High energy description of dark energy in an approximate 3-brane Brans–Dicke cosmology

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Abstract

We consider a Brans–Dicke cosmology in five-dimensional space–time. Neglecting the quadratic and the mixed Brans–Dicke terms in the Einstein equation, we derive a modified wave equation of the Brans–Dicke field. We show that, at high energy limit, the 3-brane Brans–Dicke cosmology could be described as the standard one by changing the equation of state. Finally as an illustration of the purpose, we show that the dark energy component of the universe agrees with the observations data.

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1. Introduction

Theories, in which scalar fields are coupled directly to the curvature, are termed scalar–tensor gravity, such as the low-energy effective string theory [1] and quintessence cosmologies [2–5]. However, the simplest and the best known is the Brans–Dicke (BD) theory [6]. The BD theory, which is a generalization of the general relativity, must recover the latter as the BD parameter $\omega$ goes to infinity [7]. From timing experiments using the Viking space probe [8], $\omega$ must exceed 500. This constraint ruled out many of extended inflation [9,10] and provides a succession of improved models of extended inflation [11–14].

Superstring theory suggests that the space–time of our universe might be of higher dimension [15,16], in which the extra dimensions are compactified and only the 4-dimensional (4D) is observed experimentally. Recently, a great deal of interest has been done in cosmological scenario in which matter field are confined to a 3-brane world embedded in a 5-dimensional (5D) bulk space–time [17–19].

Dark energy, distributed homogeneously in the universe, is a component of the critical density of our universe as showed by the cosmic microwave background (CMB) and type Ia supernovae (SNe) observations [20–22]. Using type Ia SNe [21] as standard candles to gauge the expansion of the universe shows that the dark energy causes the expansion of the universe to speed up. These two experiments are in agreement with $\Omega_\Lambda \simeq 0.7$ and $\Omega_m \simeq 0.3$. In a first attempt, the dark energy was described by a cosmological constant. Alternatively, a scalar field with some self-interaction potential could also describe this kind of matter/energy [2–5].

In this Letter, the dynamical system and dark energy of the universe are studied using the BD field in the 3-brane world. In this approach the BD field is used, with no potential, in the context of a nonzero cosmological constant $\Lambda_4$ [23–26]. In 3-brane world, a cosmological constant arises and depends on the (5D) bulk space–time parameters [19] as will be seen in Section 2.2. In brane cosmology [17–19], at the very early time, i.e. at high energy limit, dynamical evolution of the universe is modified by the extra terms in the Einstein equations, otherwise by the square of the matter density on the brane. In this way, we generalize the 4D BD theory to the 5D one by considering that BD field is sensitive only to the physical 3-brane. So it is described by the same 4D action and must recover the standard BD cosmology at low energy. To this aim, we add simply a BD stress–energy tensor to the modified Einstein equations, $4G_{\mu\nu}$, in 5D bulk space–time by neglecting all quadratic and mixed terms of this stress–energy tensor. To illustrate our con-
sideration we show, following first Kolitch’s work [23], that at high energy limit the 5D space–time could be described by 4D space–time cosmology, i.e. the information contained in the extra dimensions are now involved in the equation of state. The $\gamma$ factor characterizing the matter content of the universe in standard BD cosmology is equal to twice of the one in the 3-brane. Second, following Çalik’s work [27], we show the contribution of the dark energy in the dynamical evolution of the universe.

Let us notice that the choice of the high energy limit enables us to consider two stages of the evolution of the universe. The very early stages, i.e. the high energy limit, where the scalar field do not dominates the other forms of matter/energy and the second one, i.e. the recent stages or the low energy limit, where the scalar field—especially with negative pressure—dominates all forms of matter/energy.

The Letter is organized as follows: a short review of the standard BD cosmology with vacuum cosmological constant as well as the 3-brane world with BD field is presented in Section 2. Section 3 is devoted to resolve the dynamical system of the universe, while in Section 4 we relate the cosmological parameters to the dark energy. A conclusion is given in Section 5.

### 2. BD cosmology with cosmological constant

#### 2.1. 4-dimensional BD cosmology

Brans–Dicke cosmology with a nonzero cosmological constant, $\Lambda_4$, was studied by many authors [23–26]. In this section, we follow the notations and the work of the author [23] where we have:

$$ S = \int d^4x \sqrt{-g} \left( \phi (R - 2\Lambda_4) - \frac{\omega}{\phi} \partial_\mu \phi \partial^\mu \phi - 16\pi \mathcal{L}_m \right). \tag{1} $$

By varying this action with respect to the metric and BD field $\phi$, the homogeneous and isotropic Friedmann–Robertson–Walker equation with scale factor $a(t)$ and spatial curvature index $k$ and the wave equation of the BD field are:

$$ \left( \frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi} \right)^2 + \frac{k}{a^2} = \frac{3}{12} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{8\pi \rho}{3\phi} + \frac{\Lambda_4}{3} \tag{2} $$

and

$$ -\frac{1}{a^3} \frac{d(\dot{a}a^3)}{dt} = \frac{8\pi}{3} (3\gamma - 4)\rho - \frac{\Lambda_4}{4\pi} \tag{3} $$

where $\omega$ is the BD parameter, the dot denotes the derivative with respect to the time and $T_{\mu\nu} = \text{diag}(\rho, p, p, p)$ is the stress–energy tensor of a perfect fluid in an orthonormal frame. From the conservation equation of $T_{\mu\nu}$ we have:

$$ \dot{\rho} = -3\frac{\dot{a}}{a} (\rho + p). \tag{4} $$

Eqs. (2), (3) and (4) could be rewritten, as in [23,24]:

$$ \dot{X} = \frac{3}{2} \frac{Y^2}{A} (\gamma/2 - 1) - \frac{3YX^2}{2} + \frac{XY}{2A} + \frac{\Lambda_4}{2} \gamma - \frac{A_4}{6} (1 - 3\gamma/2). \tag{5} $$

where the new variables are defined as:

$$ X = \left( \frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi} \right), \tag{6} $$

$$ Y = A \frac{\dot{\phi}}{\phi}; \quad \text{and} \quad A = (3 + 2\omega)/12. \tag{7} $$

The equation of state is given by

$$ p = (\gamma - 1)\rho. \tag{8} $$

The solutions of this planar dynamical system have previously been examined in [23,24,27]. In the next subsection we will be interested in this dynamical system but in an extra dimension particularly in the 3-brane world.

#### 2.2. 5-dimensional BD cosmology

The modified Einstein equations on the 3-brane, derived from 5-dimensional bulk space–time, have the form [19]

$$ G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + k_4^2 \tau_{\mu\nu} + k_5^4 \Pi_{\mu\nu} - E_{\mu\nu} \tag{9} $$

where $\Lambda_4 = \frac{1}{4} k_4^2 (\Lambda_5 + \frac{3}{8} k_5^2 \lambda_2^2)$ is the 4D cosmological constant, $k_4^2 = 8\pi G_N = \frac{k_4^2}{\mathcal{L}}$ and $k_5^2$ are the 4D and the 5D gravitational constants, respectively ($G_N$ is the Newton’s constant of gravity) and the quadratic tensor $\Pi_{\mu\nu}$ is given by:

$$ \Pi_{\mu\nu} = -\frac{1}{4} \tau_{\mu\nu} \tau^{\alpha\bar{\nu}} + \frac{1}{12} \tau_{\mu\nu} + \frac{1}{8} q_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{24} q_{\mu\nu} \tau^2. \tag{10} $$

$E_{\mu\nu}$ is a part of the 5D Weyl tensor which we take equal to zero. $\Lambda_5$ is the cosmological constant of the bulk space–time. $\lambda$, $\tau_{\mu\nu}$ and $q_{\mu\nu}$ are the tension, the energy–momentum tensor and the metric, respectively, confined on the brane world.

Our proposal to generalize the gravitational equation (10) in 3-brane world is done as follows. First, to obtain the equivalent 4D BD of the field equations in the 5-dimensions universe, we consider that the behaviour of BD field is sensitive only to physical 3-brane, so it is described by the same action as in 4-dimension, Eq. (1). Second, to recover the BD cosmology at low energy, we propose to add simply a BD stress–energy tensor to the Einstein equation (10) in 5D bulk space–time, where all quadratic and mixed terms of this stress–energy tensor are neglected. So the modified Einstein equations are then written as:

$$ G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + 8\pi G_N \tau_{\mu\nu} + k_5^4 \Pi_{\mu\nu} - E_{\mu\nu} $$

$$ + \frac{\omega}{\phi^2} \left( \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\lambda} \phi^{,\lambda} \right) $$

$$ + \frac{1}{\phi} (\phi_{,\mu} q_{,\nu} - q_{\mu\nu} \Box \phi). \tag{11} $$
The BD field equations obtained by varying the action $S$, Eq. (1), with $E_{\mu\nu} = 0$ and $G_N = \frac{1}{8\pi}$, are:

$$\left(\frac{\dot{a}}{a} + \frac{1}{2} \frac{\dot{\phi}}{\phi}\right)^2 + \frac{k}{a^2} = \frac{3 + 2\omega}{12} \left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{8\Lambda}{3\phi} + \frac{k^2}{36\phi^2} + \frac{A_4}{3},$$

(13)

$$-\frac{1}{a^3} \frac{d(\dot{a}a^3)}{dt} = \frac{8\pi}{3 + 2\omega} \left(\frac{3}{4\pi} \left(3\gamma - 4\right) - \frac{k^2}{48\pi} \left(3\gamma - 2\right) \rho^2 - \frac{A_4\phi}{4\pi}\right),$$

(14)

$$\dot{\rho} = -3\frac{\dot{a}}{a} \gamma \rho.$$  

(15)

The wave equation of the BD field, (14), in the 3-brane world differs from the one, Eq. (3), in the standard BD cosmology. This means that the extra dimensions affect not only the Einstein equations but also the wave equation and the equation of state as will be stated later.

So at low energy limit, $\rho \gg \rho^2$, the field equations are the same as (2) and (3); or equivalently in terms of the variables $X$ and $Y$, we recover Eqs. (6) and (5).

At high energy limit, $\rho \ll \rho^2$, the field equations become:

$$\left(\frac{\dot{a}}{a} + \frac{1}{2} \frac{\dot{\phi}}{\phi}\right)^2 + \frac{k}{a^2} = \frac{3 + 2\omega}{12} \left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{k^2}{36\phi^2} + \frac{A_4}{3},$$

(16)

$$-\frac{1}{a^3} \frac{d(\dot{a}a^3)}{dt} = \frac{8\pi}{3 + 2\omega} \left(\frac{k^2}{48\pi} \left(3\gamma - 2\right) \rho^2 - \frac{A_4\phi}{4\pi}\right),$$

(17)

$$\dot{\rho} = -3\frac{\dot{a}}{a} \gamma \rho.$$  

(18)

In term of the variables $X$ and $Y$, we recover the following dynamical system:

$$X = 3\frac{Y^2}{A}(\gamma - 1) - 3\gamma X^2 + \frac{XY}{2A} + A_4 \gamma,$$

(19)

$$Y = X^2 \left(1 - \frac{3}{2} \gamma\right) - \frac{Y^2}{2A} (1 - 3\gamma) - 3XY - \frac{A_4}{6} (1 - 3\gamma)$$

(20)

which are the same as (6) and (5) with $\gamma$ in the standard case is equal to twice of the one in 5D bulk space–time. Therefore the 3-brane BD theory at high energies limit, could be described by the 4D one with the following equation of state

$$p = (2\gamma - 1)\rho.$$  

(21)

A physical consequence of this equation is to stress that a fluid with negative pressure could dominates the recent past, while in the very early time this fluid do not dominates the other forms of matter/energy. For example in a 3-brane world, at low energy, a fluid with $\gamma = \frac{1}{2}$ describes a negative pressure while at high energy it is described by a pressureless dust. So the origin of the acceleration of the present universe, caused by the so-called dark energy, could be explained by considering the low and the high energy limits of this approach.

3. Dynamical system of the universe

To show how the dark energy contributes to the dynamical system of the universe, and how changes appear from 4D to 5D; we linearize the dynamical system about the stable cosmological non-vacuum solution with flat space and show how the Hubble parameter varies with the scale factor $a(t)$.

3.1. Equilibrium solutions for a flat space

To study the dark energy in four- and five-dimensional BD cosmology using the Hubble parameter, we follow the work of [27] by introducing the variables, $H = \frac{\dot{a}}{a}$ and $F = \frac{\dot{\phi}}{\phi}$ rather than $X$ and $Y$. The field equations (2) and (3) become:

$$aH(2\omega + 3)H' = -3(\gamma(\omega + 2))F^2 - \omega(3\gamma - 4)HF - \frac{k}{a^2} \left(\omega(3\gamma - 2) + 3\right) + A_4(\gamma\omega + 1),$$

(22)

$$aH(2\omega + 3)F' = -3(\gamma - 4)F^2 - 4\omega + 3 + \frac{3\omega^2}{2}$$

(23)

where the prime denotes the derivative with respect to the scale factor.

Since the term $k/a^2$ decreases dramatically as $a(t)$ increases with the expansion of the universe, we drop it in the next by taking $k = 0$.

Let $(H_\infty, F_\infty)$ be the equilibrium points for a flat space ($k = 0$), which are obtained by setting $H'$ and $F'$ equal to zero in Eqs. (22) and (23). The solutions of the resulting equations are:

$$H_\infty [1, F_\infty [1] = \frac{1}{6A_4} \left(6A_4 + 6\omega\gamma - 3\omega^2\gamma^2\right)(-1/3, \gamma)$$

(24)

and

$$H_\infty [2, F_\infty [2] = \frac{2A_4}{(2\omega + 3)(3\omega + 4)}(\omega + 1, 1)$$

(25)

the ($\infty$) index means that the equilibrium is taken for a late time expansion.

The first solution is unstable while the second one is stable. In the next we will be interesting only in the second solution and we must have $\omega > -4/3$ or $\omega < -3/2$, and in the limit $\omega \rightarrow +\infty$ we have:

$$H_\infty \approx \sqrt{\frac{A_4}{3}} \approx \omega F_\infty.$$  

(26)

3.2. Linearized dynamical system

To solve the dynamical system (22) and (23), we linearize the solution as in [27]:

$$H = H_\infty + h(a),$$  

(27)
\[ F = F_\infty + f(a), \]
\[ \text{where } h(a) \text{ and } f(a) \text{ are linearized perturbation functions to be determined later.} \]

Putting (27) and (28) into the field equations (22) and (23) and neglecting higher terms in \( h(a) \) and \( f(a) \), one obtains the following system:

\[
\begin{align*}
\left( \frac{h'}{f'} \right) &= -\frac{H_\infty}{\omega + 1} \left( 3\gamma \omega + 4 (\gamma - 1)\omega \frac{9}{(3\gamma - 3) + (3\gamma + 5)} \right) \left( h \right) \\
- &\frac{\gamma}{a^2} \left( \frac{3}{(\omega+3) \gamma^2 - 2a} \right).
\end{align*}
\]

This system becomes

\[
\begin{align*}
\left( \frac{\dot{x}}{y} \right) &= -\frac{3a^2 + 1}{\omega + 1} \left( 3\gamma \omega + 4 (\gamma - 1)\omega \frac{9}{(3\gamma - 3) + (3\gamma + 5)} \right) \left( x \right) \\
+ &\left( \frac{3a^2 + 1}{\omega + 1} \frac{3^\gamma}{a^2 \gamma} \right) \left( y \right),
\end{align*}
\]

with

\[ h(a) = \frac{1}{3} (x\omega - x) \]

and

\[ f(a) = (x + y). \]

Therefore, the linearized solutions (27) and (28), for \( \omega \to \infty \), in the form as:

\[
\begin{align*}
H &= H_\infty - \frac{k}{2a^3} \left( \frac{a_0}{a} \right)^2 + H_0 C_1 \left( \frac{a_0}{a} \right)^{3+1/\omega} \\
&+ H_0 C_2 \left( \frac{a_0}{a} \right)^{3+1/\omega},
\end{align*}
\]

\[ F = F_\infty + H_0 C_1 \left( \frac{a_0}{a} \right)^{3+1/\omega} + H_0 C_2 \left( \frac{a_0}{a} \right)^{3+1/\omega}, \]

where the subscript ‘0’ indicates the present value. \( C_1 \) and \( C_2 \) and \( C_2' = \frac{C_2}{a^3} \) are dimensionless integration constants.

Eq. (33) shows that the Hubble parameter varies with the scale factor as in [27] but with an extra term whose exponent depends on the \( \gamma \)-parameter.

4. The cosmological parameters and dark energy

In this section we show that the Brans–Dicke theory is successful in explaining the dark energy which we relate to the most important cosmological parameters.

The Hubble parameter today, \( H_0 = \dot{a}/a(t_0) \), is used to estimate the order of magnitude for the present size and the age of the universe, has the value:

\[ H_0 \equiv 100h \text{ km s}^{-1} \text{ Mpc}^{-1}. \]

In particular, we can define the individual ratios \( \Omega_i \equiv \rho_i / \rho_c \), for matter, radiation, cosmological constant and even curvature, today. And from the standard Friedmann equations we have

\[ \frac{H^2}{H_0^2} = \Omega_\Lambda + \Omega_R \left( \frac{a_0}{a} \right)^4 + \Omega_M \left( \frac{a_0}{a} \right)^3 + \Omega_k \left( \frac{a_0}{a} \right)^2 \]

with

\[ \Omega_\Lambda = \frac{\Lambda}{3H_0^2}; \quad \Omega_k = -\frac{k}{a_0^2 H_0^2}; \]

\[ \Omega_M = \frac{8\pi G \rho_M}{3H_0^2}. \]

Now inserting the solution (33) in Eq. (36) one gets, in 4D space–time, the expression of the constants \( C_1 \) and \( C_2 \) by comparing respectively the expressions of \( \Omega_i \) in 4D with the BD ones in (33) for \( \omega \to \infty \).

First, we mention that all forms of matter/energy are possible. However, we are interested in the \( \gamma \)-parameter of the equation of state in order to recover the different exponents of Eq. (36). From (26) we obtain the following result:

\[ \left( \frac{H_\infty}{H_0} \right)^2 = \Omega_\Lambda = \frac{A_4}{3H_0^2}. \]

Eqs. (24) and (25) show that \( H_\infty \) depends on the BD parameter and on the cosmological constant. However, for large values of \( \omega \), \( H_\infty \) depends only on the cosmological constant (Eq. (26)). This equation shows explicitly the dependence of the dark energy on the cosmological constant while Eqs. (24) and (25) show implicitly its dependence on the BD parameter (i.e. for low values of \( \omega \)). In addition, the integration constants \( C_1 \) and \( C_2 \), (Eqs. (33) and (34)), depend on the model used here and, as will be shown below, these quantities are decisive to explore the acceleration era.

The integration constants \( C_1 \) and \( C_2 \) for different values of the \( \gamma \)-parameter are summarized as follows:

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>(-1/3,0,1/3,1/2,2/3,1,4/3,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>( \Omega_M ), ( \Omega_M - C_1 )</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( \Omega_M - 2C_1 )</td>
</tr>
</tbody>
</table>

we notice that in the cases \( \gamma = \frac{1}{2} \) (extended inflation) and \( \gamma = 1 \) (dust universe) the integration constants \( C_1 \) and \( C_2 \) depend on each other. The symbol \( \forall \) means that all values of \( C_2 \) are possible.

In the 3-brane, one obtains the same results by replacing the preceding \( \gamma \) by \( \frac{\gamma}{2} \). Hence, \( C_2 \) and \( C_1 \) are related to each other for \( \gamma = \frac{1}{4} \) and \( \gamma = \frac{1}{2} \).

To compare ours results with the experimental data, we use the deceleration parameter \( q_0 = \Omega_R + \frac{1}{2} \Omega_M - \Omega_\Lambda \) [7,28,29]. Neglecting \( \Omega_\Lambda \), one can parameterize the matter/energy content of the universe with just two components: the matter, characterized by \( \Omega_M \), and the vacuum energy by \( \Omega_\Lambda \).

Except the cases \( \gamma = \frac{1}{2},1 \), where \( C_1 \) and \( C_2 \) are not independent,
the line $\Omega_A = \frac{\Omega_M}{2}$, separating accelerating from decelerating universe, corresponds to:

$$C_1 = \frac{\Omega_M}{2\sqrt{\Omega_A}} = \sqrt{\Omega_A}. \quad (38)$$

$C_1 < \sqrt{\Omega_A}$ corresponds to an accelerating universe, while $C_1 > \sqrt{\Omega_A}$ corresponds to a decelerating universe.

- the line $\Omega_A = 1 - \Omega_M$, separating an open from a closed universes for a flat universe ($k = 0$), corresponds to

$$C_1 = \frac{\Omega_M}{2\sqrt{\Omega_A}} = \frac{1 - \Omega_A}{\sqrt{\Omega_A}}. \quad (39)$$

$C_1 < \frac{1-\Omega_A}{2\sqrt{\Omega_A}}$ corresponds to an open universe, while $C_1 > \frac{1-\Omega_A}{2\sqrt{\Omega_A}}$ corresponds to a closed universe.

Except the cases where $C_1$ and $C_2$ are not independent, experimental data give $C_1 = \frac{\Omega_M}{2\sqrt{\Omega_A}} \simeq 0.15$. So the $(\Omega_M, \Omega_A)$ plane shows that we live in accelerating flat universe, since $C_1 < \sqrt{\Omega_A}$, which is in accordance with the experimental data of Ia SNe [21].

While in the 3-brane the exception is for $\gamma = \frac{1}{4}$ and $\gamma = \frac{1}{2}$. In this approach, the (4D) BD cosmology and its generalization to the 3-brane world show that the accelerating universe could be explained by either a fluid with negative pressure ($\gamma = \frac{1}{2}$), a radiation ($\gamma = \frac{4}{3}$) or a stiff matter ($\gamma = 2$).

The non-vacuum dark energy, $\gamma = -\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{2}$ and $\frac{2}{3}$ in 4-dimension and $\gamma = -\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{2}$, $\frac{3}{5}$ and $\frac{2}{3}$ in the 3-brane contribute strongly to the behaviour of the universe as was seen before in the dynamical of the universe, we notice that the $\gamma$ spectrum is large in the 3-brane compared to the one of the 4D space–time.

We also claim that the 3-brane allows a much wider class of matter/energy than the 4D cosmology.

5. Conclusions

In summary, we have shown that at high energy the dynamical systems are the same in both cases with $\gamma$ (4-dimension) replaced by $2\gamma$ (5-dimension). However, at low energy the 5D BD theory coincides with the 4D one. This means that in vacuum era there is no difference between the 3-brane world and the 4-dimensional universe. While in “stiff” matter ($\gamma = 1$), pressureless dust ($\gamma = 1$) and radiation ($\gamma = \frac{2}{3}$) era in 4-dimension correspond to the dust, extended inflation ($\gamma = \frac{1}{2}$) and ($\gamma = \frac{2}{3}$) era in 3-brane world. So in the 3-brane universe, the era ($\gamma = 1$), extended inflation era ($\gamma = \frac{1}{2}$, $\frac{2}{3}$) could be studied at high energy limit as stiff matter, pressureless and radiation era respectively in standard BD cosmology. From the values of the $\gamma$-parameter the non vacuum dark energy could be constituted by an exotic form of matter/energy or by a combined of ordinary form of matter/energy. This work opens a new perspective in this field of research, namely studying the extra dimensions of the universe as standard one with a modified equation of state. Finally it is also interesting to know what happens at intermediate energy limit, this will be a subject of a forthcoming paper.

References