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Multicoloring and Mycielski construction ☆

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Abstract

The generalized Mycielskians of graphs (also known as cones over graphs) are the natural generalization of the Mycielskians of graphs (which were first introduced by Mycielski in 1955). Given a graph G and any integer $p \geqslant 0$, one can transform G into a new graph $\mu_p(G)$, the p-Mycielskian of G. In this paper, we study the kth chromatic numbers χ_k of Mycielskians and generalized Mycielskians of graphs. We show that $\chi_k(G) + 1 \leqslant \chi_k(\mu(G)) \leqslant \chi_k(G) + k$, where both upper and lower bounds are attainable. We then investigate the kth chromatic number of Mycielskians of cycles and determine the kth chromatic number of p-Mycielskian of a complete graph K_n for any integers $k \geqslant 1$, $p \geqslant 0$ and $n \geqslant 2$. Finally, we prove that if a graph G is a/b-colorable then the p-Mycielskian of G, $\mu_p(G)$, is $(at + b^{p+1})/bt$-colorable, where $t = \sum_{i=0}^{p}(a-b)^i b^{p-i}$. And thus obtain graphs G with $m(G)$ grows exponentially with the order of G, where $m(G)$ is the minimal denominator of a a/b-coloring of G with $\chi_f(G) = a/b$.

Keywords: k-Tuple coloring; a/b-Coloring; Fractional chromatic number; Homomorphism; Mycielskian; p-Mycielskian; Kneser graph

1. Introduction

A *k-tuple coloring* of a graph G is an assignment of k distinct colors to each vertex of G so that adjacent vertices receive no colors in common. The kth *chromatic number* of G, denoted by $\chi_k(G)$, is the smallest number of colors needed to give G a k-tuple coloring. Clearly $\chi_1(G)$ is the ordinary chromatic number of G.

Given two positive integers a and b with $b \leqslant a$, a *a/b-coloring* is a mapping which assigns each vertex of G a *b-element* subsets of $\{1, 2, \ldots, a\}$ in such a way that adjacent vertices are assigned disjoint subsets. A graph G is *a/b-colorable* if G has a a/b-coloring. The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is the infimum of all fractions a/b such that G is a/b-colorable. The infimum is always achieved so that $\chi_f(G)$ is always rational. Following is a different but equivalent definition of the fractional chromatic number of a graph G.

A mapping c from the collection Γ of independent sets of a graph G to the interval [0, 1] is a fractional coloring if for every vertex x of G we have $\sum_{S \in \Gamma, s.t. x \in S} c(S) \geqslant 1$. The value of a fractional coloring c is $\sum_{S \in \Gamma} c(S)$. The fractional chromatic number $\chi_f(G)$ of G is the infimum of the values of fractional colorings of G. There are several equivalent definitions of the fractional chromatic number of a graph, see [12,13].

A *fractional clique* of a graph G is a mapping $g : V(G) \to [0, 1]$ such that for any independent set S of vertices in $V(G)$, $\sum_{v \in S} g(v) \leqslant 1$. The sum $\sum_{v \in V(G)} g(v)$ is called the value of the fractional clique g of G, denoted by $v_g(G)$.

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The *fractional clique number* $\omega_f(G)$ of G is equal to $\sup\{v_g(G)\}$, where the supremum is taken over all fractional cliques g of G. Also, the supremum in the definition is attainable.

It is not difficult to see that the linear programming computing $\omega_f(G)$ is actually the dual of that computing $\chi_f(G)$. Thus, by the duality theorem of linear programming, $\omega_f(G) = \chi_f(G)$.

Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and edge set E^0. Given an integer $p \geqslant 1$ the *p-Mycielskian* of G, denoted by $\mu_p(G)$, has vertex set

$$V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^p \cup \{u\},$$

where $V^i = \{v_j^i : v_j^0 \in V^0\}$ is the ith distinct copy of V^0 for $i = 1, 2, \ldots, p$, and edge set

$$E^0 \cup \left(\bigcup_{i=0}^{p-1} \{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\} \right) \cup \{v_j^p u : v_j^p \in V^p\}.$$

We define $\mu_0(G)$ to be the graph obtained from G by adding a universal vertex u. When $p = 1$, $\mu_1(G)$ of a graph G is exactly the Mycielskian of G which was first introduced by Mycielski [19] in 1955.

The Mycielskians of graphs were studied in many papers [1,2,4–7,10,11,15,18,19]. The generalized Mycielski's graphs were also studied in [14,16,17], and in [22,23] under the name cones over graphs.

Larsen et al. [15] showed that $\chi_f(\mu(G)) = \chi_f(G) + (1/\chi_f(G))$ for any graph G. Tardif [23] then generalized this result, he proved that, for any graph G and any nonnegative integer p, $\chi_f(\mu_p(G)) = \chi_f(G) + (1/(\sum_{k=0}^{p}(\chi_f(G)-1)^k)$. Tardif's proof is based on the second definition of the fractional chromatic number and the definition of the fractional clique.

Let $m(G)$ denote the minimal denominator of a a/b-coloring of G with $\chi_f(G) = a/b$. It was shown by Fisher [5] that $m(G)$ may not be the denominator of the minimal fractional representation of $\chi_f(G)$, though it is always a multiple of this denominator. How large can $m(G)$ be? Let n denote the vertex number of a graph G. Chvátal et al. [3] proved that $m(G) \leqslant n^{n/2}$. This seems vastly overestimate $m(G)$. However, they also gave examples where $m(G)$ is asymptotically $e^{\sqrt{n \ln(n)/2}}$. And Fisher [5] gave examples where $m(G)$ is asymptotically λ^n ($\lambda \approx 1.34619$) by using Larsen's result [15] mentioned above.

For two integers i and j with $i \leqslant j$, by $[i, j]$ we denote the set of integers $i, i+1, \ldots, j$.

Chromatic number, fractional chromatic number and circular chromatic number of Mycielskians and generalized Mycielskians of graphs have been studied extensively. So it is natural to investigate the kth chromatic number of such kind of graphs. In Section 2, we show that $\chi_k(G) + 1 \leqslant \chi_k(\mu(G)) \leqslant \chi_k(G) + k$ and give examples for which upper or lower bound is attained. We then investigate the kth chromatic number of Mycielskians of cycles in Section 3. Section 4 determines the kth chromatic number of p-Mycielskian of a complete graph K_n for any integers $k \geqslant 1$, $p \geqslant 0$ and $n \geqslant 2$. Finally, in Section 5, we investigate the a/b-colorings of generalized Mycielskians of graphs. We prove that if G is a/b-colorable, then $\mu_p(G)$ is $(at + b^{p+1})/bt$-colorable, where $t = \sum_{i=0}^{p}(a-b)^i b^{p-i}$. Using this result, we then easily get infinitely many graph sequences whose $m(G)$ grows exponentially with the order of G. The exact growth rates are also given.

2. Bounds for kth chromatic numbers of Mycielskians

Let I^m denote the set of all positive integers not greater than m and I_k^m the family of all subsets of I^m of cardinality k. The *Kneser graph* $G_{m,k}$ has I_k^m as its vertex set and two vertices are adjacent if and only if the corresponding subsets are disjoint. Suppose G and H are graphs. A *homomorphism* from G to H is a mapping f from $V(G)$ to $V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. If there is a homomorphism from G to H, we say G is *homomorphic* to H. If G is homomorphic to H and H is homomorphic to G then G and H are called *hom-equivalent*. A k-tuple coloring of G with m colors can be viewed as a homomorphism from G to $G_{m,k}$ (see [21]). And $\chi_k(G)$ is just the smallest integer m for which G is homomorphic to $G_{m,k}$. It is clear that if G is homomorphic to H then $\chi_k(G) \leqslant \chi_k(H)$, and if G is hom-equivalent to H then $\chi_k(G) = \chi_k(H)$.

It was proved in [12] that $\chi_f(G_{m,k}) = m/k$, $(k \geqslant 1, m \geqslant 2k)$. We now can reformulate the fractional chromatic number of a graph G as

$$\chi_f(G) = \inf\{m/k : G \text{ is homomorphic to } G_{m,k}\}.$$

For simplicity, we write $\mu_1(G)$ as $\mu(G)$. Mycielski [19] proves that $\chi(\mu(G)) = \chi(G) + 1$. And it is obvious that $\chi_k(\mu_0(G)) = \chi_k(G) + k$ for any $k \geqslant 1$. Then what is about $\chi_k(\mu(G))$? The following theorem answers this question.

Theorem 1. *For any graph G and any integer* $k \geqslant 1$, $\chi_k(G) + 1 \leqslant \chi_k(\mu(G)) \leqslant \chi_k(G) + k$.

Proof. Suppose $\chi_k(G) = m$. Let $c : V(G) \to I_k^m$ be a k-tuple coloring of G. We define a k-tuple coloring of $\mu(G)$ $h : V(\mu(G)) \to I_k^{m+k}$ as follows: for each $j = 1, 2, \ldots, n$, $h(v_j^0) = h(v_j^1) = c(v_j^0)$ and $h(u) = \{m+1, m+2, \ldots, m+k\}$. It is easy to see that h is a proper k-tuple coloring of $\mu(G)$. Hence $\chi_k(\mu(G)) \leqslant \chi_k(G) + k$.

On the other hand, if h is any proper k-tuple coloring of $\mu(G)$, let k_0 be a color in $h(u)$, we define a k-tuple coloring c of G as follows: for each $j = 1, 2, \ldots, n$, let

$$c(v_j^0) = \begin{cases} h(v_j^0) & \text{if } k_0 \notin h(v_j^0), \\ h(v_j^1) & \text{if } k_0 \in h(v_j^0). \end{cases}$$

It is straightforward to check that c is a proper k-tuple coloring of G that does not use the color k_0. So each k-tuple coloring of G uses fewer colors than that of $\mu(G)$. It follows that $\chi_k(\mu(G)) \geqslant \chi_k(G) + 1$. □

We would like to indicate that both lower and upper bounds are attainable. Theorem 7 in Section 4 shows that $\chi_k(\mu_p(K_n)) = nk + \lceil (n-2)k/((n-1)^{p+1} - 1) \rceil$ for any integers $p \geqslant 0$, $n \geqslant 3$ and $k \geqslant 1$. When $p = 1$, we have $\chi_k(\mu(K_n)) = kn + \lceil k/n \rceil$ for any $k \geqslant 1$ and $n \geqslant 3$. If $n \geqslant k$ then $\chi_k(\mu(K_n)) = kn + 1$, attaining the lower bound $\chi_k(K_n) + 1$ in Theorem 1.

Next we show that, for $m \geqslant 5$, $\chi_2(\mu(G_{m,2}))$ attains the upper bound $\chi_2(G_{m,2}) + 2$ in Theorem 1. We first deal with the graphs $\mu(G_{5,2})$ and $\mu(G_{6,2})$.

Lemma 2. $\chi_2(\mu(G_{5,2})) = 7$.

Proof. By Theorem 1, $6 = \chi_2(G_{5,2}) + 1 \leqslant \chi_2(\mu(G_{5,2})) \leqslant \chi_2(G_{5,2}) + 2 = 7$. Suppose to the contrary that the lemma is not true, then $\chi_2(\mu(G_{5,2})) = 6$. Let h be a 2-tuple coloring of $\mu(G_{5,2})$ using colors 1, 2, 3, 4, 5, 6. Denote by v_{ij}^0 ($\{i, j\} \in I_2^5$ and $i < j$) the vertex of $G_{5,2}$. Then the vertices of $\mu(G_{5,2})$ are v_{ij}^0, v_{ij}^1 ($\{i, j\} \in I_2^5$ and $i < j$), and u. We first make two observations.

Observation 1. For any odd cycle of $\mu(G_{5,2})$ contained in V^0, there exists a vertex on this cycle such that the color set it receives is disjoint from $h(u)$.

Without loss of generality, assume $h(u) = \{1, 2\}$. Let C be an odd cycle in V^0. If, for all vertices v in C, $h(v)$ intersects $h(u)$, then it is easy to see that C is bipartite, a contradiction. Thus, Observation 1 holds.

Observation 2. Let v be any vertex in the graph $G_{5,2}$. We notice that any vertex that is not in $N(v)$ is adjacent to some vertex of $N(v)$.

By Observation 1, there exists some vertex in V^0, say v_{12}^0, such that $h(v_{12}^0) \cap h(u) = \emptyset$. Without loss of generality, assume $h(v_{12}^0) = \{5, 6\}$. Then it is clear that $h(v_{34}^1) = h(v_{35}^1) = h(v_{45}^1) = \{3, 4\}$, and $h(v_{34}^0)$, $h(v_{35}^0)$, $h(v_{45}^0)$ are contained in $\{1, 2, 3, 4\}$. By Observation 2 and the structure of $\mu(G_{5,2})$, if $ij \neq 12, 34, 35, 45$, then v_{ij}^0 is adjacent to some vertex of $\{v_{34}^1, v_{35}^1, v_{45}^1\}$, and so $h(v_{ij}^0) \subseteq \{1, 2, 5, 6\}$. Since u is adjacent to each vertex of V^1, we have $h(v_{ij}^1) \subseteq \{3, 4, 5, 6\}$.

Notice that $C_6 = v_{13}^0 v_{24}^0 v_{15}^0 v_{23}^0 v_{14}^0 v_{25}^0 v_{13}^0$ is a 6-cycle of $\mu(G_{5,2})$, and $v_{15}^0 v_{34}^0$, $v_{14}^0 v_{35}^0$, $v_{13}^0 v_{45}^0$, $v_{25}^0 v_{34}^0$, $v_{24}^0 v_{35}^0$, $v_{23}^0 v_{45}^0$ are edges of $\mu(G_{5,2})$. If $h(v_{13}^0) = \{5, 6\}$ (or $\{1, 2\}$), then it is easy to see that $h(v_{13}^0) = h(v_{14}^0) = h(v_{15}^0) = \{5, 6\}$ (or $\{1, 2\}$) and $h(v_{23}^0) = h(v_{24}^0) = h(v_{25}^0) = \{1, 2\}$ (or $\{5, 6\}$). It follows that $h(v_{34}^0) = h(v_{35}^0) = h(v_{45}^0) = \{3, 4\}$. From Observation 2 and the structure of $\mu(G_{5,2})$, we have $h(v_{ij}^1) = \{5, 6\}$ ($ij \neq 34, 35, 45$). However, this is a contradiction since $v_{13}^0 v_{24}^1$ and $v_{23}^0 v_{15}^1$ are edges of $\mu(G_{5,2})$. Thus, $h(v_{13}^0) \neq \{5, 6\}$ or $\{1, 2\}$.This implies that $|h(v_{13}^0) \cap \{1, 2\}| = |h(v_{13}^0) \cap \{5, 6\}| = 1$. Then it follows that, for each vertex $v_{ij}^0 \in C_6$, $|h(v_{ij}^0) \cap \{1, 2\}| = |h(v_{ij}^0) \cap \{5, 6\}| = 1$.

If $h(v^0_{34}) = h(v^0_{35}) = h(v^0_{45}) = \{3, 4\}$, then similar to the above argument, $h(v^1_{ij}) = \{5, 6\}$ for $ij \neq 34, 35, 45$. This is a contradiction since v^0_{13} is adjacent to v^1_{24} and $|h(v^0_{13}) \cap \{5, 6\}| = 1$. So, there exists some vertex $v^0_{pq} \in \{v^0_{34}, v^0_{35}, v^0_{45}\}$ such that $|h(v^0_{pq}) \cap h(u)| \neq 0$. Note that there is a 5-cycle among the vertices of $\{v^0_{ij} | \{i, j\} \in I^5_2, ij \neq 12, 34, 35, 45\} \cup \{v^0_{pq}\}$. This contradicts Observation 1. And the lemma follows. □

It is well known that $\chi(G_{m,k}) = m - 2k + 2$, see [9]. We shall use this fact in the proof of the following lemma.

Lemma 3. $\chi_2(\mu(G_{6,2})) = 8$.

Proof. By Theorem 1, $7 = \chi_2(G_{6,2}) + 1 \leqslant \chi_2(\mu(G_{6,2})) \leqslant \chi_2(G_{6,2}) + 2 = 8$. Suppose to the contrary that the lemma is not true, then $\chi_2(\mu(G_{6,2})) = 7$. Let h be a 2-tuple coloring of $\mu(G_{6,2})$ using colors 1, 2, 3, 4, 5, 6, 7. Without loss of generality, assume $h(u) = \{1, 2\}$.

Observe that every two adjacent vertices v^0 and w^0 in V^0 have a common neighbor in V^0, say x^0. It follows that $h(v^0)$ or $h(w^0)$ intersects $h(u)$, since otherwise the vertex x^1 cannot be colored properly. Thus, we can conclude that the vertices v^0 in V^0 with $h(v^0) \cap h(u) = \emptyset$ form an independent set. Therefore, V^0 can be partitioned into three independent sets: $\{v^0 : h(v^0) \cap h(u) = \emptyset\}$, $\{v^0 : 1 \in h(v^0) \cap h(u)\}$, and $\{v^0 : h(v^0) \cap h(u) = \{2\}\}$. This is a contradiction since $\chi(G_{6,2}) = 4$. □

Theorem 4. *Suppose m is an integer greater than or equal to 5. Then $\chi_2(\mu(G_{m,2})) = m + 2$.*

Proof. By induction on m. According to Lemmas 2 and 3, the theorem is true for $m = 5, 6$. Assume $\chi_2(\mu(G_{m-2,2})) = m$ for $m \geqslant 7$. We shall prove by contradiction that $\chi_2(\mu(G_{m,2})) = m + 2$. Suppose to the contrary it is $m + 1$. Let c be a 2-tuple coloring of $\mu(G_{m,2})$ using $m + 1$ colors with $c(u) = \{1, 2\}$. It can be shown that there is a vertex v in V^0 such that $c(v)$ is disjoint from $\{1, 2\}$, for otherwise $G_{m,2}$ is bipartite.

Without loss of generality, assume $c(v) = \{m, m + 1\}$. Note, the neighborhood of v in $G_{m,2}$, induces a subgraph isomorphic to $G_{m-2,2}$. The restriction of c on the subgraph induced by the neighborhood of v in $\mu(G_{m,2})$ and u is isomorphic to $\mu(G_{m-2,2})$ using colors $\{1, 2, \ldots, m - 1\}$, contradicting the inductive hypothesis. □

Since $G_{4,2}$ is the disjoint union of three K_2, it is clear that $\chi_2(\mu(G_{4,2})) = 5 < 6 = \chi_2(G_{4,2}) + 2$. Thus, the condition $m \geqslant 5$ in Theorem 4 is necessary.

It will be very interesting to get results similar to Theorem 4 for $\chi_k(\mu(G_{m,k}))$ with $k > 2$.

3. The kth chromatic number of $\mu(C_{2q+1})$

This section studies the kth chromatic number of Mycielskians of cycles. The kth chromatic number of odd cycles were determined by Stahl [21].

Lemma 5 (*Stahl [21]*). *Let k and q be positive integers. Then $\chi_k(C_{2q+1}) = 2k + 1 + \lfloor (k - 1)/q \rfloor$.*

It is quite easy to determine the kth chromatic number of $\mu(G)$ for the Mycielskian of even cycle C_{2q} since $\mu(C_{2q})$ is hom-equivalent to C_5 and $\chi_k(C_5)$ is determined in Lemma 5.

Theorem 6. *Let k and q be positive integers. Then $\chi_k(\mu(C_{2q+1})) = 2k + 1 + \lceil k/2 \rceil$ if k is even and $q \geqslant k$ or k is odd and $k \leqslant q \leqslant (3k - 1)/2$.*

Proof. We first show that if $q \geqslant k$ then $\chi_k(\mu(C_{2q+1})) \leqslant 2k + 1 + \lceil k/2 \rceil$. Since $\mu(C_{2q+1})$ is homomorphic to $\mu(C_{2k+1})$ when $q \geqslant k$, we only need to show that $\chi_k(\mu(C_{2k+1})) \leqslant 2k + 1 + \lceil k/2 \rceil$. We do this by giving a k-tuple coloring f of $\mu(C_{2k+1})$ using colors $0, 1, \ldots, 2k + \lceil k/2 \rceil$.

Let $C_{2k+1} = v_0 v_1 \ldots v_{2k} v_0$. For $j = 0, 1, \ldots, 2k$, let

$$f(v^0_j) = \{jk, jk + 1, \ldots, jk + k - 1\}.$$

The colors which are in the above set are reduced modulo $2k+1$ and lie in the set $\{0, 1, 2, \ldots, 2k\}$. Let $f(v_j^1)$ be any k-subset of the following set F_j,

$$F_j = (f(v_j^0)\backslash\{0, 1, \ldots, \lceil k/2\rceil - 1, k, k+1, \ldots, k+\lceil k/2\rceil - 1\}) \cup \{2k+1, 2k+2, \ldots, 2k+\lceil k/2\rceil\},$$

and let $f(u)$ be any k-subset of $\{0, 1, \ldots, \lceil k/2\rceil - 1, k, k+1, \ldots, k+\lceil k/2\rceil - 1\}$.

We now show that f is a proper k-tuple coloring of $\mu(C_{2k+1})$. It is easy to see that the restriction of f on V^0 is a k-tuple coloring of C_{2k+1}. From the definition of $f(v_j^0)$, it is not difficult to check that $|F_j| \geqslant k$ for each $j \in [0, 2k]$. On the other hand, we clearly have $F_j \cap f(v_{j-1}^0) = F_j \cap f(v_{j+1}^0) = \emptyset$. Hence all vertices v_j^1 are properly colored. Since each F_j is disjoint from $\{0, 1, \ldots, \lceil k/2\rceil - 1, k, k+1, \ldots, k+\lceil k/2\rceil - 1\}$, the vertex u is also colored properly.

Next we show that $\chi_k(\mu(C_{2q+1})) \geqslant 2k+1+\lceil k/2\rceil$ if k is even and $q \geqslant k$ or k is odd and $k \leqslant q \leqslant (3k-1)/2$. Let $C_{2q+1} = v_0 v_1 \ldots v_{2q} v_0$. Suppose the contrary that $\chi_k(\mu(C_{2k+1})) \leqslant 2k + \lceil k/2\rceil$. Let h be a k-tuple coloring of $\mu(C_{2q+1})$ using less than $2k+1+\lceil k/2\rceil$ colors. Denote by C the set of colors used by h. We shall deduce contradictions.

Since $v_j^0 v_{j+1}^1$ and $v_{j+1}^1 u$ are edges (where "+" in subscription is taken modulo $2q+1$) and since we have only $2k+\lceil k/2\rceil$ colors, $|h(v_j^0) \cap h(u)| \geqslant \lfloor k/2\rfloor$, for $j = 0, 1, \ldots, 2q$. If there exists some j such that $|h(v_j^0) \cap h(u)| > \lceil k/2\rceil$, then as v_j^0 is adjacent to v_{j+1}^0 we have $|h(v_{j+1}^0) \cap h(u)| < \lfloor k/2\rfloor$, a contradiction. Thus,

$$\lfloor k/2\rfloor \leqslant |h(v_j^0) \cap h(u)| \leqslant \lceil k/2\rceil, \quad j = 0, 1, \ldots, 2q. \tag{1}$$

If k is even then $|h(v_j^0) \cap h(u)| = k/2$ for $j = 0, 1, \ldots, 2q$. It follows that $h(v_{j+2}^0) \cap h(u) = h(v_j^0) \cap h(u)$ for $j = 0, 1, \ldots, 2q$. This is a contradiction since v_0^0 is adjacent to v_{2q}^0. Thus, $\chi_k(\mu(C_{2q+1})) \geqslant 2k+1+\lceil k/2\rceil$ if k is even and $q \geqslant k$.

Next we deal with the case k is odd and $k \leqslant q \leqslant (3k-1)/2$. By Theorem 1 and Lemma 5, the theorem is true for $k = 1$. So we assume $k \geqslant 3$.

It is well known that $\chi_f(C_{2q+1}) = 2 + 1/q$. Larsen et al. [15] showed that $\chi_f(\mu(G)) = \chi_f(G) + (1/\chi_f(G))$ for any graph G. So $\chi_f(\mu(C_{2q+1})) = 2 + (1/q) + (1/(2+1/q)) = 2 + ((q^2+2q+1)/(2q^2+q))$. According to the definition of the fractional chromatic number of a graph, we know that $\chi_k(G)/k \geqslant \chi_f(G)$ for any positive integer k. Thus, $\chi_k(\mu(C_{2q+1})) \geqslant k\chi_f(\mu(C_{2q+1})) = 2k + ((q^2+2q+1)k/(2q^2+q))$. If $k \leqslant q \leqslant (3k-1)/2$ then it is easy to check that $2k + ((q^2+2q+1)k/(2q^2+q)) > 2k + ((k+1)/2)$. It follows that $\chi_k(\mu(C_{2q+1})) > 2k + ((k+1)/2)$. Since $\chi_k(G)$ is an integer, we conclude that $\chi_k(\mu(C_{2q+1})) \geqslant 2k+1+((k+1)/2)$. □

Since $\mu(C_{2q+1})$ contains a five cycle C_5, we have $\chi_k(\mu(C_{2q+1})) \geqslant \chi_k(C_5) = 2k+1+\lfloor (k-1)/2\rfloor$ (by Lemma 5). If k is odd then $\chi_k(\mu(C_{2q+1})) \geqslant 2k+((k+1)/2)$. Note that if $q \geqslant k$ then $\chi_k(\mu(C_{2q+1})) \leqslant \chi_k(\mu(C_{2k+1})) = 2k+1+((k+1)/2)$. We now conclude that if k is odd and $q \geqslant ((3k+1)/2)$ then $\chi_k(\mu(C_{2q+1})) \in \{2k+((k+1)/2), 2k+1+((k+1)/2)\}$. However, it seems difficult to decide if $\chi_k(\mu(C_{2q+1})) = 2k + ((k+1)/2)$.

By Theorem 7 in the next section, we see that $\chi_k(\mu(C_3)) = 3k + \lceil k/3\rceil$. Therefore, if $1 \leqslant q \leqslant k-1$ then $2k + 1 + \lceil k/2\rceil \leqslant \chi_k(\mu(C_{2q+1})) \leqslant 3k + \lceil k/3\rceil$. Again it is difficult to determine the exact value for $\chi_k(\mu(C_{2q+1}))$ when $1 \leqslant q \leqslant k-1$.

4. The kth chromatic number of the generalized Mycielskians of complete graphs

In this section, we determine the kth chromatic number of the generalized Mycielskians of complete graphs. Let K_n denote the complete graph on n vertices.

Theorem 7. *Let $p \geqslant 0$, $n \geqslant 2$ and $k \geqslant 1$ be integers. Then*

$$\chi_k(\mu_p(K_n)) = \begin{cases} nk + \left\lceil \dfrac{k}{p+1} \right\rceil & \text{if } n = 2, \\[2ex] nk + \left\lceil \dfrac{(n-2)k}{(n-1)^{p+1} - 1} \right\rceil & \text{if } n \geqslant 3. \end{cases}$$

Proof. If $p = 0$ then the conclusion is obviously right. So we assume $p \geqslant 1$. Suppose c is a proper k-tuple coloring of $\mu_p(K_n)$. Let C denote the set of all colors used by c. For a vertex v of $\mu_p(K_n)$, $c(v)$ will denote the set of colors assigned to v by c. For $i = 0, 1, \ldots, p$, denote by $c(V^i)$ the set of all colors that are assigned to vertices of V^i by c. Let $A_i = C \backslash c(V^i)$ and a_i the cardinality of A_i.

Let $m = nk + \lceil (n-2)k/((n-1)^{p+1}-1) \rceil$ if $n \geqslant 3$ and let $m = nk + \lceil k/(p+1) \rceil$ if $n = 2$. We first show that m colors are enough to give a k-tuple coloring of $\mu_p(K_n)$. We construct such a k-tuple coloring c of $\mu_p(K_n)$ using the given m colors as follows. First assign k distinct colors $c_{j,1}, c_{j,2}, \ldots, c_{j,k}$ to each vertex v_j^0, $j = 1, 2, \ldots, n$, such that no two vertices of V^0 share a common color. Clearly $a_0 = m - nk$. We then assign the color set $A_0 \cup \{c_{j,a_0+1}, c_{j,a_0+2}, \ldots, c_{j,k}\}$ to each vertex v_j^1, $j = 1, 2, \ldots, n$. Clearly $A_1 = \bigcup_{j=1}^n \{c_{j,1}, c_{j,2}, \ldots, c_{j,a_0}\}$ and $a_1 = na_0 = (n-1)a_0 + a_0$. The vertices of V^2 will be colored in the same pattern, say v_j^2 $(j = 1, 2, \ldots, n)$ is assigned the color set $A_1 \cup \{c_{j,a_1+1}, c_{j,a_1+2}, \ldots, c_{j,k}\}$. Then $A_2 = A_0 \cup (\bigcup_{j=1}^n \{c_{j,a_0+1}, c_{j,a_0+2}, \ldots, c_{j,a_1}\})$. Thus, $a_2 = na_1 - a_1 + a_0 = (n-1)a_1 + a_0$. In general, suppose V^{i-1} has been colored and there are a_{i-1} colors in A_{i-1} not used in V^{i-1}. If $a_{i-1} < k$ then we color each vertex v_j^i $(j = 1, 2, \ldots, n)$ of V^i with the color set $A_{i-1} \cup \{c_{j,a_{i-1}+1}, c_{j,a_{i-1}+2}, \ldots, c_{j,k}\}$. From the pattern we color the vertices, we know that, for $1 \leqslant i \leqslant p$, if $a_{i-1} < k$ then $m = nk + a_0 = a_{i-1} + n(k - a_{i-1}) + a_i$ and so $a_i = (n-1)a_{i-1} + a_0$. Thus, for $1 \leqslant i \leqslant p$, if $a_{i-1} < k$ then

$$
\begin{aligned}
a_i &= (n-1)a_{i-1} + a_0 \\
&= (n-1)[(n-1)a_{i-2} + a_0] + a_0 \\
&\quad \cdots\cdots \\
&= (n-1)^{i-1}a_1 + (n-1)^{i-2}a_0 + \cdots + (n-1)a_0 + a_0 \\
&= (n-1)^i a_0 + (n-1)^{i-1}a_0 + \cdots + (n-1)a_0 + a_0 \\
&= \begin{cases} (i+1)a_0 = (i+1)\left\lceil \dfrac{k}{p+1} \right\rceil & \text{if } n = 2, \\ \dfrac{((n-1)^{i+1}-1)a_0}{(n-2)} = \dfrac{((n-1)^{i+1}-1)}{(n-2)} \left\lceil \dfrac{(n-2)k}{(n-1)^{p+1}-1} \right\rceil & \text{if } n \geqslant 3. \end{cases}
\end{aligned}
$$

Consequently, if $a_{p-1} < k$ then

$$
a_p = \begin{cases} (p+1)\left\lceil \dfrac{k}{p+1} \right\rceil \geqslant k & \text{if } n = 2, \\ \dfrac{((n-1)^{p+1}-1)}{(n-2)} \left\lceil \dfrac{(n-2)k}{(n-1)^{p+1}-1} \right\rceil \geqslant k & \text{if } n \geqslant 3. \end{cases}
$$

Let i be the integer such that $a_{i-1} < k$ and $a_i \geqslant k$. Clearly $1 \leqslant i \leqslant p$. Let K be a k-subset of A_i and let R be a k-subset of $C \backslash K$. By alternatively assigning the color sets K and R to vertices in $V^{i+1}, V^{i+2}, \ldots, V^p, \{u\}$, we get a proper k-tuple coloring of $\mu_p(K_n)$. Thus, $\chi_k(\mu_p(K_n)) \leqslant m$.

Next we shall show that $\chi_k(\mu_p(K_n)) \geqslant m$. Suppose c is a proper k-tuple coloring of $\mu_p(K_n)$. Let $c(V^i)$, A_i and a_i be the same as we defined in the beginning of the proof. For $i = 0, 1, \ldots, p$, let B_i and D_i denote the set of colors each of which is assigned to exactly one vertex of V^i and the set of colors each of which is assigned to at least two vertices of V^i, respectively. Clearly $c(V^i) = B_i \cup D_i$ for $i = 0, 1, \ldots, p$.

Note that, in the graph $\mu_p(K_n)$, each vertex of V^i is adjacent to exactly $n-1$ vertices of V^{i-1} $(1 \leqslant i \leqslant p)$. It is easy to see that if $j \in B_{i-1}$ then at most one vertex of V^i is assigned the color j, and if $j \in D_{i-1}$ then $j \notin c(V^i)$. However, if $j \in A_{i-1}$ then each vertex of V^i may receive the color j. It follows that $c(V^i) \subseteq A_{i-1} \cup B_{i-1}$. Let $x_1 = |c(V^i) \cap A_{i-1}|$ and $x_2 = |c(V^i) \cap B_{i-1}|$. Then $|c(V^i)| = x_1 + x_2$ and $0 \leqslant x_1 \leqslant a_{i-1}$. Since each vertex in V^i receives at least $k - x_1$ colors from B_{i-1}, we have $x_2 \geqslant n(k - x_1)$. It follows that $|c(V^i)| = x_1 + x_2 \geqslant x_1 + n(k - x_1) \geqslant a_{i-1} + n(k - a_{i-1})$. Therefore, $a_i = |A_i| \leqslant nk + a_0 - (a_{i-1} + n(k - a_{i-1})) = (n-1)a_{i-1} + a_0$. By a similar calculation as above, we have $a_p \leqslant (p+1)a_0$ if $n = 2$ and $a_p \leqslant (((n-1)^{p+1} - 1)a_0/(n-2))$ if $n \geqslant 3$. Since c is a proper k-tuple coloring of $\mu_p(K_n)$, the k colors of $c(u)$ cannot be used in V^p. So $a_p \geqslant k$. Then we must have $a_0 \geqslant \lceil k/(p+1) \rceil$ if $n = 2$ and $a_0 \geqslant \lceil (n-2)k/((n-1)^{p+1} - 1) \rceil$ if $n \geqslant 3$. Thus, c uses at least m colors. This proves that $\chi_k(\mu_p(K_n)) = m$. $\square$

Note that $\mu_p(K_2)$ is exactly the odd cycle C_{2p+1} ($p\geqslant 0$), the above theorem implies Lemma 5 (a result in [21]).

It will be interesting to determine the kth chromatic number of $\mu_p(G_{m,l})$ for $l>1$.

Remark. Recently, Pan and Zhu [20] showed that, for any graph G, if $p\geqslant 2\lceil k/s\rceil$ then $\chi_k(\mu_p(G_{m,k}))\leqslant m+s$. This implies that, for any graph G, if $p\geqslant 2k$ then $\chi_k(\mu_p(G))\leqslant \chi_k(G)+1$. Thus, if $p\geqslant 2k$, then $\chi_k(\mu_p(G))$ equals $\chi_k(G)$ or $\chi_k(G)+1$. They indicated that both values are possible for $\chi_k(\mu_p(G))$ and gave a sufficient condition under which a graph G has $\chi_k(\mu_p(G))=\chi_k(G)$ for sufficiently large p.

5. a/b-Colorings of generalized Mycielskians

Tardif [23] proved that, for any graph G and any nonnegative integer p, $\chi_f(\mu_p(G))=\chi_f(G)+(1/\sum_{k=0}^{p}(\chi_f(G)-1)^k)$. So if $\chi_f(G)=a/b$ then $\chi_f(\mu_p(G))=((at+b^{p+1})/bt)$, where $t=\sum_{i=0}^{p}(a-b)^i b^{p-i}$. However, this does not imply if G is a/b-colorable then $\mu_p(G)$ is $(at+b^{p+1})/bt$-colorable, since, as indicated by Fisher [5], we only know that $m(G)$ is a multiple of the denominator of the minimal fractional representation of $\chi_f(G)$ but do not know which multiple $m(G)$ is. The following theorem deals with this question.

Lemma 8. *Suppose G is a graph and p a nonnegative integer. If G is a/b-colorable, then $\mu_p(G)$ is $(at+b^{p+1})/bt$-colorable, where $t=\sum_{i=0}^{p}(a-b)^i b^{p-i}$.*

Proof. Suppose $V(G)=\{v_1^0,v_2^0,\ldots,v_n^0\}$ and we have a proper a/b-coloring of G. Let the a colors of the a/b-coloring of G are $1,2,\ldots,a$, and each vertex v_i^0 of G has color set $\{i_1,i_2,\ldots,i_b\}$. Imagine that each color $i\in[1,a]$ has t offspring, namely, $c_{i1},c_{i2},\ldots,c_{it}$. Together with b^{p+1} new colors $c_1,c_2,\ldots,c_{b^{p+1}}$, we have $at+b^{p+1}$ colors. Following we shall produce a $(at+b^{p+1})/bt$-coloring of $\mu_p(G)$ using these $at+b^{p+1}$ colors.

We color the vertices of $\mu_p(G)$ in the order $V^0,V^1,\ldots,V^p,\{u\}$, and only one of the V^i's or u is colored at each step. Denote by A_l the set of all colors which are not used in V^l, and a_l the cardinality of A_l, $l=0,1,\ldots,p$. First assign the color set $\bigcup_{j=1}^{b}\{c_{i_j1},c_{i_j2},\ldots,c_{i_jt}\}$ to each vertex v_i^0, $i=1,2,\ldots,n$. Clearly each vertex of V^0 receives bt colors and V^0 is properly colored. Also it is clear that $A_0=\{c_1,c_2,\ldots,c_{b^{p+1}}\}$ and $a_0=b^{p+1}$. Now color each vertex v_i^1 of $V^1(i=1,2,\ldots,n)$ with the color set $A_0\cup(\bigcup_{j=1}^{b}\{c_{i_j(b^p+1)},c_{i_j(b^p+2)},\ldots,c_{i_jt}\})$. Since v_i^1 and v_i^0 have the same neighborhood in V^0 and that V^0 has been properly colored, we conclude that this coloring of $V^0\cup V^1$ is proper. It is obvious that $A_1=\bigcup_{i=1}^{a}\{c_{i1},c_{i2},\ldots,c_{ib^p}\}$ and $a_1=(a_0/b)a=b^pa$. To each vertex v_i^2 $(i=1,2,\ldots,n)$ of V^2, we assign the color set $A_1\cup(\bigcup_{j=1}^{b}\{c_{i_j(a_1/b)+1},c_{i_j(a_1/b)+2},\ldots,c_{i_jt}\})$. Again, this coloring of $V^0\cup V^1\cup V^2$ is proper. Since all colors of A_0 are not used in V^2 and all colors of A_1 are used in V^2, we have $a_2=(a_1/b)a-a_1+a_0=(a/b-1)a_1+a_0$. Generally, suppose V^{l-1} $(1\leqslant l-1\leqslant p-1)$ has been colored and there are a_{l-1} colors of A_{l-1} not used in V^{l-1}, where $a_{l-1}=(a/b-1)a_{l-2}+a_0$ and $a_{l-1}<bt$. We shall color each vertex $v_i^l(i=1,2,\ldots,n)$ of V^l with the color set $A_{l-1}\cup(\bigcup_{j=1}^{b}\{c_{i_j(a_{l-1}/b)+1},c_{i_j(a_{l-1}/b)+2},\ldots,c_{i_jt}\})$. At this time $\bigcup_{j=0}^{l}V^j$ is properly colored. From the way we color the vertices, we know that, for $1\leqslant l\leqslant p$, $at+a_0=a_{l-1}+a(t-a_{l-1}/b)+a_l$ and so $a_l=(a/b-1)a_{l-1}+a_0$. Thus

$$\begin{aligned}
a_l &= \left(\frac{a}{b}-1\right)a_{l-1}+a_0\\
&= \left(\frac{a}{b}-1\right)\left[\left(\frac{a}{b}-1\right)a_{l-2}+a_0\right]+a_0\\
&\cdots\cdots\\
&= \left(\frac{a}{b}-1\right)^{l-1}a_1+\left(\frac{a}{b}-1\right)^{l-2}a_0+\cdots+\left(\frac{a}{b}-1\right)a_0+a_0\\
&= \left(\frac{a}{b}-1\right)^{l-1}\frac{a}{b}a_0+\left(\frac{a}{b}-1\right)^{l-2}a_0+\cdots+\left(\frac{a}{b}-1\right)a_0+a_0\\
&= a_0\sum_{i=0}^{l}\left(\frac{a}{b}-1\right)^{i}=b\sum_{i=0}^{l}(a-b)^i b^{p-i}\leqslant bt.
\end{aligned}$$

It is easy to see that if $l < p$ then $a_l < bt$ and $a_p = bt$. So the coloring process can continue until all vertices in V^p have been colored. Finally, we can color the vertex u of $\mu_p(G)$ with the $a_p = bt$ colors of A_p. This gives a $(at + b^{p+1})/bt$-coloring of $\mu_p(G)$. And the lemma follows. □

Tardif's result and Lemma 8 imply the following.

Theorem 9. *Suppose G is a graph and p a nonnegative integer. If $\chi_f(G) = a/b$ and G is a/b-colorable, then $\chi_{bt}(\mu_p(G)) = at + b^{p+1}$, where $t = \sum_{i=0}^{p}(a-b)^i b^{p-i}$.*

Lemma 10. *Let p be any nonnegative integer. If $\gcd(a, b) = 1$, then $\gcd(at + b^{p+1}, bt) = 1$, where $t = \sum_{i=0}^{p}(a-b)^i b^{p-i}$.*

Proof. Suppose the lemma is not true. Then for some prime k, we have $k|(bt)$ and $k|(at + b^{p+1})$. Since k is prime, either $k|b$ or $k|t$. If $k|t$, then $k|b$ since $k|(at + b^{p+1})$ and k is prime. If $k|b$, then $k|(at)$ since $k|(at + b^{p+1})$. As $\gcd(a, b) = 1$, we conclude that $k|t$. Thus, in both cases we have $k|b$ and $k|t$. As $t = b^{p+1} + (a-b)b^p + (a-b)^2 b^{p-1} + \cdots + (a-b)^p b + (a-b)^{p+1}$, $k|b$ and $k|t$ imply that $k|(a-b)$ and $k|a$. This is a contradiction since $\gcd(a, b) = 1$. □

Lemma 10 and Tardif's result imply the following.

Corollary 11. *Let G be a graph and p a nonnegative integer. Suppose $\chi_f(G) = a/b$ and $\gcd(a, b) = 1$. If $m(G) = b$ then $m(\mu_p(G)) = bt$, where $t = \sum_{i=0}^{p}(a-b)^i b^{p-i}$.*

We would like to point out that if $m(G)$ is not the denominator of the minimal representation of $\chi_f(G)$ then we can say nothing about $m(\mu_p(G))$.

Given two positive integers a and b such that $a > 2b$ and $\gcd(a, b) = 1$. The graph G_a^b has vertex set $\{0, 1, \ldots, a-1\}$ and two vertices i and j are adjacent if and only if $b \leqslant |i-j| \leqslant a-b$. It is well known that $\chi_f(G_a^b) = a/b$ (see [8]). By assigning each color $i \in [1, a]$ to each vertex in $\{(i-1)b, (i-1)b+1, \ldots, ib\}$ (The colors which are in this set are reduced modulo a), we get a a/b-coloring of G_a^b. Thus, $\chi_b(G_a^b) = a$. Now for $p = 0, 1, \ldots$, consider the graph $\mu_p(G_a^b)$. Let n denote its vertex number. Then $n = (p+1)a + 1$ and $p = (n-1)/a - 1$. According to Corollary 11, we have

$$m(\mu_p(G_a^b)) = b\sum_{i=0}^{p}(a-b)^i b^{p-i} = \frac{b^{p+2}}{a-2b}[(a/b-1)^{p+1} - 1].$$

Thus, if p is properly large then

$$\frac{b^{p+2}}{a-2b}(a/b-1)^p \leqslant m(\mu_p(G_a^b)) \leqslant \frac{b^{p+2}}{a-2b}(a/b-1)^{p+1},$$

and so

$$\frac{b^2}{(a-2b)(a-b)}[(a-b)^{1/a}]^{(n-1)} \leqslant m(\mu_p(G_a^b)) \leqslant \frac{b}{(a-2b)}[(a-b)^{1/a}]^{(n-1)}.$$

Corollary 12. *Suppose a and b are two positive integers with $a > 2b$. Let $\lambda = (a-b)^{1/a}$, $C_1 = (b^2/((a-2b)(a-b)))$ and $C_2 = (b/(a-2b))$. If p is properly large then*

$$C_1\lambda^{n-1} \leqslant m(\mu_p(G_a^b)) \leqslant C_2\lambda^{n-1}.$$

Since $a > 2b$, λ is greater than 1. Thus, for any two positive integers a and b with $a > 2b$, $m(\mu_p(G_a^b))$ is exponential in n with growth rate $\lambda = (a-b)^{1/a} > 1$.

For $b = 1$, the graph G_a^b is just the complete graph on a vertices. Simple calculations show that λ is approximately equal to 1.2599, 1.3161, 1.3195, 1.3077, 1.2917, 1.2754, 1.2599 for $b = 1$ and $a = 3, 4, 5, 6, 7, 8, 9$. The growth rate appears to be the largest when $b = 1$ and $a = 5$ and the growth rate is approximately equal to 1.3195. It is a little smaller than the growth rate 1.346193 obtained by Fisher [5]. However, the proof here is simple and we easily get the exact growth rates.

$m(\mu_p(K_n))$ and its growth rate λ can also be derived directly from Tardif's result and Theorem 7. For graphs G other than K_n, since we do not know the value of $\chi_k(\mu_p(G))$ for any k and p, we cannot get $m(\mu_p(G))$ from Tardif's result. Corollaries 11 and 12 provide a way to compute $m(\mu_p(G))$ and its growth rate for many graphs G other than K_n.

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