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Strongly Regular (α, β)-Geometries

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In this paper we introduce strongly regular (α, β)-geometries. These are a class of geometries that generalise semipartial geometries. Like semipartial geometries the underlying point graph is strongly regular and this is part of the motivation for studying the geometries. In the paper several necessary conditions for existence are given. Strongly regular (α, β)-reguli are defined, and it is shown how they may be used to construct strongly regular (α, β)-geometries. This generalises similar results by J. A. Thas in (1980, *European J. Combin.* **1**, 189–192) constructing semipartial geometries. Several constructions of strongly regular (α, β)-geometries are given, and possible parameters of existence for small cases are listed. © 2001 Academic Press

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1. INTRODUCTION

A (finite) (α, β)-*geometry* with parameters (s, t) is an incidence structure $(\mathscr{P}, \mathscr{L})$ with $\mathscr{P}$ a finite non-empty set called *points* together with a family $\mathscr{L}$ of subsets of $\mathscr{P}$ called *lines*, incidence being containment such that

(i) Any two distinct points are incident with at most one line;

(ii) Each line is incident with exactly $s+1$ points;

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(iii) Each point is incident with exactly $t+1$ lines;

(iv) For any point $x \in \mathscr{P}$ and any line $L \in \mathscr{L}$ not containing x there are exactly α or β points on L collinear with x.

We call a line L not incident with a point x an α-*line* with respect to x if there are exactly α lines joining x to a point of L. β-lines are defined similarly. (α, β)-geometries were first defined in [9] where the problem of linear embedding was examined.

We say an (α, β)-geometry is *strongly regular* if there exist integers p and r such that the following conditions are satisfied.

(a) If points x and y are on some line L then there exist p lines on x that are α-lines with respect to y.

(b) If points x and y are not collinear then there exist r lines on x that are α-lines with respect to y.

We will call s, t, p and r the *parameters* of the strongly regular (α, β)-geometry.

A simple count shows that the number of lines of the geometry is $b = v(t+1)/(s+1)$, where v is the number of points.

LEMMA 1.1. *The point graph of a strongly regular (α, β)-geometry, $\alpha \neq \beta$, with parameters s, t, p, and r is strongly regular with $k = s(t+1)$, $\lambda = (\alpha-1)\,p + (\beta-1)(t-p) + s - 1$, $\mu = \alpha r + \beta(t+1-r)$ and $v = ((k(k-\lambda-1))/\mu) + k + 1$.*

Proof. For a given point x there are $t+1$ lines on that point, each of which contain s points not equal to x. Hence $k = s(t+1)$.

Consider points x and y of the graph that are not adjacent, i.e., they are not both on some line of the geometry. By definition, r of the lines on x are incident with α lines on y, giving αr points on these lines that are collinear with both x and y. Each of the remaining $t+1-r$ lines on x are incident with β lines on y, giving $\beta(t+1-r)$ points on these lines that are collinear with both x and y. Hence there are exactly $\mu = \alpha r + \beta(t+1-r)$ points collinear with both x and y.

Similarly, x and y adjacent gives $\lambda = (\alpha-1)\,p + (\beta-1)(t-p) + s - 1$, and the point graph is strongly regular. For given k, λ and μ the number of vertices of a strongly regular graph is well known to be given by the above formula. ∎

Note that for a strongly regular (α, β)-geometry, $\alpha \neq \beta$, with given λ and μ, p and r may be calculated by

$$p = \frac{(\beta-1)\,t + s - \lambda - 1}{\beta - \alpha} \qquad \text{and} \qquad r = \frac{\beta(t+1) - \mu}{\beta - \alpha}.$$

In fact it is easy to show that if an (α, β)-geometry has a point graph that is strongly regular, then the constants p and r exist as in (a) and (b) above and the geometry is strongly regular.

Remark. If $\alpha = \beta$ then $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ is a partial geometry, and if $\alpha = 0$ it is semipartial geometry. See [7] for more on partial geometries, semipartial geometries and strongly regular graphs.

In this paper we outline the basic properties of strongly regular (α, β)-geometries. To a certain extent we follow the literature on semipartial geometries and prove similar results. Often the proofs follow those of the semipartial geometry case. In the next section neccessary conditions for existence are given. In Section 3 strongly regular (α, β)-reguli are defined, and it is shown how they may be used to construct strongly regular (α, β)-geometries. This generalises similar results by J. A. Thas in [19] constructing semipartial geometries. In Section 4 several constructions of strongly regular (α, β)-geometries are given. Also in this section is a list of possible parameters for strongly regular (α, β)-geometries arising from strongly regular (α, β)-reguli. Surprisingly many of the possible parameter sets do have strongly regular (α, β)-geometries with those parameters, showing that the neccessary conditions of Section 2 are strong. The paper is concluded with some general remarks.

2. NECESSARY CONDITIONS FOR EXISTENCE

THEOREM 2.1. *Let $\mathscr{K}$ be a strongly regular (α, β)-geometry with parameters as in the previous section then the following conditions hold*

(i) $\mu \mid k(k-\lambda-1)$ *and* $(s+1) \mid v(t+1)$.

(ii) *The point graph has 3 eigenvalues, one of which is k, the other two* $u_1 < u_2$ *and their multiplicities* f_1, f_2, *satisfy*

$$u_i^2 + (\mu-\lambda)\, u_i - (k-\mu) = 0, \qquad f_1 + f_2 = v - 1, \qquad u_1 f_1 + u_2 f_2 = -k$$

with f_1 *and* f_2 *positive integers.*

(iii) *For* $v > 5$, *one of the following occurs*:

(a) $v = 2k+1$, $\lambda = k/2 - 1$ *and* $\mu = k/2$, *where* v *is the sum of two squares and the eigenvalues* u_1 *and* u_2 *are* $(1 \pm \sqrt{v})/2$.

(b) u_1 *and* u_2 *are integers with* $u_1 < 0 < u_2$. *Hence* $(\mu - \lambda)^2 + 4(k-\mu)$ *is a square and* $(u_2 - u_1) \mid (u_2(v-1) + k)$.

(iv) $(u_1+1)(u_1+2\mu-k) \leqslant (k+u_1)(u_2+1)^2$ *and* $2v < f_1(f_1+3)$.

(v) *For* $\alpha \neq \beta$, $(\beta-\alpha) \mid (\beta(v-s-1) - st(s+1))$.

(vi) *For* $\alpha \neq \beta$, $v \leqslant b$ *and* $s \leqslant t$.

(vii) *If* $\beta = s+1$ *then* $(s+1-\alpha) \mid (t+1)(s-\alpha)(t-p)$, *for* $\alpha \neq s+1$.

Proof. (i) v and b must be integers.

(ii) The graph is strongly regular.

(iii) It is known that the eigenvalues of the strongly regular graph are integers with the possible exception of the conference graphs [7, Theorem 10.2].

(iv) These are the Krein conditions and the absolute bound for strongly regular graphs [7, p. 439].

(v) Let L be some line of the geometry. Then the points of the geometry not on L are of two types: those that are joined to α points of L by lines of the geometry; and those that are joined to β points of L by lines of the geometry. Suppose there are u of the former and w of the later. Then

$$u + w + s + 1 = v$$

$$\alpha u + \beta w = (s+1)\, ts$$

$$\binom{\alpha}{2} u + \binom{\beta}{2} w = \binom{s+1}{2} (\lambda - s + 1)$$

The first equation arises from counting all of the points in the geometry. The second from counting pairs (x, M) such that x is a point not on L, and M is a line joining x to a point of L. The third arises from counting triples (x, M_1, M_2) such that x is a point not on L, and $M_1 \neq M_2$ are lines joining x to points of L. These equations are then equivalent to

$$u + w = v - s - 1$$

$$\alpha u + \beta w = s(s+1)\, t$$

$$\alpha^2 u + \beta^2 w = s(s+1)(t + \lambda - s + 1)$$

which have a solution if and only if

$$\begin{vmatrix} 1 & 1 & v-s-1 \\ \alpha & \beta & s(s+1)\, t \\ \alpha^2 & \beta^2 & s(s+1)(t+\lambda-s+1) \end{vmatrix} = 0.$$

Hence either $\alpha = \beta$ or

$$\alpha\beta(v-s-1) = s(s+1)((\alpha+\beta-1)\, t + s - \lambda - 1)$$

in which case

$$u = \frac{\beta(v-s-1) - s(s+1)\,t}{\beta - \alpha}.$$

(vi) Brouwer, Haemers and Tonchev have shown that if a strongly regular graph is the collinearity graph of a partial linear space then either the partial linear space is a partial geometry or $b > v$ [4].

(vii) Suppose $\beta = s+1$. Choose a pair of non-collinear points x and y.

We wish to count the set of point-line pairs

$$\{(z, L) \mid z \in L, z \sim x, z \nsim y, L \text{ a } (s+1)\text{-line w.r.t. } x, L \text{ a } \alpha\text{-line w.r.t. } y\},$$

where $\sim$ denotes collinearity.

Since y is not collinear with x, and $\beta = s+1$, every line on x must be an α-line w.r.t. y. Hence the number of points z that are collinear with x, but not y is $(t+1)(s-\alpha)$. On such a point z there are by definition of p exactly $t-p$ $(s+1)$-lines to x. And since $z \nsim y$, such a line must be an α-line with respect to y. So counting the pairs of the above set in one way gives $(t+1)(s-\alpha)(t-p)$. Now given a line L that is an α-line w.r.t. x, and a $(s+1)$-line w.r.t. y, there are exactly $s+1-\alpha$ points z collinear with x but not with y. Hence the (integer) number of such lines is $(t+1)(s-\alpha)(t-p)/(s+1-\alpha)$. ∎

COROLLARY 2.1. *If the dual of a strongly regular (α, β)-geometry $\mathcal{K}$ is strongly regular then either $s = t$ or $\mathcal{K}$ is a partial geometry.*

Proof. Follows from (vi) of the theorem. ∎

See also [7] for conditions specific to partial and semipartial geometries.

3. STRONGLY REGULAR (α, β)-REGULI

In [19], J. A. Thas defined SPG reguli which were then used to construct semipartial geometries. In this section we generalise the definition of SPG reguli to strongly regular (α, β)-reguli, and then show that strongly regular (α, β)-reguli give rise to strongly regular (α, β)-geometries.

A *strong regular* (α, β)*-regulus* is a collection $\mathcal{R}$ of m-dimensional subspaces of $\mathrm{PG}(n, q)$, $|\mathcal{R}| > 1$, satisfying

(i) $\Sigma_i \cap \Sigma_j = \varnothing$ for every $\Sigma_i, \Sigma_j \in \mathcal{R}$, $\Sigma_i \neq \Sigma_j$.

(ii) If an $(m+1)$-dimensional subspace contains some $\Sigma_i \in \mathcal{R}$, then it has a point in common with α or β, subspaces of $\mathcal{R} - \{\Sigma_i\}$. Such an

$(m+1)$-dimensional subspace that meets α elements of $\mathscr{R}-\{\Sigma_i\}$ is said to be an α-*secant* to $\mathscr{R}$ at Σ_i, similarly for β-*secants*

(iii) If a point of $\mathrm{PG}(n, q)$ is contained in an element Σ of $\mathscr{R}$ then it is contained in a constant number p of α-secant $(m+1)$-dimensional spaces on elements of $\mathscr{R}-\{\Sigma\}$.

(iv) If a point of $\mathrm{PG}(n, q)$ is contained in no element of $\mathscr{R}$ then it is contained in a constant number r of α-secant $(m+1)$-dimensional spaces of $\mathscr{R}$.

Note that if $\alpha=0$, then a strongly regular (α, β)-regulus is an SPG regulus as defined in [19].

THEOREM 3.1. *Let $\mathscr{R}$ be a strongly regular (α, β)-regulus in* $\mathrm{PG}(n, q)$, *the elements of $\mathscr{R}$ being of dimension m. Embed* $\mathrm{PG}(n, q)$ *as a hyperplane of* $\mathrm{PG}(n+1, q)$, *and define an incidence structure $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ of points and lines as follows.*

(i) *The point set $\mathscr{P}$ is the set of points of* $\mathrm{PG}(n+1, q)-\mathrm{PG}(n, q)$.

(ii) *The lines set $\mathscr{L}$ is the set of $(m+1)$-dimensional subspaces of* $\mathrm{PG}(n+1, q)$ *that meet* $\mathrm{PG}(n, q)$ *in an element of $\mathscr{R}$.*

(iii) *Incidence $\mathscr{I}$ is containment.*

Then $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ is a strongly regular (α, β)-geometry with parameters $s=q^{m+1}-1$, $t=|R|-1$, p and r.

Proof. Clearly the number of points on a line of $\mathscr{L}$ is q^{m+1}, and the number of lines on a point is $|\mathscr{R}|$, hence $s=q^{m+1}-1$ and $t=|\mathscr{R}|-1$.

Let $x \in \mathscr{P}$ and $L \in \mathscr{L}$ such that $x \notin L$, and let $R \in \mathscr{R}$ be the element of the regulus that L contains. In $\mathrm{PG}(n+1, q)$ consider the $(m+2)$-dimensional subspace Π_{m+2} given by the span of x and L. Then Π_{m+2} meets $\mathrm{PG}(n, q)$ in a subspace Π_{m+1} of dimension $m+1$ that contains R. Since $\mathscr{R}$ is a strongly regular (α, β)-regulus, it follows that Π_{m+1} meets α or β elements $\mathscr{R}_{\alpha, \beta}$ of $\mathscr{R}-\{R\}$ in a single point. Clearly the lines of $\mathscr{L}$ given by the span of x and elements of $\mathscr{R}_{\alpha, \beta}$ are exactly those that join x to a point of L, and there are either α or β of them. Hence $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ is an (α, β)-geometry. It remains to be shown that it is strongly regular.

Suppose that $x, y \in \mathscr{P}$, such that x and y are not both on a line of $\mathscr{L}$. Hence the line joining x and y in $\mathrm{PG}(n+1, q)$ meets $\mathrm{PG}(n, q)$ in a point u not contained in any element of $\mathscr{R}$. Now by definition there are exactly r α-secant $(m+1)$-dimensional subspaces each containing u and an element of $\mathscr{R}$. Consider one such α-secant Π_{m+1} on u, which contains an element $R \in \mathscr{R}$. Let $\mathscr{A}$ be the set of α points that Π_{m+1} meets elements of $\mathscr{R}-R$ in. Let Π_{m+2} be the subspace given by the span of x (or y) and Π_{m+1}. Let L be the line of $\mathscr{L}$ determined by the span of x and R. The subspace corresponding to L is of dimension $m+1$ and is contained in Π_{m+2}. Hence a line of $\mathrm{PG}(n+1, q)$ in Π_{m+2} joining y to one of the α points of $\mathscr{A}$ will meet L in a unique point z of $\Pi_{m+2}-\Pi_{m+1}$. The line yz of $\mathrm{PG}(n+1, q)$ is then contained in a (unique) line of $\mathscr{L}$ that is on y and meets L. Hence the α choices for elements of $\mathscr{A}$ give rise to exactly α lines of $\mathscr{L}$ that are on y and meet L. We have r choices for L. Hence we have shown that given a point y not collinear with x, there are exactly r lines of our geometry on x, such that for each line there are α lines joining y to the line. Hence we have property (b) that is required for the geometry to be strongly regular. Property (a) follows in a similar manner. ▮

Notice from the details of the proof that if we have a collection of subspaces that satisfies all of the conditions required for the definition of a strongly regular (α, β)-regulus except possibly conditions (iii) and (iv), then that set gives an (α, β)-geometry.

Also note that the strongly regular graph arising from the strongly regular (α, β)-regulus can be seen as follows. Let the points of the graph be the points of $\mathrm{PG}(n+1, q)-\mathrm{PG}(n, q)$. Two distinct points are then collinear if and only if the line of $\mathrm{PG}(n+1, q)$ joining them meets $\mathrm{PG}(n, q)$ in a point contained in some element of the strongly regular (α, β)-regulus. It follows that the union of the points contained in elements of a strongly regular (α, β)-regulus has exactly two intersection sizes with respect to hyperplanes in $\mathrm{PG}(n, q)$, [5].

In Section 5 possible parameter sets for strongly regular (α, β)-geometries with $v \leqslant 4096$ from (α, β)-reguli are listed.

4. CONSTRUCTION OF STRONGLY REGULAR (α, β)-GEOMETRIES

4.1. *Sets of Type* (m, n) *in* $\mathrm{PG}(2, q)$

In a projective plane π of order q, a set $\mathscr{K}$ of type (m, n), $m<n$, is a subset of points such that any line of π meets $\mathscr{K}$ in either m or n points, for some integers m and n. See [15] for properties and examples of sets of type (m, n), also called two character sets with respect to lines. Note that the complement is also a two character set with respect to lines.

It follows immediately that the points of a set $\mathscr{K}$ of type (m, n) in $\mathrm{PG}(2, q)$ is a strongly regular (α, β)-regulus. In the following theorem we give the parameters of the geometries and graphs, but first some constants associated with $\mathscr{K}$ are calculated.

Let θ be the number of m-secants to $\mathscr{K}$ on a point not of $\mathscr{K}$ then $\theta m+(q+1-\theta)\, n=|\mathscr{K}|$ giving $\theta=((q+1)\, n-|\mathscr{K}|)/(n-m)$. Similarly let γ be the number of m-secants to $\mathscr{K}$ on a point of $\mathscr{K}$ then $\gamma(m-1)+(q+1-\gamma)(n-1)=|\mathscr{K}|-1$ giving $\gamma=((q+1)(n-1)-|\mathscr{K}|+1)/(n-m)$.

THEOREM 4.1. *Let $\mathscr{K}$ be a set of type (m, n) in* PG$(2, q)$. *Then the resulting geometry has $s = q-1$, $t = |\mathscr{K}|-1$, with $p = \alpha\gamma$, $r = (\alpha+1)\,\theta$, where $\alpha = m-1$ and $\beta = n-1$, for $m > 0$, and $\alpha = \beta = n-1$ for $m = 0$.*

The associated strongly regular graph has $v = q^3$ and $k = |\mathscr{K}|(q-1)$, $\lambda = (\alpha-1)\,p + (\beta-1)(|\mathscr{K}|-1-p) + q-1$ and $\mu = \alpha r + \beta(|\mathscr{K}|-r)$.

That sets of type (m, n) give strongly regular graphs in this way was noted in [5].

Some examples of sets of type (m, n):

(i) Maximal arcs, $(m, n) = (0, n)$, $|K| = q(n-1)+n$, exist in PG$(2, q)$, q even, for all $n \mid q$ [13]. These give partial geometries with $s = q-1$, $t = (q+1)(n-1)$ and $\alpha = n-1$ $(=\beta)$.

(ii) Unitals, $(m, n) = (1, q^{1/2})$, $|\mathscr{K}| = q^{3/2}+1$, exist in PG$(2, q)$ for every q a square [15]. These give strongly regular $(0, q^{1/2})$-geometries, i.e., semipartial geometries, with $s = q-1$ and $t = q^{3/2}$.

(iii) Baer subplanes, $(m, n) = (1, q^{1/2})$, $|\mathscr{K}| = q + q^{1/2} + 1$, exist in PG$(2, q)$ for every q a square. These give strongly regular $(0, q^{1/2})$-geometries with $s = q-1$ and $t = q + q^{1/2}$.

(iv) Disjoint unions of u Baer subplanes, $(m, n) = (u, q^{1/2}+u)$, $|\mathscr{K}| = u(q + q^{1/2}+1)$, exist in PG$(2, q)$ for every q a square, $s \in \{1, 2, ..., (q-q^{1/2})\}$ [10]. These give strongly regular $(u-1, q^{1/2}+u-1)$-geometries with $s = q-1$ and $t = u(q+q^{1/2}+1)-1$.

See also [11], [5], [17], [12], [1], [14] and [2] for other constructions of sets of type (m, n) in projective planes.

4.2. *Complements of Quadrics*

Suppose $\mathscr{Q}$ is a non-degenerate hyperbolic or elliptic quadric in PG$(2n+1, q)$. Then a plane of PG$(2n+1, q)$ may meet $\mathscr{Q}$ in either a conic, a line pair, a single line, or a single point or be entirely contained in $\mathscr{Q}$ [16]. Let L be a line disjoint from $\mathscr{Q}$, then a plane π on L must meet Q in either a conic or a single point, and so π meets the complement of $\mathscr{Q} \cup L$ in either q^2-q-1 or q^2-1 points.

It follows that a partition of the complement of $\mathscr{Q}$ in PG$(2n+1, q)$ into lines is a strongly regular (α, β)-regulus with $\alpha = q^2-q-1$ and $\beta = q^2-1$. Conditions (i) and (ii) of the definition of strongly regular (α, β)-regulus follow immediately, and condition (iii) follows since $\mathscr{Q}$ has two intersection sizes with respect to hyperplanes.

For such a partition to exist it is necessary that $q+1$ divides $|\text{PG}(2n+1, q) - \mathscr{Q}|$, which is equivalent to $q+1$ dividing $|\mathscr{Q}|$ (since $q+1$ divides $|\text{PG}(2n+1, q)|$). Now $|Q^+(2n+1, q)| = (q^n+1)(q^{n+1}-1)/(q+1)$ and $|Q^-(2n+1, q)| = (q^{n+1}+1)(q^n-1)/(q+1)$, and so $q+1$ divides

$|Q^+(2n+1, q)|$ if and only if n is odd, and $q+1$ divides $|Q^-(2n+1, q)|$ if and only if n is even. We summarise the above in the following two theorems.

THEOREM 4.2. *A partition of the points of* $\mathrm{PG}(4n+1, q) - Q^-(4n+1, q)$ *into lines is a strongly regular* (α, β)*-regulus and gives rise to a strongly regular* (α, β)*-geometry with parameters* $s = q^2 - 1$, $t = q^{2n}(q^{2n+1}+1)/(q+1) - 1$, $\alpha = q^2 - q - 1$, $\beta = q^2 - 1$, $p = q^{2n-1}(q^2-q-1)(q^{2n}-1)/(q+1)$ *and* $r = q^{4n} - q^{4n-1}$.

The associated strongly regular graph has parameters $v = q^{4n+2}$, $k = (q^{2n+1}+1)(q^{2n}-1)$, $\lambda = -2 + q^{2n} - q^{2n+1} + q^{4n}$ *and* $\mu = q^{4n} - q^{2n}$.

THEOREM 4.3. *A partition of the points of* $\mathrm{PG}(4n+3, q) - Q^+(4n+3, q)$ *into lines is a strongly regular* (α, β)*-regulus and gives rise to a strongly regular* (α, β)*-geometry with parameters* $s = q^2 - 1$, $t = q^{2n+1}(q^{2n+2}-1)/(q+1) - 1$, $\alpha = q^2 - q - 1$, $\beta = q^2 - 1$, $p = q^{2n}(q^2-q-1)(q^{2n+1}+1)/(q+1)$ *and* $r = q^{4n+2} - q^{4n+1}$.

The associated strongly regular graph has parameters $v = q^{4n+4}$, $k = (q^{2n+1}+1)(q^{2n+2}-1)$, $\lambda = -2 - q^{2n+1} + q^{2n+2} + q^{4n+2}$ *and* $\mu = q^{4n+2} + q^{2n+1}$.

To obtain partitions of the points not on a quadric into lines we use a construction of J. A. Thas originally given to construct lines spreads of quadrics [20, p. 64]. Let $Q^-(4n+1, q)$ be a non-singular elliptic quadric in $\mathrm{PG}(4n+1, q)$. Consider a quadratic extension $\mathrm{GF}(q^2)$ of $\mathrm{GF}(q)$, and the corresponding extension $Q^+(4n+1, q^2)$ of $Q^-(4n+1, q)$. Then there exist a subspace Π_{2n} of dimension $2n$ contained in $Q^+(4n+1, q^2)$ but disjoint from $\mathrm{PG}(4n+1, q)$. On each point x of $Q^-(4n+1, q)$ there is a unique line L of $\mathrm{PG}(4n+1, q^2)$ that meets both Π_{2n} and its conjugate Π'_{2n} with respect to the field extension. The line L meets $Q^-(4n+1, q)$ in a line. Taking the set of all such lines on points of $Q^-(4n+1, q)$ gives a partition of $Q^-(4n+1, q)$ into lines. Further the points of intersection of the lines L with Π_{2n} are the points of a Hermitean variety $U(2n, q^2)$ of Π_{2n}.

In fact the lines of the partition of $Q^-(4n+1, q)$ are just the lines obtained by taking the line joining a point of $U(2n, q^2)$ to its conjugate and intersecting it with $Q^-(4n+1, q)$. But if instead of taking points of $U(2n, q^2)$ we take points of $\Pi_{2n} - U(2n, q^2)$, join them to their conjugates and intersect the resulting line with $\mathrm{PG}(4n+1, q)$ it is clear that we obtain a partition of $\mathrm{PG}(4n+1, q) - Q^-(4n+1, q)$ into lines as required.

A similar construction works for $Q^+(4n+3, q)$, with the modification that we have a disjoint subspace Π_{2n+1} of dimension $2n+1$ containing a Hermitean variety $U(2n+1, q^2)$.

Hence we can always partition the points not on $Q^+(4n+3, q)$ or $Q^-(4n+1, q)$ into lines, and so get strongly regular (α, β)-geometries.

However, this construction gives (α, β)-geometries isomorphic to some arising from the Hermitean spaces $U(2n+1, q^2)$ and $U(2n, q^2)$. It is readily verified that the points of $\mathrm{PG}(2n+1, q^2) - U(2n+1, q^2)$ or $\mathrm{PG}(2n, q) - U(2n, q^2)$ are strongly regular (α, β)-reguli since every line meets such sets in either $q^2 - q$ or q^2 points. The details of the above construction make it clear that the strongly regular (α, β)-geometries arising from the Hermitean spaces are isomorphic to those of the above constructions for strongly regular (α, β)-reguli of lines in the complements of quadrics. But it is possible to "derive" the (α, β)-reguli of lines in the complements of quadrics to obtain new strongly regular (α, β)-geometries as follows.

Consider the case above where $Q^-(4n+1, q)$ is a non-singular elliptic quadric in $\mathrm{PG}(4n+1, q)$, with the quadratic extension to $\mathrm{PG}(4n+1, q^2)$ and the subspace Π_{2n} of dimension $2n$ disjoint from $\mathrm{PG}(4n+1, q)$ and containing the Hermitean variety $U(2n, q^2)$. Let $\mathscr{L}$ be the resulting (α, β)-regulus. As above a point of $U(2n, q^2)$ corresponds to a line of $Q^-(4n+1, q)$, and a point not on $U(2n, q^2)$ corresponds to a line disjoint from $Q^-(4n+1, q)$. Further a Baer subline of Π_{2n} corresponds to a set of $q+1$ lines of $\mathrm{PG}(4n+1, q)$ that partition the points of a $Q^+(3, q)$, i.e., the $q+1$ lines form a regulus $\mathscr{R}$. It is well known that $Q^+(3, q)$ has exactly two classes of lines $\mathscr{R}$ and $\mathscr{R}_{opp}$, both of which partition the point set of $Q^+(3, q)$. So we can switch one class for the other and still cover exactly the same point set. In particular if we take a Baer subline in Π_{2n} disjoint from $U(2n, q^2)$ that corresponds to a regulus $\mathscr{R}$, then we may define a new partition of $Q^-(4n+1, q)$ into lines by $\mathscr{L} - \mathscr{R} \cup \mathscr{R}_{opp}$, and hence get a new strongly regular (α, β)-regulus. There are many Baer sublines disjoint from $U(2n, q^2)$, and any collection of these that are pairwise disjoint may be used to "multiply derive" the (α, β)-reguli. Essentially the same process can be applied for the $Q^+(4n+3, q)$ case. Hence new geometries are obtained.

4.3. *Oval Constructions*

In a projective plane of order q, a *maximal arc* $\mathscr{K}$ is a set of $q(n-1)+n$ points such that every line of the plane meets the set in 0 or n points for some integer n [15]. If the plane is Desarguesian, examples are known for all n dividing q, q even [13].

It is well known that $\mathscr{K}$ defines a partial geometry as follows [18]. Let the set of points $\mathscr{P}$ of the partial geometry be the points of $\pi - \mathscr{K}$, and the set of lines $\mathscr{L}$ of the partial geometry be the secant lines to $\mathscr{K}$. Then $s = q - n$, $t = q - q/n$ and $\alpha = q - q/n - n + 1$.

Now consider a line L of π that does not meet $\mathscr{K}$. It follows that every line of $\mathscr{L}$ meets L in exactly one point. Form a new incidence structure with pointset $\mathscr{P}' = \mathscr{P} - \{x \mid x \in L\}$ and lineset $\mathscr{L}' = \mathscr{L}$. Then clearly every point of $\mathscr{P}'$ is on $q - q/n + 1$ lines of $\mathscr{L}'$, and every line of $\mathscr{L}'$ contains $q - n$ points of $\mathscr{P}'$. Further it is clear that given a point $x \in \mathscr{P}'$ and a line $M \in \mathscr{L}'$, with x not on M there will either be $q-q/n-n$ or $q-q/n-n+1$ lines of $\mathscr{L}'$ joining a point of M to x (the two cases correspond to whether the lines joining $M \cap L$ to x is secant or external to $\mathscr{K}$ respectively). Hence $(\mathscr{P}', \mathscr{L}')$ is an (α, β)-geometry with $s' = q-n-1$, $t' = q-q/n$, $\alpha = q-q/n-n$ and $\beta = q-q/n-n+1$. For one class of examples the (α, β)-geometry is strongly regular.

THEOREM 4.4. *Let $\mathscr{K}$ be a degree $q/2$ maximal arc in π, $q = 2^h$, then $(\mathscr{P}', \mathscr{L}')$ is a strongly regular (α, β)-geometry with $s' = q/2-1$, $t' = q-2$, $\alpha = q/2-2$, $\beta = q/2-1$, $p = q-4$ and $r = q-2$. The point graph has parameters $v = (q+1)\,q/2$, $k = (q-1)(q/2-1)$, $\lambda = (q-3)(q/2-2)$ and $\mu = (q-3)(q/2-1)$ (the complement of a triangular graph).*

Proof. With notation as above we show that the underlying point graph is strongly regular and the result follows. The calculation of v and k are obvious. Now consider points x and y of $\mathscr{P}'$, such that x and y are on no line of $\mathscr{L}'$, but are on line M in π. Now since $\mathscr{K}$ is a degree $q/2$ maximal arc, on x there is one further line $M_x \neq M$ on x and exterior to $\mathscr{K}$, and on y there is one further line $M_y \neq M$ on y and exterior to $\mathscr{K}$. Now $M_x \cap M_y$ cannot be a point of L for if it was there would be three exterior lines (M_x, M_y and L) on the point $M_x \cap M_y$. Now there are $q-1$ secant lines to $\mathscr{K}$ on lines through x, each secant line contains $q+1-1-1-q/2 = q/2-1$ points of $\pi - \{x\} \cup L \cup \mathscr{K}$. Of these $(q-1)(q/2-1)$ points on secant lines through x, exactly $q-2$ of them are on M_y. Hence the number of points collinear with both x and y in $\mathscr{L}'$ is $(q-1)(q/2-1)-(q-2) = (q-3)(q/2-1)$. Hence $\mu = (q-3)(q/2-1)$. The calculation for λ is essentially the same. ∎

4.4. $2-(v, k, 1)$ *Designs*

In this section two constructions of strongly regular (α, β)-geometries arising from Steiner 2-designs are given.

THEOREM 4.5. *Let $\mathscr{D}$ be a $2-(v, k, 1)$ design with pointset $\mathscr{P}$ and blockset $\mathscr{B}$. Let x be some point of $\mathscr{D}$, B be the set of blocks incident with x, and $r_{\mathscr{D}}$ the number of blocks on a point in $\mathscr{D}$. Define a new incidence structure $\mathscr{D}'$ with points $\mathscr{P}' = \mathscr{P} - \{x\}$, blocks $\mathscr{B}' = \mathscr{B} - B$, and incidence as in $\mathscr{D}$. Then $\mathscr{D}'$ is a strongly regular $(k-1, k)$-geometry with $s = k-1$, $t = r_{\mathscr{D}}-2$, $\lambda = (r_{\mathscr{D}}-2)(k-1)$ and $\mu = \lambda + k - 1$.*

Proof. The proof is straightforward. ∎

THEOREM 4.6. *Let $\mathscr{D}$ be a $2-(v, k, 1)$ design that admits a spread $\mathscr{S}$, i.e. $\mathscr{S}$ is a partition of the point set of $\mathscr{D}$ into disjoint lines. Let $\mathscr{B}$ be the set of blocks of $\mathscr{D}$ and $r_{\mathscr{D}}$ be the number of blocks on a point. Define a new* *incidence structure* $\mathscr{D}'$ *with point those of* $\mathscr{D}$, *blocks* $\mathscr{B}' = \mathscr{B} - \mathscr{S}$, *and incidence as in* $\mathscr{D}$. *Then* $\mathscr{D}'$ *is a strongly regular* $(k-1, k)$*-geometry with* $s = k-1$, $t = r_{\mathscr{D}} - 2$, $\lambda = v - 2k$ *and* $\mu = \lambda + k$.

Proof. The proof is straightforward. ∎

4.5. *Sporadic Constructions*

In this section two constructions of strongly regular (α, β)-geometries that appear to be sporadic are given.

In [6], a set $\mathscr{R}$ of 21 lines in PG(5, 3) is given that is a strongly regular (0, 2)-regulus. The geometry arising from the regulus is a partial geometry with $s = 8$, $t = 20$ and $\alpha = 2$.

In fact it is possible to partition the 252 points not on an elliptic quadric $Q^{-}(5, 3)$ in PG(5, 3) into 3 disjoint copies of these 21 lines each, hence giving a partition of the points not on the quadric into lines. By the results of Section 4.2 these lines then form a strongly regular (5, 8)-regulus, and so a strongly regular (5, 8)-geometry with parameters $s = 8$, $t = 62$, $p = 30$ and $r = 54$.

In Table I the points of this set of 63 lines in PG(5, 3) is described as follows. A point of PG(5, 3) is given as a triple *abc* where *a*, *b* and *c* are in the range 0 to 8. Taking the base 3 representation of each digit then gives a vector of length 6 over GF(3). In the following, the first 4 columns give the points of the first copy of the points of $\mathscr{R}$, the second 4 columns give the points of the second copy of $\mathscr{R}$, and the third 4 give the third copy of $\mathscr{R}$. The rows give the points on each of the lines of the partition.

It may be that this partition of $\mathrm{PG}(5, 3) - Q^{-}(5, 3)$ into lines can be obtained by the method of derivation described in Section 4.2, but it is not clear to the authors how this might be done.

The second construction of a sporadic strongly regular (α, β)-geometry is not associated with a regulus. It is a strongly regular (3, 5)-geometry with parameters $s = 4$, $t = 14$, $p = 6$ and $r = 15$. It arises from the Steiner system $\mathscr{S}$ which is a $3-(22, 6, 1)$ design with automorphism group M_{22} acting 3-transitively on the 22 elements and transitive on the 77 blocks. It is well known ([3, Section 10]) that $\mathscr{S}$ is quasi-symmetric with block intersections 0 and 2. The geometry lives in the strongly regular 2-intersection block graph with parameters srg(77, 60, 47, 45).

We define a new geometry $\mathscr{G}$ as follows. The points of $\mathscr{G}$ are the blocks of $\mathscr{S}$, and the lines of $\mathscr{G}$ are the pairs of elements of $\mathscr{S}$, incidence is by inclusion. Clearly, since each pair of points of $\mathscr{S}$ is in 5 blocks and each block has 15 pairs, $\mathscr{G}$ has 5 points on a line and 15 lines on point. Moreover, since each triple of elements is in a unique block we see that two lines of $\mathscr{S}$ have at most one point in common. It remains to be shown that $\mathscr{G}$ has the (α, β)-property.

TABLE I

A Strongly Regular (5, 8)-Regulus in PG(5, 3)

300	100	700	400	030	324	384	354	130	411	371	541
330	103	763	433	230	013	173	243	010	667	687	677
630	203	563	833	110	804	724	614	210	167	287	077
310	201	821	511	740	733	023	573	040	367	357	317
610	101	421	711	840	033	423	873	770	601	251	471
440	707	387	247	070	664	624	644	870	861	021	441
540	137	827	377	303	721	427	124	003	271	277	274
470	134	684	514	863	581	627	154	603	031	337	634
570	404	254	674	643	151	417	764	533	141	817	374
713	831	157	344	743	851	117	364	063	561	507	534
813	531	657	144	543	731	527	074	463	061	807	434
343	741	407	184	473	431	087	814	413	841	437	054
843	351	717	264	273	341	467	524	513	551	047	734
373	451	177	704	523	011	727	504	143	751	617	864
773	641	267	424	653	481	737	114	443	651	517	164
723	381	867	174	553	281	637	414	673	241	567	824
823	671	187	564	683	301	647	084	153	681	837	714
353	881	537	214	183	181	007	244	753	521	767	044
453	761	347	224	483	771	327	234	253	171	257	004
383	571	227	834	283	001	147	284	853	781	237	314
583	461	217	654	883	811	057	464	783	871	127	334

Consider an antiflag (p, L) of $\mathcal{G}$, where L is induced by the pair x, y of points of $\mathcal{S}$. The first case to consider is when in $\mathcal{S}$ neither x or y are contained in p. Now p is a line of $\mathcal{S}$ containing 6 points of $\mathcal{S}$. Further, since $\mathcal{S}$ is a 3-design, any point of $\mathcal{S}$ on p is contained in a unique block of $\mathcal{S}$ containing x and y. And such a block meets the block p in two points (since every pair of blocks of $\mathcal{S}$ meet in 0 or 2 points). Hence there are exactly $6/2 = 3$ blocks of $\mathcal{S}$ containing x and y and meeting p in two points. In $\mathcal{G}$ this says that there are exactly 3 points on L that are collinear with p.

The second case to consider is when exactly one (x say) of the points x and y of $\mathcal{S}$ are contained in p. In this case there are 5 points of $p - \{x\}$, and for each such point there is a unique block of $\mathcal{S}$ containing that point as well as x and y. It follows that in $\mathcal{G}$ there are exactly 5 points of $\mathcal{G}$ on L that are collinear with p.

In fact the above argument works for any quasi-symmetric $3-(v, k, 1)$ design such that some pair of blocks is disjoint. However, such designs are classified (see [8, 37.12]), and the only cases that exist are the above $3-(22, 6, 1)$ quasi-symmetric design, and a quasi-symmetric $3-(8, 4, 1)$

design giving a strongly regular (2, 3)-geometry with $s=2$ and $t=5$. The second design is the unique extension of the Fano plane, its blocks are the points of the geometry and blocks containing pairs of elements are the lines.

5. PARAMETERS FROM STRONGLY REGULAR (α, β)-REGULI

In this section we list possible parameter sets for strongly regular (α, β)-geometries with $v \leqslant 4096$ from strongly regular (α, β)-reguli.

First note that for a strongly regular (α, β)-reguli to exist, the number of points, v, of the associated strongly regular (α, β)-geometry must be a power of some prime. It follows that $s+1$ must also be a power of that prime. Possible parameters were found by only considering cases when v and $(s+1)$ were powers of the same prime and applying the necessary existence conditions from Section 2.

In Table II, s, t, α, β, p and r are the parameters of the (possibly non-existent) strongly regular (α, β)-geometry. The associated strongly regular graph having parameters v, k, λ, μ with eigenvalues u_1, u_2 with multiplicities f_1 and f_2.

The final column of the table gives a construction method of the strongly regular (α, β)-reguli if it is known to exist. The sets $Q^+(7, 2)\ cl$, $Q^-(9, 2)\ cl$ and $Q^+(11, 2)\ cl$ denote strongly regular (α, β)-reguli formed by partitioning complements of quadrics as in Section 4.2. The remaining examples arise from sets of type (m, n) in PG$(2, q)$. The notation U_q denotes the (α, β)-reguli arising from a unital in PG$(2, q)$; B_q denotes the (α, β)-reguli arising from a Baer subplane in PG$(2, q)$; $S_q(x, y)$ is a set of type (x, y) in PG$(2, q)$; A number n prefixing the set denotes a disjoint union of n copies of that set; A "c" as a suffix denotes taking the complement of the set; Thus $3S_{16}(0, 4)\ c$ denotes taking the complement of the disjoint union of 3 sets of type (0, 4) in PG(2, 16). Constructions of these sets can be found in the references given at the end of Section 4.1.

Note that we do not include partial geometries $(\alpha=\beta)$ or semipartial geometries $(\alpha=0)$ in the list.

We conclude this section with some simple non-existence results for several of the parameter sets given in Table II.

LEMMA 5.1. *The plane* PG(2, 3) *does not contain a* (*non-empty*, *proper*) *m-set of type* (0, 2, 4).

Proof. Suppose such a set did exist and let t_i be the number of lines of PG(2, 3) that meet the set in i points for $i=0, 2, 4$. Then standard counting

TABLE II

Parameters from Strongly Regular (α, β)-reguli

s	t	α	β	p	r	v	k	λ	μ	u_1	u_2	f_1	f_2	Comments
3	11	1	3	2	8	64	36	20	20	4	-4	27	36	U_4c
3	13	1	3	1	6	64	42	26	30	2	-6	42	21	B_4c
3	14	2	4	6	15	64	45	32	30	5	-3	18	45	$S_4(0,2)c$
2	19	1	3	10	20	81	40	19	20	4	-5	40	40	†
2	64	1	3	28	65	243	130	73	65	13	-5	60	182	†
3	33	1	3	15	30	256	102	38	42	6	-10	153	102	
3	39	1	3	12	32	256	120	56	56	8	-8	120	135	$Q^+(7,2)cl$
3	49	1	3	7	30	256	150	86	90	6	-10	150	105	
3	54	2	4	30	55	256	165	104	110	5	-11	165	90	
7	29	5	7	5	14	256	210	170	182	2	-14	210	45	
6	37	2	5	2	10	343	228	147	160	4	-17	266	76	†††
7	44	4	8	28	45	512	315	202	180	27	-5	70	441	$S_8(0,4)c$
7	62	6	8	30	63	512	441	380	378	9	-7	196	315	$S_8(0,2)c$
4	51	1	3	21	42	625	208	63	72	8	-17	416	208	††
4	77	2	5	52	78	625	312	155	156	12	-13	312	312	††
4	113	3	5	63	114	625	456	329	342	6	-19	456	168	††
8	25	1	4	5	16	729	208	67	56	19	-8	208	520	$2B_9$
8	34	1	4	2	10	729	280	103	110	10	-17	448	280	$S_9(2,5)$
8	34	1	5	4	20	729	280	127	95	37	-5	80	648	
8	38	2	5	8	21	729	312	135	132	15	-12	312	416	$3B_9$
2	163	1	3	100	164	729	328	127	164	4	-41	656	72	†
8	41	2	5	6	18	729	336	153	156	12	-15	392	336	$S_9(3,6)$
2	181	1	3	91	182	729	364	181	182	13	-14	364	364	†
8	48	3	6	12	28	729	392	211	210	14	-13	336	392	$S_9(3,6)c$
2	199	1	3	82	200	729	400	235	200	40	-5	72	656	†
8	51	3	6	9	24	729	416	235	240	11	-16	416	312	$4B_9$
8	55	4	6	10	20	729	448	262	296	4	-38	648	80	
8	55	4	7	20	40	729	448	277	272	16	-11	280	448	$S_9(2,5)c$
8	62	5	8	30	54	729	504	351	342	18	-9	224	504	U_9c
8	64	4	7	8	25	729	520	367	380	7	-20	520	208	$5B_9$
8	69	6	9	42	70	729	560	433	420	20	-7	168	560	
8	77	5	8	5	24	729	624	531	552	3	-24	624	104	$6B_9$
3	109	1	4	77	110	1024	330	98	110	10	-22	693	330	
3	175	1	3	40	128	1024	528	272	272	16	-16	495	528	$Q^-(9,2)cl$
3	185	1	3	35	126	1024	558	302	306	14	-18	558	465	
3	214	2	4	126	215	1024	645	392	430	5	-43	903	120	
3	230	2	4	110	231	1024	693	472	462	21	-11	330	693	
7	123	3	7	3	28	1024	868	732	756	4	-28	868	155	
15	30	6	10	18	25	1024	465	212	210	17	-15	465	558	
15	54	10	14	12	30	1024	825	668	650	25	-7	198	825	
15	61	13	15	13	30	1024	930	842	870	2	-30	930	93	
2	532	1	3	280	533	2187	1066	505	533	13	-41	1640	546	†
6	99	1	3	36	72	2401	600	131	156	12	-37	1800	600	†††
6	159	2	5	84	140	2401	960	389	380	29	-20	960	1440	†††
6	199	3	7	150	200	2401	1200	599	600	24	-25	1200	1200	†††
6	327	5	7	180	328	2401	1968	1607	1640	8	-41	1968	432	†††
4	402	2	5	252	403	3125	1612	855	806	62	-13	520	2604	††
3	461	1	4	301	462	4096	1386	482	462	42	-22	1386	2709	
3	649	1	3	187	510	4096	1950	926	930	30	-34	2145	1950	
3	671	1	3	176	512	4096	2016	992	992	32	-32	2016	2079	$Q^+(11,2)cl$
3	713	1	3	155	510	4096	2142	1118	1122	30	-34	2142	1953	
7	129	1	3	45	90	4096	910	174	210	14	-50	3185	910	
7	194	2	5	114	171	4096	1365	440	462	21	-43	2730	1365	
7	170	2	8	138	171	4096	1197	368	342	45	-19	1197	2898	
7	259	3	7	189	252	4096	1820	804	812	28	-36	2275	1820	
15	41	1	5	11	30	4096	630	134	90	54	-10	630	3465	$2B_{16}$
15	62	2	6	20	42	4096	945	244	210	49	-15	945	3150	$3B_{16}$
15	77	1	5	2	12	4096	1170	314	342	18	-46	2925	1170	$S_{16}(2,6)$
15	83	3	7	27	52	4096	1260	404	380	44	-20	1260	2835	$4B_{16}$
15	90	2	6	6	21	4096	1365	440	462	21	-43	2730	1365	$S_{16}(3,7)$
15	103	3	7	12	32	4096	1560	584	600	24	-40	2535	1560	$2S_{16}(0,4)$
15	104	4	8	32	60	4096	1575	614	600	39	-25	1575	2520	$5B_{16}$
15	116	4	8	20	45	4096	1755	746	756	27	-37	2340	1755	$3S_{16}(0,4)c$
15	125	5	9	35	66	4096	1890	874	870	34	-30	1890	2205	$6B_{16}$
15	129	5	9	30	60	4096	1950	926	930	30	-34	2145	1950	$S_{16}(6,10)$
15	142	6	10	42	77	4096	2145	1124	1122	33	-31	1950	2145	$S_{16}(6,10)c$
15	146	6	10	36	70	4096	2205	1184	1190	29	-35	2205	1890	$7B_{16}$
15	152	8	16	120	153	4096	2295	1334	1224	119	-9	270	3825	$S_{16}(0,8)c$
15	155	7	11	56	96	4096	2340	1340	1332	36	-28	1755	2340	$3S_{16}(0,4)$
15	167	7	11	35	72	4096	2520	1544	1560	24	-40	2520	1575	$8B_{16}$
15	168	8	12	72	117	4096	2535	1574	1560	39	-25	1560	2535	$2S_{16}(0,4)c$
15	181	5	11	5	22	4096	2730	1794	1870	10	-86	3640	455	
15	181	9	13	90	140	4096	2730	1826	1806	42	-22	1365	2730	$S_{16}(3,7)c$
15	188	8	12	32	72	4096	2835	1954	1980	19	-45	2835	1260	$9B_{16}$
15	194	10	14	110	165	4096	2925	2096	2070	45	-19	1170	2925	$S_{16}(2,6)c$
15	207	11	15	132	192	4096	3120	2384	2352	48	-16	975	3120	$U_{16}c$
15	209	9	13	27	70	4096	3150	2414	2450	14	-50	3150	945	$10B_{16}$
15	220	12	16	156	221	4096	3315	2690	2652	51	-13	780	3315	$S_{16}(0,4)c$
15	230	10	14	20	66	4096	3465	2924	2970	9	-55	3465	630	$11B_{16}$
15	233	9	14	9	42	4096	3510	2998	3066	6	-74	3744	351	
15	251	11	15	11	60	4096	3780	3484	3540	4	-60	3780	315	$12B_{16}$
15	254	14	16	126	255	4096	3825	3572	3570	17	-15	1800	2295	$S_{16}(0,2)c$
31	125	29	31	29	62	4096	3906	3722	3782	2	-62	3906	189	

arguments give that $t_0 + t_2 + t_4 = 13$, $2t_2 + 4t_4 = 4m$ and $2t_2 + 12t_4 = m(m-1)$. Solving the last two equations for t_4 gives $8t_4 = m^2 - 5m$, so $8 \mid m^2 - m$, forcing $m = 5$ or $m = 8$. The case $m = 5$ implies $t_4 = 0$, i.e., the set is a hyperoval in PG(2, 3), which does not exist. Similarly $m = 8$ gives that the set is the complement of a hyperoval, and so does not exist. ∎

COROLLARY 5.1. *The cases marked with † in the table can not arise from strongly regular (α, β)-reguli.*

Proof. Consider the case $s = 2$ with $t = 19$ in the table. Suppose a strongly regular (1, 3)-geometry arising from a strongly regular (1, 3)-regulus existed with these parameters. Since $v = 81 = 3^4$ the (1, 3)-regulus would exist in PG(3, 3). Since $s = 2$ the elements of the (1, 3)-regulus in PG(3, 3) would be points. It follows that the (1, 3)-regulus would be a collection of $(t + 1) = 20$ points in PG(3, 3) such that every line met the set in 0, $\alpha + 1 = 2$ or $\beta + 1 = 4$ points. In particular, any plane in PG(3, 3) would meet the set in a set of type (0, 2, 4). But such a set does not exist by the previous lemma, and the strongly regular (1, 3)-regulus does not exist. Essentially the same argument may be used to rule out the other three cases marked with a † in the table. ∎

Similarly no non-empty proper subsets of PG(2, 5) have type (0, 2, 4), (0, 3, 6) or (0, 4, 6) with respect to lines, and so the cases marked †† do not occur. Also no non-empty proper subsets of PG(2, 7) have type (0, 2, 4), (0, 3, 6), (0, 4, 8) or (0, 6, 8) with respect to lines, and so the cases marked ††† do not occur.

6. REMARKS

In this paper we have extended the study of semipartial geometries to strongly regular (α, β)-geometries in a straightforward way. The intension has been to provide a foundation for further study of strongly regular graphs via strongly regular (α, β)-geometries, as well as to study the geometries in their own right. But while we have given new constructions of strongly regular (α, β)-geometries the underlying strongly regular graphs were in each case previously known. It would be interesting to have constructions that give new strongly regular graphs.

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