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The travelling preacher, projection, and a lower bound for the stability number of a graph[☆]

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In loving memory of George Dantzig

Abstract

The coflow min–max equality is given a travelling preacher interpretation, and is applied to give a lower bound on the maximum size of a set of vertices, no two of which are joined by an edge.

Keywords: Network flow; Circulation; Projection of a polyhedron; Gallai's conjecture; Stable set; Total dual integrality; Travelling salesman cost allocation game

1. Coflow and the travelling preacher

An interpretation and an application prompt us to recall, and hopefully promote, an older combinatorial min–max equality called the Coflow Theorem.

Let G be a digraph. For each edge e of G, let d_e be a non-negative integer. The capacity $d(C)$ of a dicircuit C means the sum of the d_e's of the edges in C. An instance of the Coflow Theorem (1982) [2,3] says:

Theorem 1. *The maximum cardinality of a subset S of the vertices of G such that each dicircuit C of G contains at most $d(C)$ members of S equals the minimum of the sum of the capacities of any subset $\mathcal{H}$ of dicircuits of G plus the number of vertices of G which are not in a dicircuit of $\mathcal{H}$.*

The Coflow Theorem in greater generality says:

Theorem 2. *For any digraph $G = (V, E)$ and any numbers $(d, a, b) = (d_e, a_v, b_v : v \in V, e \in E)$ (where a_v may be ∞ and b_v may be $-\infty$), the following system in variables $x = (x_v : v \in V)$ is TDI:*

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(1.1) $\forall x \in V, b_v \leq x_v \leq a_v$;
(1.2) $\forall$ dicircuit C in G,
$x(C \cap V) \equiv \sum\{x_v : v \in (C \cap V)\} \leq d(C \cap E) \equiv \sum\{d_e : e \in (C \cap E)\}$.

TDI (totally dual integral) means that whether or not (d, a, b) is integer-valued, if $(w_v : v \in V)$ is integer-valued, then the linear programming dual of the LP

(1.0) maximize $\{wx : x$ satisfies (1.1)–(1.2)$\}$

has an integer-valued optimum solution provided it has an optimum solution. This implies that if (d, a, b) is integer-valued, then the LP (1.0) has an integer-valued optimum solution provided it has an optimum solution. See [10,11]. (Of course, the primal optimum is equal to the dual optimum, and this is the more general coflow min–max equality.)

We learned from discussions that Sándor Fekete and Bill Pulleyblank posed the following memorable word problem (see [5]):

A travelling preacher wishes to charge x_v to the churches, $v \in V$, which he serves, in order to maximize his income $wx = \sum\{w_v x_v : v \in V\}$, where x is subject to $b \leq x \leq a$ depending on the amount of sin and holiness at the various churches, and also subject to $x(C \cap V) \leq d(C \cap E)$ for every dicircuit C in digraph $G = (V, E)$.

The reason for the latter constraint is that, for every dicircuit C, $d(C \cap E)$ is the most any preacher can charge the churches in C without the churches in C arranging to hire a different preacher. This is related to n-church game theory. See [5,4,7].

One way to find the maximum wx subject to (1.1) and (1.2) would be to check the feasibility of any given x by applying an algorithm which determines if there is a dicircuit C such that $d(C \cap E) - x(C \cap V)$ is negative. This is easy. And use that together with the "optimization $=$ separation" approach provided by the ellipsoid method. This is polytime but not so easy. See [8]. Fekete and Pulleyblank [5] use "optimization $=$ separation" in the same way for an undirected variant of the problem.

A much more efficient approach to maximizing wx is the way we prove Theorem 2. Briefly, by a slight massaging, we get the problem into the form:

(2.0) maximize wx subject to
(2.1) $x \geq \underline{0}$;
(2.2) $\forall$ dicircuit C,
$x(C \cap E) \equiv \sum\{x_e : e \in (C \cap E)\} \leq d(C \cap E) \equiv \sum\{d_e : e \in (C \cap E)\}$

using a slightly different G and d. The x of (1.1)–(1.2) is part of the x of (2.1)–(2.2), although it is indexed by new edges rather than vertices.

The dual of this LP has a variable $y_C \geq 0$ for each dicircuit C. However, we can represent a circulation in G, given as a flow, y_C, around dicircuits, as flows in edges, and vice versa. We thus get a Hoffman circulation problem [9]. The LP dual of that circulation problem has, besides the variables x_e of (2.1)–(2.2), an additional new variable, say η_v, for each vertex $v \in V$. The dual circulation problem is:

(3.0) maximize $\sum\{w_e x_e : e \in E\}$ subject to
(3.1) $\forall e \in E, x_e - \eta_{t(e)} + \eta_{h(e)} \leq d_e$;
(3.2) $\forall e \in E, x_e \geq 0$.

For each dicircuit C, by adding up the inequalities (3.1) for $e \in C$, we get $x(C \cap E) \leq d(C \cap E)$. We can solve the Hoffman circulation problem and its dual by standard methods to get an optimum (x, η). We can forget the values of the variables η_v; that is, these are projected away, to get an optimum solution of (2.0). For further details, see [3].

This was originally discovered in the first author's Ph.D. work [2], as a response to the challenge by her advisor, the second author, to find interesting instances of solving a combinatorial optimization problem by projecting away "don't care" variables of another combinatorial optimization problem.

Perhaps the first application of projection to solving a combinatorial optimization problem was treating a capacitated b-matching problem with parity constraints (for example, the "Chinese Postman Problem") as a projection of a b-matching polytope with loops at vertices of the graph and each edge of the graph replaced by three edges in series. See [11]; in particular, pages 600–605.

2. A lower bound on the stability number of a graph

A *stable set* in a graph or digraph is a set of vertices, no two of which are joined by an edge. The maximum size of a stable set in a graph or digraph G is called the stability number of G and is denoted $\alpha(G)$. Recently, Bessy and Thomassé [1] proved the following theorem, conjectured by Gallai [6] in 1963. A digraph is called *strongly connected* if each edge and each vertex is in a dicircuit.

Theorem 3. *For any strongly connected digraph G, $\alpha(G)$ is greater than or equal to the minimum number of dicircuits which together cover all the vertices.*

Note that Theorem 3 provides a lower bound on the stability number of an undirected graph by considering any orientation.

A *feedback set* in a digraph G is a subset F of its edges such that $G - F$ has no dicircuits. A feedback set F is called *coherent* if every edge of G is in some dicircuit which contains at most one member of F.

Bessy and Thomassé [1] proved the following wonderful lemma.

Theorem 4. *Every strongly connected digraph has a coherent feedback set.*

Applying Theorem 1 to a strongly connected digraph G with a coherent feedback set F and setting $d_e = 1$ for each e in F and letting the other d_e's be 0, yields Theorem 5 below.

Theorem 5. *Let G be a strongly connected digraph and F a coherent feedback set in G. The maximum size of a set of vertices of G which intersects each dicircuit at most $|C \cap F|$ times equals the minimum of $\sum\{|C \cap F| : C \in \mathcal{H}\}$ over sets $\mathcal{H}$ of dicircuits of G which cover all the vertices.*

Note that since G is strongly connected, and F is a coherent feedback set, a set S of vertices of G which intersects each dicircuit of C at most $|C \cap F|$ times is a stable set. Also, for any dicircuit C, $|C \cap F| \geq 1$. Thus Theorem 5 immediately yields Theorem 3.

For related ideas, see [12].

References

[1] Stéphane Bessy, Stéphan Thomassé, Spanning a strong digraph by α circuits: A proof of Gallai's conjecture, Combinatorica (in press).

[2] Kathie Cameron, Polyhedral and Algorithmic Ramifications of Antichains, Ph.D. Thesis, University of Waterloo, 1982.

[3] Kathie Cameron, Jack Edmonds, Coflow polyhedra, Discrete Math. 101 (1992) 1–21.

[4] Ulrich Faigle, Sándor P. Fekete, Winfried Hochstättler, Walter Kern, On approximately fair cost allocation in Euclidean TSP games, OR Spektrum 20 (1998) 29–37.

[5] Sándor P. Fekete, William R. Pulleyblank, A note on the traveling preacher problem, Report No. 98.331, Angewandte Mathematik und Informatik, Universität zu Köln, 1998.

[6] Tibor Gallai, Problem 15, in: M. Fiedler (Ed.), Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963), p. 161.

[7] Daniel Granot, Gur Huberman, Minimum cost spanning tree games, Math. Program. 21 (1981) 1–18.

[8] Martin Grötschel, László Lovász, Alexander Schrijver, Geometric Algorithms and Combinatorial Optimization, second ed., Springer-Verlag, Berlin, 1993.

[9] A.J. Hoffman, Some recent applications of the theory of linear inequalities to extremal combinatorial analysis, in: Combinatorial Analysis, Proceedings of Symposia in Applied Mathematics, vol. X, Amer. Math. Soc., 1960, pp. 113–127.

[10] Bernhard Korte, Jens Vygen, Combinatorial Optimization: Theory and Algorithms, third ed., Springer-Verlag, Berlin, 2005.

[11] Alexander Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, Springer-Verlag, Berlin, 2003.

[12] András Sebő, Minmax relations for cyclically ordered digraphs, J. Combin. Theory Ser. B 97 (2007) 518–552.