mathbb{R}} (u_x^2 + v_x^2) \, dx \, d\tau + \int_0^t \int_{\mathbb{R}} (u_x^2 + v_x^2) \, dx \, d\tau \leq C(\delta + \delta_0),
\]
provided that \(\delta\) and \(\delta_1\) are sufficiently small.
Proof. Multiplying the first equation of (3.2) by $2u$ and the second equation of (3.2) by $2c_0v$ and integrating the resulting identity over $\mathbb{R} \times (0,t)$, we arrive at by Cauchy–Schwarz inequality

$$
\int_{\mathbb{R}} (u^2 + c_0v^2) \, dx + 2(1 - \alpha) \int_0^t \int_{\mathbb{R}} (u^2 + c_0v^2) \, dx \, d\tau + 2\alpha \int_0^t \int_{\mathbb{R}} (u^2_x + c_0v^2_x) \, dx \, d\tau
$$

$$
= \|u_0\|^2 + c_0\|v_0\|^2 - 2 \int_0^t \int_{\mathbb{R}} uv \, dx \, d\tau + 2c_0 \int_0^t \int_{\mathbb{R}} v \, dx \, d\tau
$$

$$
- 2 \int_0^t \int_{\mathbb{R}} u\tilde{\theta}_x \, dx \, d\tau - 2c_0 \int_0^t \int_{\mathbb{R}} u_x v^2 \, dx \, d\tau - 2c_0 \int_0^t \int_{\mathbb{R}} \bar{\psi}_x v^2 \, dx \, d\tau
$$

$$
+ 4c_0 \int_0^t \int_{\mathbb{R}} \tilde{\theta}_x uv \, dx \, d\tau + 2c_0 \int_0^t \int_{\mathbb{R}} v F(x, \tau) \, dx \, d\tau
$$

$$
\leq (1 + c_0)\delta_0 + \varepsilon (1 - \alpha) \int_0^t \int_{\mathbb{R}} u^2 \, dx \, d\tau + \frac{1}{\varepsilon(1 - \alpha)} \int_0^t \int_{\mathbb{R}} v^2 \, dx \, d\tau
$$

$$
+ \varepsilon \alpha \int_0^t \int_{\mathbb{R}} u^2_x \, dx \, d\tau + \frac{c_0^2 v^2}{\varepsilon\alpha} \int_0^t \int_{\mathbb{R}} v^2 \, dx \, d\tau + \delta (1 - \alpha) \int_0^t \int_{\mathbb{R}} u^2 \, dx \, d\tau
$$

$$
+ \frac{1}{\delta(1 - \alpha)} \int_0^t \int_{\mathbb{R}} \tilde{\theta}_x^2 \, dx \, d\tau + 2c_0 \left(\|u_x\|_{L^\infty} + \|\bar{\psi}_x\|_{L^\infty}\right) \int_0^t \int_{\mathbb{R}} v^2 \, dx \, d\tau
$$

$$
+ 2c_0\|\tilde{\theta}_x\|_{L^\infty} \int_0^t \int_{\mathbb{R}} (u^2 + v^2) \, dx \, d\tau + c_0\delta \int_0^t \int_{\mathbb{R}} v^2 \, dx \, d\tau
$$

$$
+ \frac{c_0}{\delta} \int_0^t \int_{\mathbb{R}} F^2(x, \tau) \, dx \, d\tau.
$$

Employing Lemma 2.2 and (3.9), we have from the above inequality

$$
\int_{\mathbb{R}} (u^2 + c_0v^2) \, dx + \left(2 - \varepsilon - \delta)(1 - \alpha) - 2C_\delta\right) \int_0^t \int_{\mathbb{R}} u^2 \, dx \, d\tau
$$

$$
+ (2 - \varepsilon)\alpha \int_0^t \int_{\mathbb{R}} u^2_x \, dx \, d\tau + \left(2(1 - \alpha) - \frac{c_0 v^2}{\varepsilon\alpha} - C(\delta_1 + \delta)\right) \int_0^t \int_{\mathbb{R}} c_0v^2 \, dx \, d\tau
$$
\[ + \left\{ 2c_0 \alpha - \frac{1}{\varepsilon(1 - \alpha)} \right\} \int_0^t \int_\mathbb{R} v^2_x \, dx \, d\tau \]
\[ \leq C\delta_0 + C\delta + \frac{c_0}{\delta} \int_0^t \int_\mathbb{R} F^2(x, \tau) \, dx \, d\tau, \]  
(3.13)

where we have used the following inequality:
\[ \frac{1}{\delta(1 - \alpha)} \int_0^t \int_\mathbb{R} \bar{\theta}^2_x \, dx \, d\tau = \frac{1}{\delta(1 - \alpha)} \int_0^t \| \bar{\theta}_x \|^2 \, d\tau \leq \frac{1}{\delta(1 - \alpha)} C\delta^2 \int_0^t e^{-2(1 - \alpha)\tau} \, d\tau \]
\[ \leq C\delta. \]  

Now we estimate the last term in the right side of inequality (3.13). In fact from Lemma 2.2, we have by Cauchy–Schwarz inequality
\[ \frac{c_0}{\delta} \int_0^t \int_\mathbb{R} F^2(x, \tau) \, dx \, d\tau = \frac{c_0}{\delta} \int_0^t \int_\mathbb{R} (\bar{\psi}^2_x + 2\bar{\psi} \bar{\theta}_x)^2 \, dx \, d\tau \]
\[ \leq \frac{C}{\delta} \int_0^t \int_\mathbb{R} (\bar{\psi}^2_x + \bar{\theta}^2_x) \, dx \, d\tau \]
\[ \leq \frac{C}{\delta} \int_0^t \int_\mathbb{R} \bar{\psi}^2_x \, dx \, d\tau \leq C\delta. \]  
(3.14)

Thus, (3.13) and (3.14) give
\[ \int_\mathbb{R} (u^2 + c_0 v^2) \, dx + \left\{ (2 - \varepsilon - \delta)(1 - \alpha) - C\delta \right\} \int_0^t \int_\mathbb{R} u^2 \, dx \, d\tau \]
\[ + (2 - \varepsilon)\alpha \int_0^t \int_\mathbb{R} u^2_x \, dx \, d\tau + \left\{ 2(1 - \alpha) - \frac{c_0}{\delta} + C(\delta + \delta_1) \right\} \int_0^t \int_\mathbb{R} c_0 v^2 \, dx \, d\tau \]
\[ + \left\{ 2c_0 \alpha - \frac{1}{\varepsilon(1 - \alpha)} \right\} \int_0^t \int_\mathbb{R} v^2_x \, dx \, d\tau \]
\[ \leq C(\delta_0 + \delta), \]  
(3.15)

which implies, with the help of (3.10),
\[ \int_\mathbb{R} (u^2 + v^2) \, dx + \int_0^t \int_\mathbb{R} (u^2 + v^2) \, dx \, d\tau + \int_0^t \int_\mathbb{R} (u^2 + v^2) \, dx \, d\tau \]
\[ \leq C(\delta_0 + \delta), \]  
(3.16)
provided that $\delta$ and $\delta_1$ are sufficiently small. This proves Lemma 3.3.

Lemma 3.4. Let the assumptions in Theorem 3.1 hold. Then the solution $(u(x, t), v(x, t))$ of (3.2), (3.3) obtained in Lemma 3.2 satisfies for any $\nu < 4\alpha(1 - \alpha)$,

$$\int_{\mathbb{R}} (u_x^2 + v_x^2) \, dx + \int_0^t \int_{\mathbb{R}} (u_{xx}^2 + v_{xx}^2) \, dx \, d\tau \leq C(\delta + \delta_0),$$  \hspace{1cm} (3.17)

provided that $\delta$ and $\delta_1$ are sufficiently small.

Proof. First, we multiply the first equation of (3.2) by $(−2u_{xx})$ and the second equation of (3.2) by $(−2c_0v_{xx})$ respectively, and add the resulting equations together. After all these, we take integration over $(x, t) \in \mathbb{R} \times (0, t)$ and reach

$$\int_{\mathbb{R}} (u_x^2 + c_0v_x^2) \, dx + 2(1 - \alpha) \int_0^t \int_{\mathbb{R}} (u_x^2 + c_0v_x^2) \, dx \, d\tau$$

$$+ 2\alpha \int_0^t \int_{\mathbb{R}} (u_{xx}^2 + c_0v_{xx}^2) \, dx \, d\tau$$

$$= \|u_0\|^2 + c_0\|v_0\|^2 - 2 \int_0^t \int_{\mathbb{R}} u_x v_x \, dx \, d\tau + 2c_0 \int_0^t \int_{\mathbb{R}} v_x u_{xx} \, dx \, d\tau$$

$$- 2 \int_0^t \int_{\mathbb{R}} u_x \bar{\theta}_{xx} \, dx \, d\tau + 2c_0 \int_0^t \int_{\mathbb{R}} u_x v_x^2 \, dx \, d\tau + 2c_0 \int_0^t \int_{\mathbb{R}} \bar{\psi}_x v_x^2 \, dx \, d\tau$$

$$- 4c_0 \int_0^t \int_{\mathbb{R}} \bar{\theta}_x u v_{xx} \, dx \, d\tau - 2c_0 \int_0^t \int_{\mathbb{R}} F(x, \tau) v_{xx} \, dx \, d\tau$$

$$\leq (1 + c_0)\delta_0 + \varepsilon(1 - \alpha) \int_0^t \int_{\mathbb{R}} u_x^2 \, dx \, d\tau + \frac{1}{\varepsilon(1 - \alpha)} \int_0^t \int_{\mathbb{R}} v_{xx}^2 \, dx \, d\tau$$

$$+ \varepsilon\alpha \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, d\tau + \frac{c_0^2}{\varepsilon\alpha} \int_0^t \int_{\mathbb{R}} v_{xx}^2 \, dx \, d\tau + \int_0^t \int_{\mathbb{R}} u_x^2 \, dx \, d\tau$$

$$+ \int_0^t \int_{\mathbb{R}} \bar{\theta}_{xx}^2 \, dx \, d\tau + 2c_0(\|u_x\|_{L^\infty} + \|\bar{\psi}_x\|_{L^\infty}) \int_0^t \int_{\mathbb{R}} v_{xx}^2 \, dx \, d\tau$$

$$+ 2c_0\|ar{\theta}_x\|_{L^\infty} \int_0^t \int_{\mathbb{R}} (u_x^2 + v_{xx}^2) \, dx \, d\tau + c_0\delta \int_0^t \int_{\mathbb{R}} v_{xx}^2 \, dx \, d\tau$$


\[
+ \frac{c_0}{\delta} \int_0^t \int_{\mathbb{R}} F^2(x, \tau) \, dx \, d\tau.
\]

(3.18)

By Lemmas 2.2 and 3.3, we have from (3.14),
\[
\int_{\mathbb{R}} (u_x^2 + c_0 v_x^2) \, dx + (2 - \varepsilon)\alpha \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dx \, d\tau
\]
\[
+ \left\{ 2c_0\alpha - \frac{1}{\varepsilon(1 - \alpha)} - C\delta \right\} \int_0^t \int_{\mathbb{R}} v_{xx}^2 \, dx \, d\tau
\]
\[
\leq C(\delta_0 + \delta).
\]

(3.19)

Thus the proof of Lemma 3.4 is completed by (3.10) and (3.19) provided that \(\delta\) and \(\delta_1\) are sufficiently small.

Lemma 3.5. Suppose that \((u(x,t), v(x,t))\) is a solution to (3.2), (3.3) obtained in Lemma 3.2 under the assumptions in Theorem 3.1, then for any \(\nu < 4\alpha(1 - \alpha)\), we have
\[
\int_{\mathbb{R}} (u_{xx}^2 + v_{xx}^2) \, dx + \int_0^t \int_{\mathbb{R}} (u_{xxx}^2 + v_{xxx}^2) \, dx \, d\tau \leq C(\delta + \delta_0),
\]

(3.20)

provided that \(\delta\) and \(\delta_1\) are sufficiently small.

Proof. Differentiating (3.2) twice with respect to \(x\), multiplying the results by \(2u_{xx}\) and \(2c_0v_{xx}\), respectively, integrating the resulting equation with respect to \((x,t)\) over \(\mathbb{R} \times (0,t)\), we get
\[
\int_{\mathbb{R}} (u_{xx}^2 + c_0 v_{xx}^2) \, dx + 2(1 - \alpha) \int_0^t \int_{\mathbb{R}} (u_{xx}^2 + c_0 v_{xx}^2) \, dx \, d\tau
\]
\[
+ 2\alpha \int_0^t \int_{\mathbb{R}} (u_{xxx}^2 + c_0 v_{xxx}^2) \, dx \, d\tau
\]
\[
= \|u_{0xx}\|^2 + c_0\|v_{0xx}\|^2 - 2 \int_0^t \int_{\mathbb{R}} u_{xx} v_{xxx} \, dx \, d\tau + 2v_0 \int_0^t \int_{\mathbb{R}} v_{xx} u_{xxx} \, dx \, d\tau
\]
\[
- 2 \int_0^t \int_{\mathbb{R}} u_{xx} \bar{\theta}_{xxx} \, dx \, d\tau + 4c_0 \int_0^t \int_{\mathbb{R}} (u v_x)_{xxx} \, dx \, d\tau
\]
\[
+ 4c_0 \int_0^t \int_{\mathbb{R}} (\bar{\psi} v_x)_{xxx} \, dx \, d\tau + 4c_0 \int_0^t \int_{\mathbb{R}} (\bar{\theta}_x v_x)_{xxx} \, dx \, d\tau
\]
\[ + 2c_0 \int_0^t \int_\mathbb{R} F_{xx}(x, \tau)v_{xx} \, dx \, d\tau \]
\[ \leq (1 + c_0) \delta_0 + \varepsilon (1 - \alpha) \int_0^t \int_\mathbb{R} u_{xx}^2 \, dx \, d\tau + \frac{1}{\varepsilon (1 - \alpha)} \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau \]
\[ + \varepsilon \alpha \int_0^t \int_\mathbb{R} u_{xxx}^2 \, dx \, d\tau + \frac{c_0^2 \varepsilon^2}{\varepsilon \alpha} \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau + \int_0^t \int_\mathbb{R} u_{xx}^2 \, dx \, d\tau \]
\[ + \int_0^t \int_\mathbb{R} \bar{\theta}_{xxx}^2 \, dx \, d\tau - 4c_0 \int_0^t \int_\mathbb{R} (uv_x)_x v_{xxx} \, dx \, d\tau \]
\[ - 4c_0 \int_0^t \int_\mathbb{R} (\bar{\psi} v_x)_x v_{xxx} \, dx \, d\tau - 4c_0 \int_0^t \int_\mathbb{R} (\bar{\theta}_x u)_x v_{xxx} \, dx \, d\tau \]
\[ - 2c_0 \int_0^t \int_\mathbb{R} F_x(x, \tau)v_{xxx} \, dx \, d\tau. \]  
(3.21)

Shuffling the terms, we get by Lemmas 2.2 and 3.4,
\[ \int_\mathbb{R} \left( u_{xx}^2 + c_0 v_{xxx}^2 \right) \, dx + (2 - \varepsilon) \alpha \int_0^t \int_\mathbb{R} u_{xxx}^2 \, dx \, d\tau \]
\[ + \left\{ 2c_0 \alpha - \frac{1}{\varepsilon (1 - \alpha)} \right\} \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau \]
\[ \leq C (\delta_0 + \delta) - 4c_0 \int_0^t \int_\mathbb{R} (uv_x)_x v_{xxx} \, dx \, d\tau - 4c_0 \int_0^t \int_\mathbb{R} (\bar{\psi} v_x)_x v_{xxx} \, dx \, d\tau \]
\[ - 4c_0 \int_0^t \int_\mathbb{R} (\bar{\theta}_x u)_x v_{xxx} \, dx \, d\tau - 2c_0 \int_0^t \int_\mathbb{R} F_x(x, \tau)v_{xxx} \, dx \, d\tau. \]  
(3.22)

Next we are devoted to estimate the terms in the right side of (3.22) as follows.
First, we obtain from (3.9), Lemmas 3.3 and 3.4 by Cauchy–Schwarz inequality
\[ - 4c_0 \int_0^t \int_\mathbb{R} (uv_x)_x v_{xxx} \, dx \, d\tau \]
\[ \leq c_0 \delta_1^2 \int_0^t \int_\mathbb{R} u_{xxx}^2 \, dx \, d\tau + \frac{8c_0}{\delta_1^2} \int_0^t \int_\mathbb{R} (u_x^2 v_x^2 + u_{xx}^2 v_{xx}^2) \, dx \, d\tau \]
\[ \leq c_0 \delta^2 \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau + \frac{8c_0}{\delta^1} \left( \|u_x\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \right) \int_0^t \int_\mathbb{R} (v_x^2 + v_{xx}^2) \, dx \, d\tau \]
\[ \leq c_0 \delta^2 \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau + C(\delta + \delta_0). \quad (3.23) \]

Similarly, we have from Lemmas 2.2 and 3.4,
\[ -4c_0 \int_0^t \int_\mathbb{R} (\bar{\psi} v_x)_x v_{xxx} \, dx \, d\tau \]
\[ = -4c_0 \int_0^t \int_\mathbb{R} \bar{\psi}_x v_x v_{xxx} \, dx \, d\tau - 4c_0 \int_0^t \int_\mathbb{R} \bar{\psi} v_{xxx} v_{xxx} \, dx \, d\tau \]
\[ \leq c_0 \delta \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau + \frac{4c_0}{\delta} \|\bar{\psi}_x\|_{L^\infty}^2 \int_0^t \int_\mathbb{R} v_x^2 \, dx \, d\tau + 2c_0 \int_0^t \int_\mathbb{R} \bar{\psi}_x v_{xx}^2 \, dx \, d\tau \]
\[ \leq c_0 \delta \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau + C(\delta + \delta_0), \quad (3.24) \]

and
\[ -4c_0 \int_0^t \int_\mathbb{R} (\bar{\theta}_x u)_x v_{xxx} \, dx \, d\tau \leq c_0 \delta \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau + C(\delta + \delta_0). \quad (3.25) \]

In addition, applying Lemmas 2.2, 3.3 and 3.4, we derive by Cauchy–Schwarz inequality
\[ -2c_0 \int_0^t \int_\mathbb{R} F_{x}(x) \, v_{xxx} \, dx \, d\tau = -2c_0 \int_0^t \int_\mathbb{R} (v \bar{\psi}_x + 2\bar{\psi} \bar{\theta}_x)_x v_{xxx} \, dx \, d\tau \]
\[ \leq c_0 \delta \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau + \frac{c_0}{\delta} \int_0^t \int_\mathbb{R} (v \bar{\psi}_{xx} + 2(\bar{\psi} \bar{\theta}_x)_x)^2 \, dx \, d\tau \]
\[ \leq c_0 \delta \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau + \frac{C}{\delta} \int_0^t \int_\mathbb{R} \bar{\psi}_{xx}^2 \, dx \, d\tau + \frac{C}{\delta} \int_0^t \int_\mathbb{R} \bar{\psi}_{xx}^2 \, dx \, d\tau \]
\[ + \frac{C}{\delta} \int_0^t \int_\mathbb{R} \bar{\psi}_{xx}^2 \, dx \, d\tau \]
\[ \leq c_0 \delta \int_0^t \int_\mathbb{R} v_{xxx}^2 \, dx \, d\tau + C(\delta + \delta_0). \quad (3.26) \]
Substituting (3.23)–(3.26) into (3.22), we get
\[
\int_{\mathbb{R}} \left( u_{xx}^2 + c_0 v_{xx}^2 \right) \, dx + (2 - \varepsilon) \alpha \int_0^t \int_{\mathbb{R}} u_{xxx}^2 \, dx \, d\tau \\
+ \left\{ 2c_0 \alpha - \frac{1}{\varepsilon (1 - \alpha)} - C(\delta_1^2 + \delta) \right\} \int_0^t \int_{\mathbb{R}} v_{xxx}^2 \, dx \, d\tau \\
\leq C(\delta + \delta_0),
\]
which proves (3.20) provided that \( \delta \) and \( \delta_1 \) are sufficiently small. \( \square \)

Thus the combination of Lemmas 3.3–3.5 implies (3.5).

Finally, we have to show that the a priori assumption (3.8) can be closed. Since, under this a priori assumption (3.8), we have deduced that (3.5) holds provided that \( \delta \) and \( \delta_1 \) are sufficiently small. Therefore the assumption (3.8) is always true provided that \( \delta \) and \( \delta_0 \) are sufficiently small.

Now we turn to show that (3.6) is true. To do this, we introduce the following lemma.

**Lemma 3.6.** If \( g(t) \geq 0, g(t) \in L^1(0, \infty) \) and \( g'(t) \in L^1(0, \infty) \), then \( g(t) \to 0 \) as \( t \to \infty \).

The proof of Lemma 3.6 can be found in [8, 11] and the details are omitted.

Taking \( g(t) = \| u_x(t) \|_2 \) in Lemma 3.6, we can conclude from (3.5) that \( g(t) \in L^1(0, \infty) \). Denote \( L^2 \)-inner product by \( \langle \cdot, \cdot \rangle \). By using the definition of \( L^2 \)-inner product and integrating by parts, we have \( g'(t) = 2(\alpha u_x, u_{xx}) = -2(\alpha u_x, u_{xx}) \). It is easy to verify from the proof of Lemma 3.4 that
\[
-\langle u_x, u_{xx} \rangle = \langle -u_x, u_{xx} \rangle = \left( 1 - \alpha \right) u_x + v_x - \alpha u_{xx} + \bar{\theta}_x, u_{xx} \in L^1(0, \infty).
\]
Hence, \( g'(t) \in L^1(0, \infty) \) which implies, by Lemma 3.6,
\[
\| u_x(t) \| \to 0 \quad \text{as} \quad t \to \infty. \tag{3.27}
\]

Applying Sobolev inequality, we have from (3.27) and (3.5),
\[
\sup |u(x, t)| \leq \left\| u(t) \right\|^{1/2} \left\| u_x(t) \right\|^{1/2} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.28}
\]

The same process is applied to \( v(x, t) \) so that
\[
\sup |v(x, t)| \leq \left\| v(t) \right\|^{1/2} \left\| v_{xx}(t) \right\|^{1/2} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.29}
\]

Similarly, taking \( g(t) = \| u_{xx}(t) \|^{2} \) and \( g(t) = \| v_{xx}(t) \|^{2} \), we have from (3.5) and Lemma 3.6,
\[
\| u_{xx}(t) \| \to 0 \quad \text{and} \quad \| v_{xx}(t) \| \to 0 \quad \text{as} \quad t \to \infty. \tag{3.30}
\]

So Sobolev inequality and (3.30) give
\[
\sup |u_x(t, t)| \leq \left\| u_x(t) \right\|^{1/2} \left\| u_{xx}(t) \right\|^{1/2} \to 0 \quad \text{as} \quad t \to \infty, \tag{3.31}
\]
\[ \sup |v_x(x,t)| \leq \|v_x(t)\|^{1/2} \|v_{xx}(t)\|^{1/2} \to 0 \quad \text{as } t \to \infty. \]  
(3.32)

Thus, (3.6) is proved by (3.28), (3.29), (3.31) and (3.32). The proof of Theorem 3.1 is completed.

4. Decay rates of solutions

In this section, we will study the decay rates of solutions to the Cauchy problem (3.2), (3.3) under a priori assumption

\[ \sum_{k=0}^{2} \|\partial^k_x u(t)\|^2 + \sum_{k=0}^{2} \|\partial^k_x v(t)\|^2 \leq e^{-lt}, \quad 0 < t < T, \]  
(4.1)

with

\[ l = \min \left\{ (2 - \epsilon)(1 - \alpha), 2(1 - \alpha) - \frac{c_0 v^2}{\kappa \alpha} \right\}, \]  
(4.2)

where \( \epsilon \) and \( c_0 \) are defined by (3.10).

By Sobolev inequality, we have from (4.1),

\[ \|(u, u_x, v, v_x)\|_{L^\infty} \leq e^{-lt/2}, \]  
(4.3)

which will be used later.

Moreover, we list the following Gronwall’s inequality which is used in the following text.

**Lemma 4.1** (Gronwall’s inequality). Let \( \eta(\cdot) \) be a nonnegative continuous function on \([0, \infty)\), which satisfies the differential inequality

\[ \eta'(t) + \lambda \eta(t) \leq \omega(t), \]

where \( \lambda \) is a positive constant and \( \omega(t) \) is a nonnegative continuous function on \([0, \infty)\). Then

\[ \eta(t) \leq \left( \eta(0) + \int_0^t e^{\lambda \tau} \omega(\tau) \, d\tau \right) e^{-\lambda t}. \]

Now we can state the main results of decay rates of solutions.

**Theorem 4.2.** Suppose that \( (u(x,t), v(x,t)) \) is a solution to problem (3.2), (3.3) under the assumptions imposed in Theorem 3.1, then when \( v < 4\alpha(1 - \alpha) \), we have for any \( t \in [0, T] \),

\[ \|\partial^k_x u(t)\|^2 + \|\partial^k_x v(t)\|^2 \leq C(\delta + \delta_0)^{1/4} e^{-lt}, \quad k = 0, 1, 2, \]  
(4.4)

provided that \( \delta \) and \( \delta_0 \) are sufficiently small, where \( l \) is defined by (4.2).
Proof. The proofs are divided into three steps.

First, \((3.2) \times 2u + (3.2) \times 2c_0v\) and integrating the resulting identities over \(x \in \mathbb{R}\), we reach by Cauchy–Schwarz inequality from Lemma 2.2,

\[
\frac{d}{dt} \int \left( u^2 + c_0 v^2 \right) dx + 2(1 - \alpha) \int \left( u^2 + c_0 v^2 \right) dx + 2\alpha \int \left( u^2 + c_0 v^2 \right) dx
\]

\[
= -2 \int u \bar{\theta} x dx + 2c_0 \int v \bar{\theta} x dx - 2 \int 2u \bar{\theta} x dx - 2c_0 \int (u_x + \bar{\psi} v) v^2 dx
\]

\[
+ 4c_0 \int \bar{\theta} x u v dx + 2c_0 \int v F(x, t) dx
\]

\[
\leq \varepsilon (1 - \alpha) \int u^2 dx + \frac{1}{\varepsilon (1 - \alpha)} \int v^2 dx + 2c_0 \int u_x v^2 dx + 2c_0 \int \bar{\theta} x v^2 dx
\]

\[
- 2 \int u \bar{\theta} x dx + 2c_0 \int \bar{\theta} x \| \bar{\theta} x \|_{L^\infty} \int u^2 dx + 2c_0 (\| \bar{\psi} x \|_{L^\infty} + \| \bar{\theta} x \|_{L^\infty}) \int v^2 dx
\]

\[
- 2c_0 \int u_x v^2 dx + 2c_0 \int v F(x, t) dx.
\]  

(4.5)

After shuffling the terms, we have by Lemma 2.2,

\[
\frac{d}{dt} \int \left( u^2 + c_0 v^2 \right) dx + (2 - \varepsilon)(1 - \alpha) \int u^2 dx + \left\{ 2(1 - \alpha) - \frac{c_0 v^2}{\varepsilon \alpha} \right\} \int c_0 v^2 dx
\]

\[
+ (2 - \varepsilon) \alpha \int u_x^2 dx + \left\{ 2c_0 \alpha - \frac{1}{\varepsilon (1 - \alpha)} \right\} \int v_x^2 dx
\]

\[
\leq C \delta e^{-(1 - \alpha)t} \int \left( u^2 + c_0 v^2 \right) dx - 2 \int u \bar{\theta} x dx - 2c_0 \int u_x v^2 dx
\]

\[
+ 2c_0 \int v F(x, t) dx.
\]  

(4.6)

Next, we will estimate the terms in the right side of (4.6).

In fact, we have from the assumption (4.1) and Lemma 2.2 by employing Cauchy–Schwarz inequality

\[
-2 \int u \bar{\theta} x dx \leq \delta e^{\|/2-(1-\alpha)t} \int u^2 dx + \frac{1}{\delta} e^{-\|/2-(1-\alpha)t} \int \bar{\theta}^2 dx
\]

\[
\leq \delta e^{\|/2-(1-\alpha)t} e^{-t} + C \delta e^{\|/2-(1-\alpha)t} e^{-2(1-\alpha)t}
\]

\[
\leq C \delta e^{\|/2+(1-\alpha)t}.
\]  

(4.7)
Moreover, we get from (3.5) by Sobolev inequality
\[
\| (u, u_x, v, v_x) \|_{L^\infty} \leq C(\delta + \delta_0)^{1/2}.
\] (4.8)

Thus we derive by Cauchy–Schwarz inequality from (4.1), (4.3) and (4.8),
\[
-2c_0 \int_R u_x v^2 \, dx \leq c_0 \left( \int_R u_x^2 |v| \, dx + \int_R |v|^3 \, dx \right) \leq c_0 \|v\|_{L^\infty} \int_R (u_x^2 + v^2) \, dx
\]
\[
= c_0 \|v\|_{L^2}^{1/2} \|v\|_{L^\infty}^{1/2} \int_R (u_x^2 + v^2) \, dx \leq C(\delta + \delta_0)^{1/4} e^{-lt/4} e^{-lt}
\]
\[
= C(\delta + \delta_0)^{1/4} e^{-lt/4}.
\] (4.9)

In addition, we deduce from Lemma 2.2 and Cauchy–Schwarz inequality
\[
2c_0 \int_R vF(x, t) \, dx \leq \delta e^{1/2 - (1-\alpha)t} \int_R v^2 \, dx + \frac{c_0^2}{\delta} e^{-[l/2 - (1-\alpha)t]} \int_R F^2(x, t) \, dx
\]
\[
\leq \delta e^{1/2 - (1-\alpha)t} e^{-lt} + \frac{c_0}{\delta} e^{-[l/2 - (1-\alpha)t]} \int_R \left( \psi_x^2 + \tilde{\theta}_x^2 \right) \, dx
\]
\[
\leq \delta e^{1/2 - (1-\alpha)t} e^{-lt} + \frac{c_0}{\delta} e^{-[l/2 - (1-\alpha)t]} \|g^2 e^{-2(1-\alpha)t} \|
\]
\[
\leq C\delta e^{-l/2 + (1-\alpha)t}.
\] (4.10)

Thus, we get from (4.6), (4.7), (4.9), (4.10) and (4.1),
\[
\frac{d}{dt} \int_R (u^2 + c_0 v^2) \, dx + (2 - \varepsilon)(1 - \alpha) \int_R u_x^2 \, dx + \left( 2(1 - \alpha) - \frac{c_0 v^2}{\varepsilon \alpha} \right) \int_R c_0 v^2 \, dx
\]
\[
\leq C \delta e^{-[(l+(1-\alpha))t]} + C \delta e^{-[l/2 + (1-\alpha)t]} + C(\delta + \delta_0)^{1/4} e^{-lt/4}
\]
\[
\leq C(\delta + \delta_0)^{1/4} e^{-lt/4} + C\delta e^{-l/(2 + (1-\alpha)t)}.
\] (4.11)

Recalling the definition (4.2) of \( l \), we have from (4.11),
\[
\frac{d}{dt} \int_R (u^2 + c_0 v^2) \, dx + l \int_R (u^2 + c_0 v^2) \, dx
\]
\[
\leq C(\delta + \delta_0)^{1/4} e^{-lt/4} + C\delta e^{-l/(2 + (1-\alpha)t)}.
\] (4.12)

Noticing that \( l/2 < 1 - \alpha \), we obtain from Lemma 4.1,
\[
\int_R (u^2 + c_0 v^2) \, dx
\]
\[
\leq \left\{ C\delta_0 + C(\delta + \delta_0)^{1/4} \int_0^t e^{-lt/4} e^{l\tau} d\tau + C\delta \int_0^t e^{-[(l/2 + (1-\alpha))t]} e^{l\tau} d\tau \right\} e^{-lt}
\]
\[
\leq C(\delta + \delta_0)^{1/4} e^{-lt},
\] (4.13)
which implies (4.4) for \( k = 0 \).

Similarly, for \( k = 1 \), if we multiply (3.2)1 by \((-2u_{xx})\) and (3.2)2 by \((-2c_0v_{xx})\), add and take integration of the resulting identities over \( x \in \mathbb{R} \), then we get by conducting the same procedure to the proofs of inequality (4.13),

\[
\int_{\mathbb{R}} (u_{xx}^2 + c_0v_{xx}^2) \, dx \leq C(\delta + \delta_0)^{1/4}e^{-lt},
\]

which proves (4.4) for \( k = 1 \).

Finally, we shall show that (4.4) is true for \( k = 2 \). In fact, differentiating (3.2) twice with respect to \( x \), multiplying the results by \( 2u_{xx} \) and \( 2c_0v_{xx} \), respectively, integrating the resulting equation over \( x \in \mathbb{R} \), we arrive at

\[
\frac{d}{dt} \int_{\mathbb{R}} (u_{xx}^2 + c_0v_{xx}^2) \, dx + 2(1 - \alpha) \int_{\mathbb{R}} (u_{xx}^2 + c_0v_{xx}^2) \, dx + 2\alpha \int_{\mathbb{R}} (u_{xxx}^2 + c_0v_{xxx}^2) \, dx
\]

\[= -2 \int_{\mathbb{R}} u_{xx}v_{xxx} \, dx + 2c_0 \int_{\mathbb{R}} v_{xx}u_{xxx} \, dx - 2 \int_{\mathbb{R}} u_{xx}\tilde{\theta}_{xxx} \, dx
\]

\[+ 4c_0 \int_{\mathbb{R}} (uv_x)_{xx}v_{xx} \, dx + 4c_0 \int_{\mathbb{R}} (\tilde{\psi}_x)_{xx}v_{xx} \, dx + 4c_0 \int_{\mathbb{R}} (\tilde{\theta}_xu)_{xx}v_{xx} \, dx
\]

\[+ 2c_0 \int_{\mathbb{R}} F_{xx}(x,t)v_{xx} \, dx\]

\[\leq \varepsilon(1 - \alpha) \int_{\mathbb{R}} u_{xx}^2 \, dx + \frac{1}{\varepsilon(1 - \alpha)} \int_{\mathbb{R}} v_{xx}^2 \, dx + \varepsilon\alpha \int_{\mathbb{R}} u_{xxx}^2 \, dx + \frac{c_0^2v^2}{\varepsilon\alpha} \int_{\mathbb{R}} v_{xx}^2 \, dx
\]

\[- 2 \int_{\mathbb{R}} u_{xx}\tilde{\theta}_{xxx} \, dx - 4c_0 \int_{\mathbb{R}} (uv_x)_{xx}v_{xx} \, dx - 4c_0 \int_{\mathbb{R}} (\tilde{\psi}_x)_{xx}v_{xx} \, dx
\]

\[- 4c_0 \int_{\mathbb{R}} (\tilde{\theta}_xu)_{xx}v_{xx} \, dx + 2c_0 \int_{\mathbb{R}} F_{xx}(x,t)v_{xx} \, dx \]

Hence

\[
\frac{d}{dt} \int_{\mathbb{R}} (u_{xx}^2 + c_0v_{xx}^2) \, dx + (2 - \varepsilon)(1 - \alpha) \int_{\mathbb{R}} u_{xx}^2 \, dx
\]

\[+ \left\{ 2(1 - \alpha) - \frac{c_0^2v^2}{\varepsilon\alpha} \right\} \int_{\mathbb{R}} c_0v_{xx}^2 \, dx + (2 - \varepsilon)\alpha \int_{\mathbb{R}} u_{xxx}^2 \, dx
\]

\[+ \left\{ 2c_0\alpha - \frac{1}{\varepsilon(1 - \alpha)} \right\} \int_{\mathbb{R}} v_{xxx}^2 \, dx
\]

\[\leq -2 \int_{\mathbb{R}} u_{xx}\tilde{\theta}_{xxx} \, dx - 4c_0 \int_{\mathbb{R}} (uv_x)_{xx}v_{xx} \, dx - 4c_0 \int_{\mathbb{R}} (\tilde{\psi}_x)_{xx}v_{xx} \, dx
\]

\[+ 2c_0 \int_{\mathbb{R}} F_{xx}(x,t)v_{xx} \, dx.
\]
Next, we are going to give the estimates of the terms in the right side of (4.16).

Indeed, using Cauchy–Schwarz inequality, we have from (4.1) and Lemma 2.2,

\[-2 \int u_{xx} \bar{\theta}_{xxx} \, dx \leq \delta e^{[-l/(2-(1-\alpha))]} \int u_{xx}^2 \, dx + \frac{1}{\delta} e^{[-l/(2-(1-\alpha))]} \int \bar{\theta}_{xxx}^2 \, dx\]

\[\leq \delta e^{[-l/(2-(1-\alpha))]} e^{-lt} + \frac{1}{\delta} e^{[-l/(2-(1-\alpha))]} C \delta e^{[-2(1-\alpha)t]}\]

\[\leq C \delta e^{[-l/(2+(1-\alpha))]} t.\]  

(4.17)

For the simplicity of notation, we define \(\lambda = 2c_0\alpha - 1/(\delta(1-\alpha))\). Then we derive from (4.3) by Cauchy–Schwarz inequality

\[-4c_0 \int (u v_x)_x v_{xxx} \, dx \leq \frac{1}{4} \int v_{xxx}^2 \, dx + \frac{32c_0^2}{\lambda} \int (u_x^2 v_x^2 + u^2 v_{xx}^2) \, dx\]

\[\leq \frac{1}{4} \lambda \int v_{xxx}^2 \, dx + \frac{32c_0^2}{\lambda} \|u_x\|_\infty^2 \int v_x^2 \, dx + \frac{32c_0^2}{\lambda} \|u\|_\infty^2 \int v_{xx}^2 \, dx\]

\[\leq \frac{1}{4} \lambda \int v_{xxx}^2 \, dx + C e^{-lt} \int (v_x^2 + v_{xx}^2) \, dx.\]  

(4.18)

Moreover, we have from Lemma 2.2 with Cauchy–Schwarz inequality

\[-4c_0 \int (\bar{\psi} v_x)_x v_{xxx} \, dx \leq \frac{1}{4} \int v_{xxx}^2 \, dx + \frac{32c_0^2}{\lambda} \int (\bar{\psi}_x^2 v_x^2 + \bar{\psi}^2 v_{xx}^2) \, dx\]

\[\leq \frac{1}{4} \lambda \int v_{xxx}^2 \, dx + \frac{32c_0^2}{\lambda} \|\bar{\psi}_x\|_\infty^2 \int v_x^2 \, dx + \frac{32c_0^2}{\lambda} \|\bar{\psi}\|_\infty^2 \int v_{xx}^2 \, dx\]

\[\leq \frac{1}{4} \lambda \int v_{xxx}^2 \, dx + C e^{-2(1-\alpha)t} \int (v_x^2 + v_{xx}^2) \, dx.\]  

(4.19)

Similarly, we have

\[-4c_0 \int (\bar{\theta} u)_x v_{xxx} \, dx \leq \frac{1}{4} \int v_{xxx}^2 \, dx + \frac{32c_0^2}{\lambda} \int (\bar{\theta}_x u_x^2 + \bar{\theta}^2 u_x^2) \, dx\]

\[\leq \frac{1}{4} \lambda \int v_{xxx}^2 \, dx + \frac{32c_0^2}{\lambda} \|\bar{\theta}_x\|_\infty^2 \int u_x^2 \, dx + \frac{32c_0^2}{\lambda} \|\bar{\theta}\|_\infty^2 \int u_x^2 \, dx\]

\[\leq \frac{1}{4} \lambda \int v_{xxx}^2 \, dx + C e^{-2(1-\alpha)t} \int (u_x^2 + u_{xx}^2) \, dx.\]  

(4.20)
On the other hand, we have by Lemma 2.2, Cauchy–Schwarz inequality and (4.1),

\[
2c_0 \int_{\mathbb{R}} F_{xx}(x,t) v_{xx} \, dx \leq \delta e^{l/2-(1-\alpha)t} \int_{\mathbb{R}} v_{xx}^2 \, dx + \frac{c_0^2}{\delta} e^{-l/2-(1-\alpha)t} \int_{\mathbb{R}} F_{xx}^2(x,t) \, dx \\
\leq \delta e^{l/2-(1-\alpha)t} e^{-lt} + \frac{C}{\delta} e^{-l/2-(1-\alpha)t} \int_{\mathbb{R}} \left( \psi_{xxx}^2 + \left[ (\psi \theta_x)_{xx} \right]^2 \right) \, dx \\
\leq \delta e^{-l/2+(1-\alpha)t} + \frac{C}{\delta} e^{-l/2-(1-\alpha)t} \int_{\mathbb{R}} \left( \psi_{xxx}^2 + \psi_{xx}^2 \theta_x^2 + \psi_x^2 \theta_{xx}^2 + \psi_{xxx}^2 \theta_{xxx}^2 \right) \, dx \\
\leq \delta e^{-l/2+(1-\alpha)t} + \frac{C}{\delta} e^{-l/2-(1-\alpha)t} l^2 e^{-2(1-\alpha)t} \leq C \delta e^{-l/2+(1-\alpha)t}.
\] (4.21)

Substituting (4.17)–(4.21) into (4.16), we get from (3.5),

\[
\frac{d}{dt} \int_{\mathbb{R}} (u_{xx}^2 + c_0 v_{xx}^2) \, dx + (2 - \varepsilon) (1 - \alpha) \int_{\mathbb{R}} u_{xx}^2 \, dx \\
+ \left\{ 2(1 - \alpha) - \frac{c_0^2}{\varepsilon \alpha} \right\} \int_{\mathbb{R}} c_0 v_{xx}^2 \, dx + (2 - \varepsilon) \alpha \int_{\mathbb{R}} u_{xxx}^2 \, dx + \frac{1}{4} \lambda \int_{\mathbb{R}} v_{xxx}^2 \, dx \\
\leq C \delta e^{-l/2+(1-\alpha)t} + C e^{-2(1-\alpha)t} \int_{\mathbb{R}} (u_x^2 + u_{xx}^2 + \psi^2 + v_{xx}^2) \, dx \\
+ C e^{-lt} \int_{\mathbb{R}} (v_x^2 + v_{xx}^2) \, dx \\
\leq C \delta e^{-l/2+(1-\alpha)t} + C (\delta + \delta_0) e^{-2(1-\alpha)t} + C e^{-lt} \int_{\mathbb{R}} (v_x^2 + v_{xx}^2) \, dx.
\] (4.22)

Recalling the definition (4.2) of \( l \), we obtain from (3.10) and (4.22),

\[
\frac{d}{dt} \int_{\mathbb{R}} (u_{xx}^2 + c_0 v_{xx}^2) \, dx + l \int_{\mathbb{R}} (u_{xx}^2 + c_0 v_{xx}^2) \, dx \\
\leq C \delta e^{-l/2+(1-\alpha)t} + C (\delta + \delta_0) e^{-2(1-\alpha)t} + C e^{-lt} \int_{\mathbb{R}} (v_x^2 + v_{xx}^2) \, dx.
\] (4.23)

Noticing \( l/2 < 1 - \alpha \), we easily deduce from (4.23) with the help of Lemma 4.1 and (3.5),

\[
\int_{\mathbb{R}} (u_{xx}^2 + c_0 v_{xx}^2) \, dx \\
\leq C \delta_0 + C \delta \int_0^t e^{-l/2+(1-\alpha)\tau} \, d\tau.
\]
The proof of Theorem 4.2 is completed by (4.13), (4.14) and (4.24).

Finally, we verify that the a priori assumption (4.1) is reasonable. Indeed, under this a priori assumption, we have showed that (4.4) holds. Therefore, the assumption (4.1) is always true provided that \( \delta \) and \( \delta_0 \) are sufficiently small.

We remark that we can obtain the \( L^p \) (\( 1 \leq p \leq \infty \)) optimal decay rate with the help of Lemma 2.2 and Theorem 4.2. Its proof is completely similar to the one in [10] and the details are omitted. Precisely, we have the following result.

**Corollary 4.3.** Suppose that \( (u(x,t), v(x,t)) \) is a solution to problem (3.2), (3.3) under the assumptions imposed in Theorem 3.1. If the initial data \( (u_0(x), v_0(x)) \in L^1(\mathbb{R}, \mathbb{R}^2) \), \( v < 4\alpha(1-\alpha) \), and \( \delta \) and \( \delta_0 \) are sufficiently small, then the solution \( (u(x,t), v(x,t)) \) has the following optimal decay rate:

\[
\sum_{k=0}^{2} \| \partial_x^k u(t) \|_{L^p} + \sum_{k=0}^{2} \| \partial_x^k v(t) \|_{L^p} \leq C(1+t)^{-1/2 + 1/(2p)} e^{-\left(1-\alpha - \frac{\nu}{4\alpha}\right)t}.
\]  

(4.25)

**Remark 4.4.** It is noticed that

\[
\frac{l}{2} < 1 - \alpha - \frac{\nu}{4\alpha}.
\]  

(4.26)

In fact, let \( k = \nu/(4\alpha(1-\alpha)) \). Since \( \alpha < 4\alpha(1-\alpha) \), then \( k \in (0, 1) \). Hence (4.6) is equivalent to

\[
\frac{l}{2} < 1 - \alpha - k(1-\alpha).
\]  

(4.27)

In what follows we prove (4.27). If \( k < \varepsilon/2 \), then

\[
\frac{l}{2} \leq (1-\alpha) - \frac{\varepsilon}{2}(1-\alpha) < 1 - \alpha - k(1-\alpha)
\]  

due to (4.2). If \( k \geq \varepsilon/2 \), we have from (3.10),

\[
2c_0 \varepsilon = 8c_0 k \alpha (1-\alpha) \geq 4c_0 \varepsilon \alpha (1-\alpha) > 2 > 2k \geq \varepsilon.
\]

This leads to

\[
\frac{l}{2} \leq (1-\alpha) - \frac{c_0 \nu^2}{2\alpha \varepsilon} = (1-\alpha) - \frac{2c_0 \nu k (1-\alpha)}{\varepsilon} < (1-\alpha) - k(1-\alpha)
\]  

gain due to (4.2). Thus (4.27) holds, which implies (4.26).
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