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Various shadowing properties for positively expansive maps

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Abstract

In this paper, various shadowing properties are considered for a positively expansive map on a compact metrizable space. We show that the Lipschitz shadowing property, the s-limit shadowing property and the strong shadowing property are all equivalent to the (usual) shadowing property for a positively expansive map. Furthermore, for a positively expansive open map, the average shadowing property is shown.

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Introduction

Let X be a compact metrizable space, and let f be a continuous map of X onto itself. Fix any metric d for X (throughout this paper, this term means that d is a metric compatible with the topology of X). As usual, a sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is called a δ*-pseudo-orbit* ($\delta > 0$) of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \geqslant 0$. We say that f has the (usual) *shadowing property* if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$, there exists $y \in X$ satisfying $d(f^i(y), x_i) < \varepsilon$ for all $i \geqslant 0$. This property is independent of a metric for X.

We say that f is *positively expansive* if there exist a metric d for X and a constant $c > 0$ such that $d(f^i(x), f^i(y)) \leqslant c$ $(x, y \in X)$ for all $i \geqslant 0$ implies $x = y$. Such a number c

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is called an *expansive constant*. This property (although not c) is also independent of a metric. It is easy to see that every one-sided shift map and every expanding differentiable map on a C^∞ closed manifold are positively expansive (see [7,13,20,21]).

These properties are very often appearing in several branches of the theory of dynamical systems, and especially, they are usually playing an important role in the investigation of the stability theory and the ergodic theory (see [5,6,8,12,15,16,21]).

We say that f has the *Lipschitz shadowing property* if there are a metric d for X and positive constants L, ε_0 such that for any $0 < \varepsilon < \varepsilon_0$ and any ε-pseudo-orbit $\{x_i\}_{i=0}^\infty$ of f, there exists $y \in X$ such that $d(f^i(y), x_i) < L\varepsilon$ for all $i \geqslant 0$ (see [11,12]).

The so-called limit shadowing property is introduced and studied in [12]. We say that f has the *limit shadowing property* if there is a metric d for X with the following property: for any sequence $\{x_i\}_{i=0}^\infty$ of points in X, if $d(f(x_i), x_{i+1}) \to 0$ as $i \to \infty$, then there exists $y \in X$ satisfying $d(f^i(y), x_i) \to 0$ as $i \to \infty$. Since there is an example of the system possessing the limit shadowing property but not possessing the shadowing property (see [12, pp. 65–66]), the property is not equivalent to the shadowing property in general.

We say that f has the *s-limit shadowing property* if there is a metric d for X with the following property: for every $\varepsilon > 0$, there is $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i\}_{i=0}^\infty$ of f, there exists $y \in X$ satisfying $d(f^i(y), x_i) < \varepsilon$ for all $i \geqslant 0$, and, if in addition, $d(f(x_i), x_{i+1}) \to 0$ as $i \to \infty$, then $d(f^i(y), x_i) \to 0$ as $i \to \infty$. The s-limit shadowing property is treated in [1], and it is proved therein that every expansive homeomorphism on a compact metric space having the shadowing property possesses the s-limit shadowing property. In this paper, we show a similar result for a positively expansive open map.

Clearly, both the Lipschitz and the s-limit shadowing properties are stronger than the shadowing property by definition.

We say that f has the *strong shadowing property* if there is a metric d for X with the following property: for every $\varepsilon > 0$, there is $\delta > 0$ such that if a sequence $\{x_i\}_{i=0}^\infty \subset X$ satisfies the inequality

$$\sum_{i=0}^{\infty} d\big(f(x_i), x_{i+1}\big) < \delta,$$

then there is a point $y \in X$ satisfying

$$\sum_{i=0}^{\infty} d\big(f^i(y), x_i\big) < \varepsilon.$$

In [9] the above pseudo-orbit, which is called a *δ-strong-pseudo-orbit* of f, is considered in the investigation of the ergodic theory of dynamical systems (see also [12, p. 70]).

As in the usual shadowing property, both the (usual) limit and s-limit shadowing properties are independent of a metric for X. Actually, suppose that D is another metric for X. Then it is easy to see that if $d(f(x_i), x_{i+1}) \to 0$ (respectively $D(f^i(y), x_i) \to 0$) as $i \to \infty$, then $D(f(x_i), x_{i+1}) \to 0$ (respectively $d(f^i(y), x_i) \to 0$) as $i \to \infty$. However, both the Lipschitz shadowing property and the strong shadowing property depend on the metric.

In this paper, we show that most of the above various shadowing properties are mutually equivalent for positively expansive maps. More precisely, the following is proved.

Theorem 1. *Let f be a positively expansive map on a compact metrizable space X. Then the following conditions are mutually equivalent*:

(1) *f is an open map,*
(2) *f has the shadowing property,*
(3) *there is a metric such that f has the Lipschitz shadowing property,*
(4) *there is a metric such that f has the s-limit shadowing property,*
(5) *there is a metric such that f has the strong shadowing property,*

An interesting example of a positively expansive map on X which is not an open map can be found in [14].

In [3–5], the average shadowing property is defined and discussed in the context of random dynamical systems for piecewise C^2-differentiable maps. Let $f:(X,d)\to(X,d)$ be a continuous map. For $\delta>0$, a sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is called a *δ-average-pseudo-orbit* of f if there is a number $N=N(\delta)>0$ such that for all $n\geqslant N$ and $k\geqslant 0$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d\big(f(x_{i+k}),x_{i+k+1}\big)<\delta.$$

The notion of average-pseudo-orbits is a certain generalization of the notion of pseudo-orbits and is arising naturally in the realizations of independent Gaussian random perturbations with zero mean etc (see [3,4] and [5, p. 368]).

We say that f has the *average shadowing property* if there is a metric d for X with the following property: for every $\varepsilon>0$, there is $\delta>0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ε-shadowed in average by some point $y\in X$; that is,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d\big(f^i(y),x_i\big)<\varepsilon.$$

This property also depends on a metric for X.

It is known that every Axiom A diffeomorphism restricted to the basic set has the average shadowing property (see [3,5]). To my best knowledge, however, it is unknown whether an expanding differentiable map on a C^∞ closed manifold admits the average shadowing property. In this paper, we give an affirmative answer for the problem.

Recall that a continuous map f on X is said to be *topologically transitive* if there is a dense orbit; that is, $X=\overline{\{f^n(x)\colon n\geqslant 0\}}$ for some $x\in X$ (see [8,15,21]). The average shadowing property is closely related to the topological transitivity for a positively expansive open map. Actually, the following is proved.

Theorem 2. *Let $f:X\to X$ be a positively expansive open map on a compact metrizable space. Then the following conditions are equivalent*:

(1) *f has the average shadowing property with respect to some metric,*
(2) *f is topologically transitive.*

Let $f:X\to X$ be a positively expansive open map. If X is connected, then f is topologically transitive (more precisely, f is topologically mixing, see [15,17]). The next corollary quickly follows from Theorem 2.

Corollary. *Let $f:X\to X$ be a positively expansive open map on a compact metrizable space. If X is connected, then f has the average shadowing property with respect to some metric.*

Let $f:X\to X$ be a continuous map on a compact metrizable space. We say that f *expands small distances* if there exist a metric d for X and constants $\delta_0>0$ and $\lambda>1$ such that $0<d(x,y)<\delta_0$ $(x,y\in X)$ implies $d(f(x),f(y))>\lambda d(x,y)$. If, in addition, f is open, then we call such f "*Ruelle expanding*". This terminology is equivalent to Ruelle's definition of expanding maps (see [15, p. 143]). Of course, every expanding differentiable map on a C^∞ closed manifold is Ruelle expanding.

It is easy to see that if f expands small distances as above, then f is positively expansive with an expansive constant $\delta_0/2$. Hence, by the corollary, every expanding differentiable map on a C^∞ closed manifold has the average shadowing property.

Remark 1. In [19], the average shadowing property is shown for an expansive homeomorphism having the shadowing property on a compact metrizable space (with respect to some metric). As a corollary, it was proved therein that if f is Ruelle expanding, then the inverse limit system $\boldsymbol{f}$ has the average shadowing property under the condition that f is Lipschitz (see [19, p. 241]). In this paper, we have "dropped" the Lipschitz assumption and proven the average shadowing property for f.

Here we say that $f:(X,d)\to(X,d)$ is *Lipschitz* if there exists a constant $K>0$ such that $d(f(x),f(y))\leqslant Kd(x,y)$ for all $x,y\in X$.

Clearly, f has the usual (respectively the limit, the s-limit) shadowing property if and only if f^n has the usual (respectively the limit, the s-limit) shadowing property for all $n>0$. On the other hand, it is easy to see that if f has the Lipschitz (respectively the strong, the average) shadowing property, then so does f^n $(n>0)$, respectively. Conversely, if f is Lipschitz with constant K (we may suppose that $K\geqslant 1$) and f^n $(n>0)$ has the Lipschitz (respectively the strong, the average) shadowing property, then so does f, respectively. Indeed, for any integer $n>0$;

- if $\sum_{i=0}^{\infty} d(f(x_i),x_{i+1})<\delta$, then $\sum_{i=0}^{\infty} d(f^n(x_i),x_{n+i+1})<K^{n-1}\delta$,
- if $\sum_{i=0}^{\infty} d(f(x_i),x_{i+1})<\delta$ and $\sum_{i=0}^{\infty} d(f^{ni}(y),x_{ni})<\varepsilon$, then

$$\sum_{i=0}^{\infty} d\big(f^i(y),x_i\big)<\big(1+K+K^2+\cdots+K^{n-1}\big)\varepsilon+\delta.$$

Hence, by Lemma 1(ii) (see the next section), we have the following.

Remark 2. For a positively expansive open map f on a compact metrizable space, f has the Lipschitz (respectively the strong, the average) shadowing property with respect to some metric if and only if f^n has the Lipschitz (respectively the strong, the average) shadowing property with respect to some metric for all $n > 0$.

1. Proof of Theorem 1

First of all in this section, we construct a special metric D for X. The next lemma, which is proved by following [13, Proof of Proposition], not only performs its duty in the proof of Theorem 1 but also plays an essential role in the proof of Theorem 2.

Lemma 1 (cf. [19]). *Let $f:(X,d)\to(X,d)$ be positively expansive. Then there exists a new metric D for X such that*

(i) *f expands small distances,*
(ii) *f is Lipschitz.*

Proof. Let $c > 0$ be an expansive constant and define a nested sequence of open symmetric neighborhoods of the diagonal, Δ (in $X \times X$), as follows. Set $V_0 = X \times X$, and for $n \geqslant 1$, let

$$V_n = \bigl\{(x,y)\in X\times X:\ d\bigl(f^i(x), f^i(y)\bigr) < c \text{ for } 0\leqslant i\leqslant n-1\bigr\}.$$

Then $\bigcap_{n=0}^{\infty} V_n = \Delta$ and $g(V_n) = V_{n-1}\cap g(V_1)$ for $n > 1$ (see [13, Construction Lemma]). Here $g = f\times f: X\times X\to X\times X$.

Since V_1 is a neighborhood of Δ, there exists $\delta > 0$ such that

$$N_\delta(\Delta) = \bigl\{(x,y)\in X\times X:\ d(x,y) < \delta\bigr\}\subset V_1.$$

Since X is compact and $\bigcap_{n=0}^{\infty} V_n = \Delta$, there is $N \geqslant 1$ such that $V_{1+N}\subset N_{\delta/3}(\Delta)$. Then $V_{1+N}\circ V_{1+N}\circ V_{1+N}\subset V_1$. Define a new sequence $\{U_n\}_{n=0}^{\infty}$ by $U_0 = V_0$, $U_n = V_{1+(n-1)N}$ for $n\geqslant 1$. By [13, Metric Lemma], there is a metric ρ for X such that

$$U_n\subset\bigl\{(x,y)\in X\times X:\ \rho(x,y) < 1/2^n\bigr\}\subset U_{n-1}\quad \text{for } n\geqslant 1.$$

Let $L = \max\{1, \operatorname{diam}_\rho(X)\}$ and put $K^N = 2^5 L$, where $\operatorname{diam}_\rho(X) = \sup\{\rho(x,y):\ x,y\in X\}$. If $\rho(x,y)\geqslant 1/2^5$, then

$$\rho\bigl(f^N(x), f^N(y)\bigr)\leqslant L\leqslant 2^5 L\cdot\rho(x,y)\leqslant K^N\rho(x,y). \tag{1}$$

Suppose that $0 < \rho(x,y) < 1/2^5$. Then there exists $n\geqslant 3$ such that $(x,y)\in U_{n+1}\setminus U_{n+2}$. Since $(x,y)\notin U_{n+2}$, we see $\rho(x,y)\geqslant 1/2^{n+3}$. On the other hand, since $(x,y)\in U_{n+1} = V_{1+nN}$, we have $d(f^i(x), f^i(y)) < c$ for all $0\leqslant i\leqslant nN$, and so $d(f^i(f^N(x)), f^i(f^N(y))) < c$ for all $0\leqslant i\leqslant (n-1)N$. Hence $(f^N(x), f^N(y))\in U_n$. Thus

$$\rho\bigl(f^N(x), f^N(y)\bigr) < \frac{1}{2^n} = \frac{2^3}{2^{n+3}} < K^N\rho(x,y). \tag{2}$$

Therefore, by (1) and (2), $\rho(f^N(x), f^N(y))\leqslant K^N\rho(x,y)$ for all $x,y\in X$.

Now, by [13, Proof of Theorem 1],

$$\rho\big(f^{3N}(x), f^{3N}(y)\big) > 2\rho(x,y) \quad \text{if } 0 < \rho(x,y) < 1/2^5. \tag{3}$$

Define a metric ρ' for X by

$$\rho'(x,y) = \sum_{i=0}^{N-1} \frac{1}{K^i} \rho\big(f^i(x), f^i(y)\big) \quad \text{for all } x, y \in X.$$

Then, it is easy to see that

$$\rho'\big(f(x), f(y)\big) \leqslant K\rho'(x,y) \quad \text{for all } x, y \in X \tag{4}$$

since $\rho(f^N(x), f^N(y)) \leqslant K^N \rho(x,y)$ for all $x, y \in X$.

Take $\nu > 0$ such that $\rho(x,y) < \nu$ $(x, y \in X)$ implies $\rho(f^i(x), f^i(y)) \leqslant 1/2^5$ for all $0 \leqslant i \leqslant 3N$. If $\rho(x,y) < \nu$ $(x, y \in X)$, then

$$\begin{aligned}
\rho'\big(f^{3N}(x), f^{3N}(y)\big) &= \sum_{i=0}^{N-1} \frac{1}{K^i} \rho\big(f^{3N}\big(f^i(x)\big), f^{3N}\big(f^i(y)\big)\big) \\
&\geqslant \sum_{i=0}^{N-1} \frac{2}{K^i} \rho\big(f^i(x), f^i(y)\big) \\
&= 2\rho'(x,y)
\end{aligned}$$

by (3). Note that $\rho'(x,y) \geqslant \rho(x,y)$ for all $x, y \in X$.

We are in a position to construct a metric D for X what we want. Put $\lambda^{3N} = 2$ and define

$$D(x,y) = \sum_{i=0}^{3N-1} \frac{1}{\lambda^i} \rho'\big(f^i(x), f^i(y)\big) \quad \text{for all } x, y \in X.$$

Then, it is easy to see that for all $x, y \in X$,

- $D(f(x), f(y)) \leqslant K D(x,y)$,
- $\lambda D(x,y) < D(f(x), f(y))$ if $0 < D(x,y) < \nu$.

Indeed, by (4)

$$\begin{aligned}
D\big(f(x), f(y)\big) &= \sum_{i=0}^{3N-1} \frac{1}{\lambda^i} \rho'\big(f\big(f^i(x)\big), f\big(f^i(y)\big)\big) \\
&\leqslant K \sum_{i=0}^{3N-1} \frac{1}{\lambda^i} \rho'\big(f^i(x), f^i(y)\big) \\
&= K D(x,y).
\end{aligned}$$

Since, $D(x,y) < \nu$ implies $\rho'(x,y) < \nu$ (recall that $D(x,y) \geqslant \rho'(x,y)$ by construction), we see

$$\begin{aligned} D\big(f(x), f(y)\big) &= \sum_{i=0}^{3N-1} \frac{1}{\lambda^i}\rho'\big(f^i\big(f(x)\big), f^i\big(f(y)\big)\big) \\ &> \sum_{i=1}^{3N-1} \frac{1}{\lambda^{i-1}}\rho'\big(f^i(x), f^i(y)\big) + \frac{2}{\lambda^{3N-1}}\rho'(x, y) \\ &= \lambda D(x, y) \end{aligned}$$

by the choice of λ. The lemma is proved. $\square$

The next lemma is essentially the same as [7, Lemma 1].

Lemma 2. *Suppose that $f:(X,d) \to (X,d)$ expands small distances; that is, there are constants $\delta_0 > 0$ and $\lambda > 1$ such that $d(f(x), f(y)) > \lambda d(x, y)$ whenever $0 < d(x, y) < \delta_0$ $(x, y \in X)$. Then the followings are equivalent*:

(i) *f is an open map,*
(ii) *there exists $0 < \delta_1 < \delta_0/2$ such that if $d(f(x), y) < \lambda\delta_1$ $(x, y \in X)$, then*

$$B_{\delta_1}(x) \cap f^{-1}(y) \neq \emptyset.$$

Proof. To see (ii) $\Rightarrow$ (i), let $0 < \delta_1 < \delta_0/2$ be as in (ii). Then, it can be easily checked that for every $y \in X$ with $d(f(x), y) < \lambda\delta_1$, there is just one point $g(y)$ in $B_{\delta_1}(x)$ satisfying $f(g(y)) = y$. Since, the map $g:\{y \in X\colon\ d(f(x), y) < \lambda\delta_1\} \to X$ is continuous (see [15, p. 144]), f is a local homeomorphism.

We can prove the converse (i) $\Rightarrow$ (ii) following the proof of [7, Lemma 1], and so the lemma is proved. $\square$

Suppose that $f:(X,d) \to (X,d)$ expands small distances. If f satisfies the above property (ii), then it is easy to see that for all $0 < \delta \leqslant \delta_1$ and $x, y \in X$,

$$d\big(f(x), y\big) < \delta \quad \text{implies} \quad B_{\delta/\lambda}(x) \cap f^{-1}(y) = \{\text{single point}\}. \tag{5}$$

This assertion will be used several times in the proofs of theorems.

The proof of Theorem 1 is divided into Propositions 1, 2 and 3. In the following three propositions, let $f: X \to X$ be a positively expansive map on a compact metrizable space X, and let d be the metric obtained by Lemma 1(i); that is, f expands small distances with constants $\delta_0 > 0$ and $\lambda > 1$ (with respect to d).

The first proposition is well-known (cf. [15,17]) and can be proved by using Bowen's method (see [6,12]). In this paper, we shall give a proof for completeness.

Proposition 1 (cf. [19]). *Under the above assumption, the following conditions are mutually equivalent*:

(i) *$f:(X,d) \to (X,d)$ is an open map,*
(ii) *$f:(X,d) \to (X,d)$ has the shadowing property,*
(iii) *$f:(X,d) \to (X,d)$ has the Lipschitz shadowing property.*

Proof. We show that if f is an open map, then f has the Lipschitz shadowing property. Let $L = 2\lambda/(\lambda - 1) = 2\sum_{k=0}^{\infty} \lambda^{-k} > 1$, and fix any $0 < \varepsilon \leqslant \delta_1/L$.

Now, let $\{x_i\}_{i=0}^{\infty}$ be given ε-pseudo-orbit of f; that is, $d(f(x_i), x_{i+1}) < \varepsilon$ for all $i \geqslant 0$. Pick any $i \geqslant 1$, and put

$$\lambda_j = \sum_{k=0}^{j-1} \lambda^{-k} \quad \text{for } j \geqslant 1$$

for simplicity. Since $d(f(x_i), x_{i+1}) < \varepsilon$, by (5), there exists $y_{i-1}^{(i)} \in B_{\varepsilon/\lambda}(x_{i-1})$ such that $f(y_{i-1}^{(i)}) = x_i$. Thus

$$d\big(f(x_{i-2}), y_{i-1}^{(i)}\big) \leqslant d\big(f(x_{i-2}), x_{i-1}\big) + d\big(x_{i-1}, y_{i-1}^{(i)}\big) \leqslant \lambda_2\varepsilon < L\varepsilon.$$

Hence, there exists $y_{i-2}^{(i)} \in B_{\lambda_2\varepsilon/\lambda}(x_{i-2})$ such that $f(y_{i-2}^{(i)}) = y_{i-1}^{(i)}$ by (5), and so

$$d\big(f(x_{i-3}), y_{i-2}^{(i)}\big) \leqslant d\big(f(x_{i-3}), x_{i-2}\big) + d\big(x_{i-2}, y_{i-2}^{(i)}\big) \leqslant \lambda_3\varepsilon < L\varepsilon.$$

By (5), there exists $y_{i-3}^{(i)} \in B_{\lambda_3\varepsilon/\lambda}(x_{i-3})$ such that $f(y_{i-3}^{(i)}) = y_{i-2}^{(i)}$. Thus $d(f(x_{i-3}), y_{i-2}^{(i)}) \leqslant \lambda_4\varepsilon < L\varepsilon$.

Repeating the process, we can find $y_0^{(i)} \in B_{\lambda_i\varepsilon/\lambda}(x_0)$ such that $f(y_0^{(i)}) = y_1^{(i)}$. By construction, $f^k(y_0^{(i)}) = y_k^{(i)}$ for all $0 \leqslant k \leqslant i$. Since X is compact, if we let $y_k = \lim_{i\to\infty} y_k^{(i)}$, then it is easy to see that $f^k(y_0) = y_k$ and $y_k \in B_{L\varepsilon}(x_k)$ for all $k \geqslant 0$. Thus f has the Lipschitz shadowing property.

To get the conclusion of this proposition, it is enough to show that if f has the shadowing property, then f is an open map. For δ_0, since f has the shadowing property, there exists $0 < \delta < \delta_0/2$ such that every $\delta\lambda$-pseudo-orbit of f is δ_0-shadowed by some point.

Now, let $d(f(x), y) < \delta\lambda$ $(x, y \in X)$, and define a $\delta\lambda$-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ of f by $x_0 = x$ and $x_i = f^{i-1}(y)$ for $i \geqslant 1$. Then, there exists $z \in X$ such that $d(f^i(z), x_i) < \delta_0$ for all $i \geqslant 0$. By construction, it is easy to see that $\lambda^{i-1} d(f(z), y) \leqslant d(f^i(z), f^{i-1}(y)) \leqslant \delta_0$ for all $i > 0$. Therefore $f(z) = y$ is concluded. Since $d(f(x), y) < \delta\lambda$ and $d(z, x) < \delta_0$, we have $d(z, x) < \delta$ because

$$\lambda d(x, z) < d\big(f(x), f(z)\big) = d\big(f(x), y\big) < \delta\lambda.$$

Hence $B_\delta(x) \cap f^{-1}(y) \neq \emptyset$, and thus f is open by Lemma 2. □

Main thing to prove in this section is the next two propositions.

Proposition 2 (cf. [1, p. 226]). *Let $f : (X, d) \to (X, d)$ be as before. If f is open, then f has the s-limit shadowing property.*

Proof. The conclusion is obtained by modifying the technique displayed in [12, p. 67]. Let f be an open map. Then, by Proposition 1, f has the Lipschitz shadowing property. Let ε_0 and $L > 0$ be two constants as in the definition of the Lipschitz shadowing property of f.

Now, fix any $0 < \varepsilon \leqslant \varepsilon_0/L$, and let $\{x_i\}_{i=0}^{\infty}$ be an given ε-pseudo-orbit of f. Then, there exists $y \in X$ such that $d(f^i(y), x_i) < L\varepsilon$ for all $i \geqslant 0$. If we assume further that $d(f(x_i), x_{i+1}) \to 0$ as $i \to \infty$, then, for any $0 < \delta < \varepsilon$ there exists $I_\delta > 0$ such that $d(f(x_i), x_{i+1}) < \delta$ for all $i \geqslant I_\delta$. Thus $\{x_i\}_{i=I_\delta}^{\infty}$ is a δ-pseudo-orbit. Since f has the Lipschitz shadowing property, we can find $y_\delta \in X$ such that

$$d\big(f^{i-I_\delta}(y_\delta), x_i\big) < L\delta \quad \text{for all } i \geqslant I_\delta. \tag{6}$$

On the other hand, since $d(f^i(y), x_i) < L\varepsilon$ for all $i \geqslant 0$, we see

$$\begin{aligned} d\big(f^{i-I_\delta}\big(f^{I_\delta}(y)\big), f^{i-I_\delta}(y_\delta)\big) &= d\big(f^i(y), f^{i-I_\delta}(y_\delta)\big) \\ &\leqslant d\big(f^i(y), x_i\big) + d\big(x_i, f^{i-I_\delta}(y_\delta)\big) \\ &< L(\varepsilon + \delta) < \delta_0 \end{aligned}$$

for all $i \geqslant I_\delta$. Since f expands small distances, we obtain $d(f^{I_\delta}(y), y_\delta) < \lambda^{-i+I_\delta}\delta_0$ for all $i \geqslant I_\delta$, and so $f^{I_\delta}(y) = y_\delta$. Therefore

$$d\big(f^i(y), x_i\big) = d\big(f^{i-I_\delta}\big(f^{I_\delta}(y)\big), x_i\big) = d\big(f^{i-I_\delta}(y_\delta), x_i\big) < L\delta$$

for all $i \geqslant I_\delta$ by (6). Since δ is arbitrary, $d(f^i(y), x_i) \to 0$ as $i \to \infty$. □

Proposition 3. *Let $f : (X, d) \to (X, d)$ be as before. Then, f has the strong shadowing property if and only if f is an open map.*

Proof. For δ_0, since f has the strong shadowing property, there exists $0 < \delta < \delta_0/2$ such that if a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies the inequality $\sum_{i=0}^{\infty} d(f(x_i), x_{i+1}) < \lambda\delta$, then there exists a point $y \in X$ with $\sum_{i=0}^{\infty} d(f^i(y), x_i) < \delta_0$.

Now, let $d(f(x), y) < \delta\lambda$ $(x, y \in X)$, and define a $\delta\lambda$-strong-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ of f by $x_0 = x$ and $x_i = f^{i-1}(y)$ for $i \geqslant 1$. Then, there exists $z \in X$ such that $\sum_{i=0}^{\infty} d(f^i(z), x_i) < \delta_0$, and so $d(f^i(z), x_i) < \delta_0$ for all $i \geqslant 0$. By construction, it is easy to see that $\lambda^{i-1} d(f(z), y) \leqslant d(f^i(z), f^{i-1}(y)) \leqslant \delta_0$ for all $i > 0$. Therefore $f(z) = y$. Since $d(f(x), y) < \delta\lambda$ and $d(z, x) < \delta_0$, we have $d(z, x) < \delta$ because

$$\lambda d(x, z) < d\big(f(x), f(z)\big) = d\big(f(x), y\big) < \delta\lambda.$$

Hence $B_\delta(x) \cap f^{-1}(y) \neq \emptyset$ so that f is open by Lemma 2.

To prove the converse, suppose that f is an open map. Let $L = 2\lambda/(\lambda - 1) = 2\sum_{k=0}^{\infty} \lambda^{-k} > 1$, and fix any $0 < \varepsilon \leqslant \delta_1/L$.

Now, let $\{x_i\}_{i=0}^{\infty}$ be any ε-strong-pseudo-orbit of f; that is, $\sum_{i=0}^{\infty} d(f(x_i), x_{i+1}) < \varepsilon$. Denote the distance $d(f(x_{i-1}), x_i)$ by ε_i, and pick any $i \geqslant 1$. Then, $0 \leqslant \varepsilon_i < \varepsilon$ for all $i \geqslant 1$. To simplify notation, put

$$\lambda_j = \sum_{k=0}^{j-1} \lambda^{-k} \quad \text{for } j \geqslant 1$$

and set

$$\mu_j = \sum_{k=0}^{j-1} \varepsilon_{i-k} \lambda^{-j+k+1} \quad \text{for } 1 \leqslant j \leqslant i - 1.$$

Then, $\mu_1 = \varepsilon_i$. Since $d(f(x_{i-1}), x_i) = \varepsilon_i$, there exists $y_{i-1}^{(i)} \in B_{\varepsilon_i/\lambda}(x_{i-1})$ such that $f(y_{i-1}^{(i)}) = x_i$ by (5). Thus

$$\begin{aligned} d\big(f(x_{i-2}), y_{i-1}^{(i)}\big) &\leqslant d\big(f(x_{i-2}), x_{i-1}\big) + d\big(x_{i-1}, y_{i-1}^{(i)}\big) \\ &\leqslant \mu_2 < L\varepsilon. \end{aligned}$$

Since there exists $y_{i-2}^{(i)} \in B_{\mu_2/\lambda}(x_{i-2})$ such that $f(y_{i-2}^{(i)}) = y_{i-1}^{(i)}$ by (5) (recall $L\varepsilon \leqslant \delta_1$),

$$\begin{aligned} d\big(f(x_{i-3}), y_{i-2}^{(i)}\big) &\leqslant d\big(f(x_{i-3}), x_{i-2}\big) + d\big(x_{i-2}, y_{i-2}^{(i)}\big) \\ &\leqslant \mu_3 < L\varepsilon. \end{aligned}$$

By (5), there exists $y_{i-3}^{(i)} \in B_{\mu_3/\lambda}(x_{i-3})$ such that $f(y_{i-3}^{(i)}) = y_{i-2}^{(i)}$. Thus $d(f(x_{i-3}), y_{i-2}^{(i)}) \leqslant \mu_4 < L\varepsilon$. Repeating the process, we can find $y_0^{(i)} \in B_{\mu_i/\lambda}(x_0)$ such that $f(y_0^{(i)}) = y_1^{(i)}$. By construction, $f^k(y_0^{(i)}) = y_k^{(i)}$ for all $0 \leqslant k \leqslant i$. Thus

$$\sum_{k=0}^{i-1} d\big(y_k^{(i)}, x_k\big) \leqslant \frac{1}{\lambda}\sum_{j=1}^{i} \mu_j = \frac{1}{\lambda}\sum_{j=1}^{i} \lambda_j \varepsilon_j \leqslant \frac{L}{2}\sum_{j=1}^{i} \varepsilon_j < \frac{L}{2}\varepsilon.$$

Since X is compact, we can set $y_k = \lim_{i\to\infty} y_k^{(i)}$. Thus, it is easy to see that $f^k(y_0) = y_k$ for all k and $\sum_{i=0}^{\infty} d(f^i(y_0), x_i) \leqslant L\varepsilon/2 < L\varepsilon$. □

2. Proof of Theorem 2

In this section, let f be a positively expansive map on a compact metrizable space X. By Lemma 1, there exist constants $K, \delta_0 > 0$ and $\lambda > 1$ such that for any $x, y \in X$,

(2.1) $0 < d(x, y) < \delta_0$ implies $\lambda d(x, y) < d(f(x), f(y))$,
(2.2) $d(f(x), f(y)) \leqslant K d(x, y)$

with respect to some metric d for X.

We may suppose that $K \geqslant \lambda > 1$. Hereafter, we fix both the above metric and the constants, and assume further that f is an open map. Then, by (5)

(2.3) for every $0 < \delta \leqslant \delta_0$, if $d(f(x), y) < \delta$, then

$$B_{\delta/\lambda}(x) \cap f^{-1}(y) = \{\text{single point}\}.$$

Let $\{x_{-i}\}_{i=0}^{\infty} \subset X$ be a *backward orbit* of f; that is, $f(x_{-i}) = x_{-i+1}$ for all $i \geqslant 0$. Denote by $\boldsymbol{X}_f$ the set of all backward orbits of f, and let $\boldsymbol{f} : (\boldsymbol{X}_f, \boldsymbol{d}) \to (\boldsymbol{X}_f, \boldsymbol{d})$ be the inverse limit system of f. Here $\boldsymbol{d}$ is the metric on $\boldsymbol{X}_f$ (see [15, pp. 143–147] for the definition and properties).

Before starting the proof, we collect some well-known dynamical properties of a positively expansive open map with an expansive constant c. Let $\Omega(f)$ be the non-wandering set of f. Then it is easy to see that

(2.4) the set of periodic points, $P(f)$, of f is dense in $\Omega(f)$

(cf. [2]), and from this, we see

(2.5) $f(\Omega(f)) = \Omega(f)$.

For $\varepsilon > 0$, define the *local stable set* of $x \in X$, $W^s_\varepsilon(x)$, by

$$W^s_\varepsilon(x) = \bigl\{y \in X\colon d\bigl(f^n(x), f^n(y)\bigr) \leqslant \varepsilon \text{ for all } n \geqslant 0\bigr\}$$

as usual. Remark that if $\varepsilon \leqslant c$, then $W^s_\varepsilon(x) = \{x\}$ for all $x \in X$. Thus, by following the proof of [2, Theorem 2] or [18, Theorem 2] we have

(2.6) $\Omega(f)$ is decomposed into a finite disjoint union of closed f-invariant sets $\{\Lambda_i\}_{i=1}^{\ell}$; that is, $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_\ell$ such that $f_{|\Lambda_j}$ is topologically transitive for $1 \leqslant j \leqslant \ell$.

Such a set Λ_j is called a *basic set* (cf. [6,8]).

(2.7) There is a Markov partition of $\boldsymbol{X}_f$ with arbitrarily small diameter with respect to $\boldsymbol{f}$

(see [8,10] and [15, p. 146] for the definition and its proof).

Under the above notation and facts, we prove the following two lemmas.

Lemma 3. *If we assume further that f is topologically transitive, then there exists a constant $B > 1$ such that for each pair $(\{x_{-i}\}_{i=0}^\infty, y) \in \boldsymbol{X}_f \times X$, there exists a backward orbit $r(\{x_{-i}\}_{i=0}^\infty, y) = \{z_{-i}\}_{i=0}^\infty \in \boldsymbol{X}_f$ satisfying*

(i) $z_0 = y$,
(ii) $d(x_{-i}, z_{-i}) \leqslant B\lambda^{-i} d(x_0, y)$ *for all* $i \geqslant 0$.

Proof. Let $\delta_0 > 0$ and $\lambda > 1$ be as above, and let $(\{x_{-i}\}_{i=0}^\infty, y) \in \boldsymbol{X}_f \times X$ be given.

Case 1. $d(x_0, y) < \delta_0$. Since $f(x_{-1}) = x_0$ and $d(f(x_{-1}), y) < \delta_0$, by (2.3), there exists $z_{-1} \in X$ such that $d(x_{-1}, z_{-1}) \leqslant \delta_0/\lambda$ and $f(z_{-1}) = y$. Especially, $\lambda d(x_{-1}, z_{-1}) < d(f(x_{-1}), f(z_{-1})) = d(x_0, y)$. Since $\delta_0/\lambda < \delta_0$ and $f(x_{-2}) = x_{-1}$, by (2.3), there exists $z_{-2} \in X$ such that $d(x_{-2}, z_{-2}) \leqslant \delta_0/\lambda^2$, $f(z_{-2}) = z_{-1}$ and $\lambda d(x_{-2}, z_{-2}) < d(x_{-1}, z_{-1})$.

Repeating the process, we can find $z_{-i} \in X$ such that $d(x_{-i}, z_{-i}) \leqslant \delta_0/\lambda^i$, $f(z_{-i}) = z_{-i+1}$ and

$$d(x_{-i}, z_{-i}) < \lambda^{-i} d(x_0, y) \quad \text{for all } i \geqslant 1.$$

Let $z_0 = y$ and set $r(\{x_{-i}\}_{i=0}^\infty, y) = \{z_{-i}\}_{i=0}^\infty$.

Case 2. $d(x_0, y) \geqslant \delta_0$. Let K be as in (2.2), and let $0 < \varepsilon = \varepsilon(\delta_0) < \delta_0$ be the number as in the definition of the shadowing property of f. Denote by $\mathcal{R}$ a Markov partition $\{R_1, \ldots, R_m\}$ on $\boldsymbol{X}_f$ with $\max_{1 \leqslant i \leqslant m} \operatorname{diam}_{\boldsymbol{d}}(R_i) \leqslant \varepsilon$ (see (2.7)). Let $\mathcal{A}$ be a $m \times m$-transition matrix of the Markov partition induced by the inverse limit system $\boldsymbol{f}$ of f.

Then, since f is topologically transitive (see [15, p. 145]), there is an integer $n_0 > 0$ such that the matrix $\mathcal{A}^{n_0}$ is strictly positive (see [8]). Therefore

$$\boldsymbol{f}^{n_0}\big(R\big(\boldsymbol{f}^{-n_0}\big(\{x_{-i}\}_{i=0}^{\infty}\big)\big)\big) \cap R\big(\{y_{-i}\}_{i=0}^{\infty}\big) \neq \emptyset.$$

Here $\{y_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ $(y_0 = y)$ and $R(\{x_{-i}\}_{i=0}^{\infty})$ is an element of $\mathcal{R}$ containing $\{x_{-i}\}_{i=0}^{\infty}$. Thus we can pick $w \in X$ such that the sequence

$$\big\{\ldots, x_{-n_0-2}, x_{-n_0-1}, w, f(w), \ldots, f^{n_0-1}(w), y, f(y), \ldots\big\}$$

is an ε-pseudo-orbit of f. Using the shadowing property we can find $\{z_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ such that $z_0 = y$ and

$$\begin{cases} d(x_{-n_0-i}, z_{-n_0-i}) < \delta_0 & \text{for all } i \geqslant 0, \\ d(f^{n_0-j}(w), z_{-j}) < \delta_0 & \text{for all } 0 \leqslant j \leqslant n_0. \end{cases}$$

Put $r(\{x_{-i}\}_{i=0}^{\infty}, y) = \{z_{-i}\}_{i=0}^{\infty}$. Then, by (2.1)

$$d(x_{-n_0-1-i}, z_{-n_0-1-i}) < \lambda^{-i} d(x_{-n_0-1}, z_{-n_0-1}) < \lambda^{-i}\delta_0$$

for all $i \geqslant 0$. Thus

$$\begin{aligned} d(x_{-i}, z_{-i}) &= d\big(f^{n_0+1}(x_{-n_0-1-i}), f^{n_0+1}(z_{-n_0-1-i})\big) \\ &\leqslant K^{n_0+1} d(x_{-n_0-1-i}, z_{-n_0-1-i}) \\ &< K^{n_0+1}\lambda^{-i}\delta_0 \\ &\leqslant K^{n_0+1}\lambda^{-i} d(x_0, y). \end{aligned}$$

Finally, we set $B = K^{n_0+1}$. The proof of the lemma is complete. □

Lemma 4. *Let $\Omega(f) = \bigcup_{j=1}^{\ell} \Lambda_j$ be as in (2.6). Then $\omega(x) \cap \Lambda_j = \emptyset$ for any $x \in B_{\delta_0}(\Lambda_j) \setminus \Omega(f)$. Here $\omega(x)$ is the ω-limit set of x.*

Proof. Let $\delta_0 > 0$ and $\lambda > 1$ be as in (2.1). Suppose that there is $x \in B_{\delta_0}(\Lambda_j) \setminus \Omega(f)$ satisfying $\omega(x) \cap \Lambda_j \neq \emptyset$. Fix $0 < \varepsilon < \delta_0/2$ such that $B_\varepsilon(x) \cap \Omega(f) = \emptyset$, and choose a backward orbit $\{y_{-i}\}_{i=0}^{\infty} \subset \Lambda_j$ with $y_0 = y$. By (2.3), we can construct $\{x_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ such that

$$x_0 = x \quad \text{and} \quad d(x_{-i}, y_{-i}) < \lambda^{-i} d(x_0, y_0) \quad \text{for all } i \geqslant 0.$$

Since $\omega(x) \cap \Lambda_j \neq \emptyset$, we may assume that $f^n(x)$ converges to some point $z \in \Lambda_j$ as $n \to \infty$. Let $0 < \delta = \delta(\varepsilon) < \delta_0$ be the number as in the definition of the shadowing property. Pick two integers $I, N > 0$ such that $d(x_{-I}, y_{-I}) < \delta/2$ and $d(f^N(x), z) < \delta/2$. Since $f_{|\Lambda_j}$ is topologically transitive, we can find $w \in \Lambda_j$ and $M > 0$ such that $d(z, w) < \delta/2$ and $d(f^M(w), y_{-I}) < \delta/2$. Thus

$$\big\{\ldots, x, f(x), \ldots, f^{N-1}(x), w, f(w), \ldots, f^{M-1}(w), x_{-I}, x_{-I+1}, \ldots, x_{-1}, x, \ldots\big\}$$

is a cyclic δ-pseudo-orbit of f. Since f is positively expansive, there exists a periodic point $f^{N+M+I}(p) = p \in B_\varepsilon(x) \cap \Omega(f)$. This is a contradiction. □

Proof of Theorem 2. Let $f : X \to X$ be an open map which expands small distances (with the constants δ_0 and λ) with respect to the metric d. Recall that such f has the Lipschitz shadowing property with constants ε_0 and L (see Proposition 1). By Lemma 3, if f is topologically transitive, then the average shadowing property will be proved by following [3, pp. 375–377, Proof of Theorem 4] (see also [5, Proof of Theorem 6.3.1]).

Indeed, let $B > 1$ be as in Lemma 3, and fix any $0 < \varepsilon \leqslant \varepsilon_0$ small enough. Let $\{x_i\}_{i=0}^{\infty} \subset X$ be an ε-average-pseudo-orbit of f. Then there exists $N = N(\varepsilon) > 0$ such that for all $n \geqslant N$ and $k \geqslant 0$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d\big(f(x_{i+k}), x_{i+k+1}\big) < \varepsilon.$$

Now, define a sequence of segments of the above $\{x_i\}_{i=0}^{\infty}$ as follows. Pick $1 < n' \leqslant N$ such that

$$\frac{1}{n'}\sum_{i=0}^{n'-1} d\big(f(x_i), x_{i+1}\big) < \varepsilon \quad \text{and} \quad \frac{1}{n'-1}\sum_{i=0}^{n'-2} d\big(f(x_i), x_{i+1}\big) \geqslant \varepsilon. \tag{7}$$

Then $d(f(x_{n'-1}), x_{n'}) < \varepsilon$. For, if $d(f(x_{n'-1}), x_{n'}) \geqslant \varepsilon$, then by (7)

$$n'\varepsilon > \sum_{i=0}^{n'-1} d\big(f(x_i), x_{i+1}\big) \geqslant (n'-1)\varepsilon + \varepsilon.$$

This is a contradiction.

Set $i_1 = n' - 1$, and let $m_1 \geqslant 0$ be the largest number such that $d(f(x_i), x_{i+1}) < \varepsilon$ for all $i_1 \leqslant i \leqslant i_1 + m_1 - 1$. Then, we put $i_2 = i_1 + m_1 + 1$, and define $m_2 \geqslant 0$ as the largest number such that $d(f(x_i), x_{i+1}) < \varepsilon$ for all $i_2 \leqslant i \leqslant i_2 + m_2 - 1$. Repeat the process, and denote the nth $(n \geqslant 1)$ segment

$$\{x_{i_n}, x_{i_n+1}, x_{i_n+2}, \ldots, x_{i_n+m_n-1}\}$$

of $\{x_i\}_{i=0}^{\infty}$ by $\boldsymbol{x}_n$. Since f has the Lipschitz shadowing property, for $\boldsymbol{x}_n$ there exists a point $y^{(n)} \in X$ such that

$$d\big(f^i\big(y^{(n)}\big), x_{i_n+i}\big) < L\varepsilon \tag{8}$$

for all $0 \leqslant i \leqslant m_n - 1$ and $n \geqslant 1$. Finally, let $\Delta_n = d(f^{m_n}(y^{(n)}), y^{(n+1)})$ for all $n \geqslant 1$.

Since $i_1 = n' - 1 < \infty$, to get the conclusion, it is enough to construct a shadowing orbit (in average) which shadows a sequence of the above segments $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$. Let us construct the average shadowing orbit inductively.

At the first step, fix any backward orbit $\{z_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ of $y^{(1)}$ with $z_0 = y^{(1)}$. By Lemma 3, for

$$\big(\{z_{-i}\}_{i=0}^{\infty} \cup \big\{f^j\big(y^{(1)}\big)\big\}_{j=1}^{m_1},\ y^{(2)}\big) \in \boldsymbol{X}_f \times X,$$

there exists $\{w_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ such that $w_0 = y^{(2)}$ and

$$d\big(f^j(w_{-m_1}), f^j\big(y^{(1)}\big)\big) \leqslant B\lambda^{-m_1+j} d\big(f^{m_1}\big(y^{(1)}\big), y^{(2)}\big) \tag{9}$$

for $0 \leqslant j \leqslant m_1 - 1$. For convenience, we set $x^{(1)} = w_{-m_1}$. Then, the orbit $\{f^j(x^{(1)})\}_{j=0}^{m_1+m_2-1}$ approximates the first two segments $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

In the second step, by Lemma 3 for

$$\big(\{w_{-i}\}_{i=m_1+1}^{\infty} \cup \big\{f^j\big(x^{(1)}\big)\big\}_{j=1}^{m_1+m_2},\ y^{(3)}\big) \in \boldsymbol{X}_f \times X,$$

there exists $\{v_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ such that $v_0 = y^{(3)}$ and

$$\begin{aligned} d\big(f^j(v_{-m_1-m_2}), f^j\big(x^{(1)}\big)\big) &\leqslant B\lambda^{-m_1-m_2+j} d\big(f^{m_1+m_2}\big(x^{(1)}\big), y^{(3)}\big) \\ &\leqslant B\lambda^{-m_1-m_2+j} d\big(f^{m_2}\big(y^{(2)}\big), y^{(3)}\big) \end{aligned} \tag{10}$$

for $0 \leqslant j \leqslant m_1 + m_2 - 1$. If we set $x^{(2)} = v_{-m_1-m_2}$, then the orbit $\{f^j(x^{(2)})\}_{j=0}^{m_1+m_2+m_3-1}$ approximates the first three segments $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. In the nth step of the procedure, we can construct the initial point $x^{(n)}$ whose orbit approximates the first $n+1$ segments from $\boldsymbol{x}_1$ to $\boldsymbol{x}_{n+1}$.

Set $L_0 = 0$ and $L_n = \sum_{j=1}^{n} m_j$, and let

$$S_n = \sum_{k=1}^{n} \sum_{j=L_{k-1}}^{L_k-1} d\big(f^j\big(x^{(n)}\big), f^{j-L_{k-1}}\big(y^{(k)}\big)\big).$$

Then, by (9) and (10) it is not hard to show that

$$\begin{aligned} S_n \leqslant \beta\big[&\big(1 + \lambda^{-m_n} + \lambda^{-m_n-m_{n-1}} + \cdots + \lambda^{-m_n-m_{n-1}-\cdots-m_2}\big)\Delta_n \\ &+ \big(1 + \lambda^{-m_{n-1}} + \lambda^{-m_{n-1}-m_{n-2}} + \cdots + \lambda^{-m_{n-1}-m_{n-2}-\cdots-m_2}\big)\Delta_{n-1} \\ &+ \cdots + \big(1 + \lambda^{-m_2}\big)\Delta_2 + \Delta_1\big]. \end{aligned}$$

Here $\beta = B/(1 - 1/\lambda)$. To simplify notation, put $\alpha_n = \lambda^{-m_{n+1}} \leqslant \lambda^{-1}$ for $n \geqslant 1$. Then we have

$$\begin{aligned} S_n \leqslant \beta\big[&(1 + \alpha_1\alpha_2\alpha_3\cdots\alpha_{n-1} + \alpha_2\alpha_3\cdots\alpha_{n-1} + \cdots + \alpha_{n-2}\alpha_{n-1})\Delta_n \\ &+ \cdots + (1 + \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3 + \alpha_3)\Delta_4 \\ &+ (1 + \alpha_1\alpha_2 + \alpha_2)\Delta_3 + (1 + \alpha_1)\Delta_2 + \Delta_1\big]. \end{aligned} \tag{11}$$

To estimate the right-hand side of (11), put

$$Q_n = 1 + \alpha_n + \alpha_n\alpha_{n-1} + \cdots + \alpha_n\alpha_{n-1}\cdots\alpha_{n-j} + \cdots.$$

Then

$$\begin{aligned} Q_n &\leqslant 1 + \lambda^{-1} + \lambda^{-2} + \cdots + \lambda^{-k} + \cdots \\ &\leqslant \lambda/(1-\lambda) < \infty. \end{aligned}$$

Notice that this estimates does not depend on the index n of the segment. Therefore

$$S_n < \beta(\Delta_1 Q_1 + \Delta_2 Q_2 + \cdots + \Delta_n Q_n) \leqslant \beta q \sum_{j=1}^{n} \Delta_j.$$

Here $q = \lambda/(1-\lambda)$.

Since $\Delta_j \leqslant d(f(x_i), x_{i+1}) + 2L\varepsilon$ for $L_{j-1} \leqslant i \leqslant L_j - 1$ (see (8)), we have

$$S_n < \beta q \sum_{i=0}^{L_n-1} \bigl(d\bigl(f(x_i), x_{i+1}\bigr) + 2L\varepsilon\bigr).$$

Hence

$$\frac{1}{L_n} \sum_{i=0}^{L_n-1} d\bigl(f^i\bigl(x^{(n)}\bigr), x_i\bigr) \leqslant \beta q \left(\frac{1}{L_n} \sum_{i=0}^{L_n-1} d\bigl(f(x_i), x_{i+1}\bigr) + 2L\varepsilon\right).$$

By the definition of the ε-average-pseudo-orbit, for n large enough, we can rewrite the last inequality as follows:

$$\frac{1}{L_n} \sum_{i=0}^{L_n-1} d\bigl(f^i\bigl(x^{(n)}\bigr), x_i\bigr) \leqslant \beta q(1 + 2L)\varepsilon.$$

If we set $y = \lim_{n\to\infty} x^{(n)}$, then, by the construction of $\{x^{(n)}\}_{n=1}^{\infty}$, it is not hard to show that the orbit of y shadows $\{x_i\}_{i=0}^{\infty}$ in average.

To show the converse, suppose f has the average shadowing property. Since f is positively expansive (with constant $c > 0$) and open, there exists a decomposition $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_\ell$ by (2.6).

Claim. *Under the above notations, we have* $\ell = 1$.

If this claim is true, then $X = \Omega(f)$ so that f is topologically transitive. To show the equality, assuming that there exists $x \notin \Omega(f)$, we shall lead a contradiction. Take $0 < \varepsilon < c/2$ such that $B_\varepsilon(x) \cap \Omega(f) = \emptyset$, and let $\delta = \delta(\varepsilon) > 0$ be the number as in the definition of the shadowing property of f. Pick any backward orbit $\{x_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ with $x_0 = x$. Then $\omega(x) \cup \alpha(\{x_{-i}\}_{i=0}^{\infty}) \subset \Omega(f)$. Here

$$\alpha\bigl(\{x_{-i}\}_{i=0}^{\infty}\bigr) = \{y \in X\colon \text{ there exists } i_n > 0 \text{ such that } x_{-i_n} \to y \text{ as } n \to \infty\}.$$

As in the proof of Lemma 4, we can construct a cyclic δ-pseudo-orbit from x to x because $\Omega(f) = \Lambda_1$ and $f : \Lambda_1 \to \Lambda_1$ is topologically transitive. Hence, by the positive expansiveness of f, there exists $p \in P(f) \cap B_\varepsilon(x)$. This is a contradiction, and Theorem 2 is proved.

To prove the claim, assuming that $\ell \geqslant 2$, we lead a contradiction (a similar argument has already used in [19] to prove an analogous result for expansive homeomorphisms with the shadowing property). For simplicity, suppose $\ell = 2$ (the other case is treated similarly). Take $\varepsilon > 0$ small enough and fix integers $n_1, n_2 \geqslant 5$ such that

$$(n_1 - 1)\varepsilon < d(U_1, \Lambda_2) \leqslant n_1\varepsilon \quad \text{and} \quad (n_2 - 1)\varepsilon < d(\Lambda_1, \Lambda_2) \leqslant n_2\varepsilon.$$

Here U_1 is a compact neighborhood of Λ_1 and $d(A, B) = \inf\{d(a, b)\colon a \in A,\ b \in B\}$ for $A, B \subset X$. Since f has the average shadowing property, there is $0 < \delta = \delta(\varepsilon) < \varepsilon$ such that every δ-average-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ε-shadowed in average by some point in X. Finally, let us fix $n_3 \geqslant 3$ such that

$$(n_3 - 1)\delta < d(\Lambda_1, \Lambda_2) \leqslant n_3\delta.$$

Take $x \in \Lambda_1$, $y \in \Lambda_2$ with $d(x,y) = d(\Lambda_1, \Lambda_2)$. Since $\Omega(f) = \overline{P(f)}$, there are $p \in \Lambda_1 \cap P(f)$ and $q \in \Lambda_2 \cap P(f)$ such that

$$\max\bigl\{d(x,p), d(y,q), d\bigl(f(x), f(p)\bigr), d\bigl(f(y), f(q)\bigr)\bigr\} < \delta.$$

Let $\ell_1, \ell_2 > 0$ be the (minimum) periods of p, q respectively; that is, $f^{\ell_1}(p) = p$, $f^{\ell_2}(q) = q$. Fix $\ell_3 > 0$ such that $\ell_i \ell_3 > n_3$ for $i = 1, 2$, and denote a cyclic sequence

$$\bigl\{\dots, y, f(q), f^2(q), \dots, f^{\ell_1\ell_2\ell_3^2-1}(q), x, f(p), f^2(p), \dots, \\ f^{\ell_1\ell_2\ell_3^2-1}(p), y, f(q), \dots\bigr\}$$

(composed of two points $\{x, y\}$ and two periodic orbits) by $\{z_i\}_{i=0}^{\infty}$ $(z_0 = y)$. Then, it is easy to see that this is a δ-average-pseudo-orbit. Indeed, for every $m > 2\ell_1\ell_2\ell_3^2$ and $k \geqslant 0$, we have

$$\frac{1}{m}\sum_{i=0}^{m-1} d\bigl(f(z_{i+k}), z_{i+k+1}\bigr) < \delta.$$

Pick $w \in X$ such that ε-shadows $\{z_i\}_{i=0}^{\infty}$ in average. If $w \in \Lambda_2$, then $f^i(w) \in \Lambda_2$ for all $i \geqslant 0$. Hence, for a sufficiently large $m > 3\ell_1\ell_2\ell_3^2$, we have

$$\frac{1}{m}\sum_{i=0}^{m-1} d\bigl(f^i(w), z_i\bigr) > \frac{(n_2 - 1)\varepsilon}{3} > \varepsilon.$$

This is a contradiction.

If $w \notin \Lambda_2$, then, by Lemma 4, there exists a number $m' > 0$ satisfying $f^i(w) \in U_1$ for all $i > m'$. Thus

$$\begin{aligned}\frac{1}{m}\sum_{i=0}^{m-1} d\bigl(f^i(w), z_i\bigr) &= \frac{1}{m}\left(\sum_{i=0}^{m'-1} d\bigl(f^i(w), z_i\bigr) + \sum_{i=0}^{m-m'-1} d\bigl(f^{m'+i}(w), z_{m'+i}\bigr)\right) \\ &> (n_1 - 1)\varepsilon/3 > \varepsilon\end{aligned}$$

if we take m $(> m')$ large enough. This is also a contradiction. □

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