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An improved multiaxial method to estimate the elastic-plastic behavior from a purely elastic solution

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Abstract

In most fatigue applications, the nominal structural behavior is dominated by elastic deformation, but the fatigue lifetime is significantly influenced by plasticity around stress concentrations and flaws. Although the elastic-plastic behavior can be modeled with finite element analysis (FEA), the computational expense may be prohibitive, especially for variable amplitude loading with multiaxial stress states. To overcome this complexity, an elastic-plastic estimate is explored that utilizes the purely elastic solution. The method is conceptually similar to previous work, but is adapted to be consistent for variable amplitude multiaxial cyclic loading histories. This approach utilizes a magnitude criterion (i.e. Neuber [1] or ESED [2]), and an elastic alignment assumption to adjust the boundary conditions applied to the local material model. In this document, the effect of alignment is explored with a multiaxial 'Box' loading path.

Keywords: multiaxial deformation; ESED method; elastic-plastic approximation

1. Introduction

Many fatigue applications benefit from improved estimates in the mechanical response, particularly for short-moderate lifetimes where plastic deformation dominates the material damage. In recent years, the finite element method has been utilized successfully in a wide variety of circumstances, including cyclic fatigue applications. However, cyclic deformation using FEM is costly, particularly for non-proportional multiaxial loading histories. Consequently, an elastic-plastic approximation based on the local elastic solution (i.e. FEM) is one practical method to approximate the elastic-plastic behaviour with minimal cost.

Much of the previous work to obtain an elastic-plastic estimate is based on stress concentration factors around notches. The most common of these estimates was developed by Neuber [1] in 1961, when he related the elastic-plastic stress and strain concentration factors to the purely elastic behaviour. Another popular approach was introduced by Molski and Glinka [2], which equates the strain energy density between the purely elastic and elastic-plastic approximation (ESED). Both of these methods have been the involved in several other investigations [3-8], but their extension to non-proportional multiaxial loading is fairly limited. For instance, applying the ESED method...
to multiaxial loading [4] can result in a multiple solutions. Consequentially, a procedure to determine the correct (or most appropriate) solution must be adopted. This ambiguity and the restriction of potential plasticity models limit the applications of typical ESED methods.

A pseudo-material approach [9-10] has been more successful in achieving appropriate estimates for general non-proportional multiaxial loadings. This method involves assuming that the notched material behaves similar to the real material (with modification of material parameters, or potentially the material model). The advantage of the pseudo-material method is its familiar construction and straightforward application to non-proportional multiaxial loading. The difficulty of these methods is choosing an appropriate pseudo-material model because either the notch geometry (or elastic solution’s mechanical behaviour) is coupled to the true material response when estimating the local strains. Although such an avenue would be ideal with limited notches geometries and sufficient experiments, its non-trivial coupling makes modifying assumptions for different applications non-trivial. For example, this coupling requires special consideration to translate the hydrostatic response, because of the elastic solution tends to over-estimate volume changes. Furthermore, the pseudo-material approach is not necessarily compatible with the familiar relationships (such as Neuber or ESED) and the solution requires solving the plasticity problem twice.

In the current investigation, a method is proposed to estimate the elastic-plastic behaviour from a purely elastic solution by applying a magnitude criterion and directional alignment as modified boundary conditions. The result is a method that maintains the multiaxial advantages of the pseudo-material approach without geometric coupling and repeating the elastic-plastic solution procedure. The proposed approximation is robust, consistent, and efficient, with straightforward choices to adjust magnitude and constraint for improved estimate capabilities.

### Nomenclature

- $\alpha^{(i)}$: Backstress tensor
- $\chi^{(i)}$: Ratcheting exponent
- $\kappa$: Bulk modulus
- $\mu$: Shear modulus
- $\sigma'$: Deviatoric stress tensor
- $\varepsilon$: Purely elastic stress tensor
- $\xi$: Nominal stress tensor
- $\Delta\varepsilon$: Total strain increment
- $\Delta\varepsilon'$: Elastic strain increment
- $\Delta\varepsilon''$: Plastic strain increment
- $\Delta\sigma$: Stress increment
- $\Delta\sigma'$: Deviatoric stress increment
- $\Delta\varepsilon'$: Purely elastic strain increment
- $\Delta\sigma'$: Purely elastic stress increment
- $c^{(i)}$: Plasticity / SED parameter
- $d\alpha^{(i)}$: Differential of backstress
- $d\varepsilon$: Differential of plastic strain
- $d\lambda$: Differential of magnitude of plastic strain
- $d\alpha^{(i)}$: Differential of the elastic SED state variable
- $f$: Yield surface
- $k$: Shear yield strength
- $n$: Exponent of power-law relationship
2. Material Modeling

In the classic notch-problem, determining the stress concentration factor (i.e., from elastic solutions, experimental techniques, or the FEM) is often the first step to estimate the local elastic-plastic behaviour. Due to the overwhelming popularity of the finite element method, it is advantageous to utilize the local purely elastic solution (rather than the nominal loading) to further generalize the elastic-plastic estimate. Although utilizing the local elastic solution greatly simplifies the notch problem, several assumptions are still required to acquire meaningful elastic-plastic approximations. For instance, consider a general local solid mechanics problem with 12 unknowns (6 stresses and 6 strains) at a single material point. The elastic-plastic estimate is obtained by approximating these unknowns through an appropriate material model (6 relationships) and the purely elastic solution (6 components). In this investigation, the material’s constitutive behaviour and geometry are considered independently to construct an approximation technique that is appropriate to multiaxial fatigue loading.

Since cyclic deformation is of primary interest, the total strain increment, $\Delta \varepsilon$, is additively constructed from the elastic, $\Delta \varepsilon^e$, and plastic, $\Delta \varepsilon^p$, strain increments, which is appropriate for small strain deformation (i.e. $|\varepsilon|<<1$)

$$\Delta \varepsilon = \Delta \varepsilon^e + \Delta \varepsilon^p$$ (1)

An incremental format was adopted because only rate independent deformation is considered in the current investigation. The elastic strain increment is related to the stress increment by isotropic linear elasticity (Hooke’s Law [11]):

$$\Delta \varepsilon^e = \frac{1}{2\mu} \Delta \sigma^e + \frac{1}{9\kappa} \text{tr}(\Delta \sigma) I$$ (2)

where $I$ is the 2nd order identity tensor and the trace, tr($x$), is defined as:

$$\text{tr}(x) = x_{11} + x_{22} + x_{33}.$$ (3)

This elasticity format was chosen because it decouples the deviatoric, $\Delta \sigma^e$, and hydrostatic, $\text{tr}(\Delta \sigma)$, stress contributions with the shear, $\mu$, and bulk, $\kappa$, moduli, respectively, which is advantageous when considering volume conserving plastic deformation. The plastic strain increment is characterized by a plastic strain magnitude, $d\lambda$, and a plastic strain direction, $N$:

$$d\varepsilon^p = d\lambda N.$$ (4)
The plastic strain direction is chosen to satisfy the normality condition of the yield locus:

$$ f = \left( \sigma' - \sum_{i=1}^3 \alpha^{(i)} \right) : \left( \sigma' - \sum_{i=1}^3 \alpha^{(i)} \right) - 2k_{M} = 0 \quad (5) $$

where the initial shear yield strength, $k_{M}$, scales the difference in the deviatoric and backstress ($\alpha^{(i)}$) terms. The backstress is additively constructed from a multilayer model suggested by Jiang et al. [12]. The backstress evolution may be expressed in the following format:

$$ d\alpha^{(i)} = r^{(i)} N - \left( \frac{k_{M}}{r^{(i)}} \right) \alpha^{(i)} e^{(i)} d\lambda \quad (6) $$

where $r^{(i)}$ represents the hardening, $e^{(i)}$ represents an inverse of critical strain, and $\dot{\chi}^{(i)}$ defines the ratcheting characteristics.

To provide some information on the local stress behaviour, the current investigation considers three basic alignment cases. The first aligns the increment in stress for the elastic and elastic-plastic solutions:

$$ \sigma\text{-aligned:} \quad \frac{\Delta^{E}\sigma}{\|\Delta^{E}\sigma\|} = \frac{\Delta\sigma}{\|\Delta\sigma\|} \quad (7) $$

where $\Delta^{E}\sigma$ is the increment of stress for the elastic solution, $\Delta\sigma$ is the increment in stress for the elastic-plastic solution, and $\|x\|$ refers to the 2nd norm of tensor, $x$, which is defined below for a symmetric 2nd order tensor:

$$ |x| = \sqrt{x : x} = \sqrt{x_{11} + x_{22} + x_{33} + 2x_{12} + 2x_{13} + 2x_{23}}. \quad (8) $$

This alignment specifies 5 degrees of freedom and is clearly consistent for purely elastic loading, where the two stress definitions should behave identically. Aligning stress direction ensures smoothness in the approximated stress field (for a continuous magnitude estimate), which would maintain equilibrium from the purely elastic solution along with any 'stress-free' boundary conditions. This stress alignment is consistent with assuming that the stress concentration factors remain proportional during the elastic-plastic estimate, which is often assumed for engineering applications. This assumption is ideal for fatigue applications, where the critical area is on (or very near) as free surface, which is considered a relatively low constraint.

To consider scenarios with increased constraint, the strain may be aligned in the elastic and elastic-plastic solutions:

$$ \varepsilon\text{-aligned:} \quad \frac{\Delta^{E}\varepsilon}{\|\Delta^{E}\varepsilon\|} = \frac{\Delta\varepsilon}{\|\Delta\varepsilon\|} \quad (9) $$

In this case, compatibility remains satisfied in the elastic-plastic approximation, resulting in higher stress estimates. One last scenario was considered to separate the transverse constraint introduced by the notch. In this case, the stresses are aligned in every direction except the transverse ($\theta$) component, where the strain is aligned.

$$ \text{mixed:} \quad \frac{\Delta^{E}\sigma}{\|\Delta^{E}\sigma\|} = \frac{\Delta\sigma}{\|\Delta\sigma\|} \quad \text{except}_{\theta} \quad \text{and} \quad \frac{\Delta^{E}\varepsilon_{\theta}}{\|\Delta^{E}\varepsilon_{\theta}\|} = \frac{\Delta\varepsilon_{\theta}}{\|\Delta\varepsilon_{\theta}\|} \quad (10) $$

This alignment condition likely provides a compromise between the two extreme cases. It should be noted that all three cases considered preserved the plane stress boundary condition appropriate for specimen considered (i.e., $\sigma_{zz}$, $\sigma_{\theta z}$, $\sigma_{\theta\theta} = 0$), but only small differences are expected if all strain components were force to be aligned, as may be interpreted by Eq. 9. Other directional alignments may also be considered, but these cases encompass the majority of potential behaviour for the 'Box' loading history presented subsequently.

With the material constitutive model and the elastic to elastic-plastic stress alignment chosen, specifying the magnitude of stress would fully characterize the mechanical behaviour. More generally, this magnitude may be interpreted as a scalar relationship between the purely elastic and elastic-plastic solutions. Fortunately even the earliest works in the notch-problem literature (i.e., Neuber [1] or ESED [2]) provide valuable insight toward constructing this relationship. For instance, consider the equivalent strain energy density (ESED) condition [2-3], which equates the strain energy in the local elastic solution to the local elastic-plastic solution:
where the purely elastic strain energy, $U_e$, is decomposed into local elastic ($U_e$) and plastic ($U_p$) contributions. Each term is defined by an integral as presented below:

$$
\int \sigma : d\varepsilon = \int \sigma : d\varepsilon' + \int \sigma' : d\varepsilon
$$

Although the above integrals are well defined for multi-axial cyclic deformation, adjustment is necessary to achieve Masing behaviour [13], which is evident in many notch experiments. Specifying Masing behaviour is analogous to utilizing a nominal 'pseudo-material' to estimate the local elastic-plastic response [9-10]. To clarify deviatoric (i.e., torsion) fully reversed ($R = -1$) behaviour in Fig. 1a. As illustrated, this Masing behaviour follows a curve analogous to stress-strain behaviour, while the monotonic curve exhibits unique mirrored behaviour.

By recognizing the similarity in character to the Masing behaviour and the stress-strain response, it is evident that one may adopt a model similar to the plasticity model (Eq. 6) to relate the Masing behaviour may be achieved for the stress-energy curve may be described by a series of linear segments, as illustrated in Fig. 1. By considering the segments independently (unlike many Armstrong-Frederick type models, which utilize a series of additive segments that are always active prior to saturation [12]), each segment may be uniquely defined in either stress or $U_e$ space. Since the change in stress determines the elastic strain energy density, the energy state variable, $q^{\text{ini}}$, is defined with respect to a stress increment.

$$
q^{\text{ini}} = 2\sqrt{\mu} (U_e)^{1/2}
$$

Consequently, the Masing behaviour may be achieved for the $U_e$ by applying a plasticity model with appropriately modified constants. Although this analogy is appropriate, it is more effective to modify the plasticity model to specify the $U_e$ from a given stress increment. In other words, it is convenient to construct a cyclic pseudo-plasticity model (with kinematic hardening) by a controlled stress increment, rather than plastic strain, or in this case $U_e$. To construct such a model, suppose that a stress-energy curve may be described by a series of linear segments, as illustrated in Fig. 1. By considering the segments independently (unlike many Armstrong-Frederick type models, which utilize a series of additive segments that are always active prior to saturation [12]), each segment may be uniquely defined in either stress or $U_e$ space. Since the change in stress determines the elastic strain energy density, the energy state variable, $q^{\text{ini}}$, is defined with respect to a stress increment.

To construct the multi-axial cyclic behaviour of a segmented stress-ESED curve (that does not soften or saturate in M terms), each segment is represented by two parameters: the inverse of the increment in deviatoric stress ($e^{\text{ini}}$) and the increment in $U_e$ ($r^{\text{ini}}$), both are labelled in Fig. 1b. Each segment is only activated after the previous segment has achieved saturation ($\|\varepsilon^{\text{ini}}\| = r^{\text{ini}}$). A ‘yield’ stress criterion is not necessary, since $U_e$ accumulates even at zero stress. Lastly, a rule to prevent saturation was adopted that represents a nearly infinite slope that continues to accumulate $U_e$ without a (inappropriate) saturation criterion. The evolution of each segmented pseudo-plastic strain was defined similarly to the Jiang et. al. model [12] with an infinite ratcheting exponent. Mathematically, this model may be concisely expressed below separating the two basic conditions:

$$
U_e = U_e + U_p
$$

(11)

where $U_e$ is replaced with the plastic strain:

$$
\sigma = K(e^{\text{ini}})^\gamma
$$

(17)
Hardening: \[ q^{(i+1)} = r^{(i+1)} \Rightarrow \left\{ \begin{align*} \|q^{(i)}\| & < r^{(i)} \quad \Rightarrow \quad dq^{(i)} = r^{(i)} c^{(i)} d\sigma^i \\ \|q^{(i)}\| & \geq r^{(i)} \quad \Rightarrow \quad dq^{(i)} = r^{(i)} c^{(i)} - c^{(i)} \|d\sigma\| q^{(i)} \end{align*} \] (18)

Saturation: \[ q^{(M)} = r^{(M)} \Rightarrow dq^{(M+1)} = 10^3 r^{(M)} c^{(M)} d\sigma^i \] (19)

where the corresponding deviatoric elastic strain energy density is defined as the sum of these state variables projected along the direction of deviatoric stress increment:

\[ U_e = \frac{d\sigma^i}{\|d\sigma\|} \sum_{i=1}^n (q^{(i)}) \] (20)

This projection is appropriate since the strain and stress increments are aligned for the elastic deformation, because if the bounded surface relates to plastic strain (instead of back-stress), resulting in non-physical behaviour during multi-axial loading. However, a 1-D equivalent model was utilized in a similar way to describe the hydrostatic elastic strain energy density. The local plastic strain energy was determined using the usual definition (the last integral in Eq. 12). This definition is appropriate since the plastic behaviour already exhibits Masing behaviour in the stress-plastic strain response and consequently the strain energy density when hardening is neglected. Furthermore, an expression analogous to Eq. 16 is not obtainable for the local plastic strain energy density, making using the usual definition a necessity.

Fig. 1. Deviatoric stress magnitude versus (a) cumulative strain energy density for the Masing and monotonic models for fully-reversed cyclic loading and (b) the elastic strain energy density, illustrating the linearly segmented material model

3. Results and Discussion

The estimation method outlined in the previous section was adopted to predict the deformation behaviour of several experiments first conducted by Barkey et. al. [9]. These experiments utilized strain gages at the notch-tip of specimens subjected to nominal axial and torsional multiaxial loading histories. To obtain the local elastic solution based on the nominal loading of these experiments, without a finite element model, one may utilize the concept of stress concentration. For plane stress bi-axial (tension-torsion) loading, the local elastic stress, \( \varepsilon \sigma \), may be related to the nominal stress, \( \xi \), by the following relationships:

\[ \varepsilon \sigma_n = K_z \xi, \quad \varepsilon \sigma_T = K_{zT} \xi, \quad \varepsilon \sigma_{\varphi} = K_{z\varphi} \xi \] (21)

where the stress concentration factors \( K_z \), \( K_{zT} \), and \( K_{z\varphi} \) characterize the notch behaviour in the axial, torsion, and
transverse directions respectively. The stress concentration factors specified are appropriate for the geometry utilized by Barkey et. al. [9]. The constitutive material behaviour was assumed to be isotropic, where the elastic and plastic constants are consistent with the parameters reported in the literature [9]. A summary of these parameters is provided in Table 1, including the Masing elastic strain energy density models, and the Jiang [6] plasticity model.

Table 1: Material parameters for SED and Plasticity models

<table>
<thead>
<tr>
<th>i</th>
<th>Hydrostatic SED</th>
<th>Deviatoric SED</th>
<th>Plasticity (χ(0) = χ)</th>
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<tr>
<td></td>
<td>r(0) (MPa)</td>
<td>c(0) (MPa⁻¹)</td>
<td>r(0) (MPa)</td>
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<td>2.790</td>
</tr>
</tbody>
</table>

To establish the effect of the various alignments described in the modelling section, the estimation method was applied to a ‘Box’ loading path and compared with the experimental results. The strain response of the ‘Box’ loading path is presented in Fig. 2a for the experiment, σ-aligned, ε-aligned, and mixed boundary conditions. The nominal stress and local stress response is presented in Fig. 2b, where the x-axis is the axial stress (σzz) and the y-axis is the non-zero torsional stress (τθθ). Although the nominal and local stresses are on the same order of magnitude, the purely elastic solution predicts stresses that are much higher, due to the stress concentration factors (Eq. 21).

Fig. 2. Multiaxial ‘Box’ loading path showing (a) shear strain vs. axial strain and (b) the shear stress vs. axial stress for the experiment [9], and various alignments

All alignment scenarios result in relatively similar stress and strain ranges, where the σ-aligned case most closely matches the character of the experiment. The agreement in the experiment and s-aligned simulation is most evident early in the experiment, before transverse hardening/softening effects contribute to the deformation. As one may expect, the ε-aligned case results in a rectangular strain path, and the σ-aligned case results in a rectangular stress path. Also, the ε-aligned case appears to be an upper bound on the stress range, while the σ-aligned case corresponds to an upper bound on the strain range. The mixed alignment case follows a similar path to the experiment (and σ-aligned case) for changes in axial stress, but due to the added constraint, it maintains a constant axial strain during the change in nominal shear stress. While this character does not agree with the experimental...
observations, it is evident that the σ-aligned case over estimates the change in axial strain during the nominal shear deformation. This suggests that the actual notch constraint is somewhere between the σ-aligned and mixed scenarios considered.

4. Conclusions

The current method estimates the elastic-plastic response from a purely elastic history by decoupling the notch geometry and local material response with a modified boundary condition approach. Specifically, the method combines a magnitude criterion with directional alignment to impose boundary conditions for an appropriate multiaxial plasticity model. The approximation is developed in a general manner, with the potential to adjust the constraint (direction alignment, i.e. stress), magnitude (i.e. ESED or Neuber), and the plasticity character. A few additional conclusions may be drawn:

- The magnitude criterion should exhibit Masing behaviour between the nominal stress and local strain to be consistent with the experimental evidence.
- The effect of alignment was explored for the ‘Box’ loading path, illustrating best agreement with the V-aligned case. Furthermore, σ-alignment provides an upper bound estimate on the strain range and ε-alignment provides an upper bound estimate on the stress range, for a given magnitude criteria.
- With appropriate modelling choices, the current approach is consistent with any arbitrary multiaxial cyclic loading. Subsequently, the elastic-plastic response may be employed to improve fatigue life or damage estimates associated with the local mechanical response.

References