On the Zeros of a Differential Polynomial and Normal Families*

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1. INTRODUCTION

Let \( f(z) \) be a meromorphic function in the complex plane. Throughout this paper we use the familiar notation of value distribution theory (see [1]). For \( \phi = f' + f^n \), W. K. Hayman [2] proved that if \( n \geq 5 \) and \( f \) is transcendental then \( \phi \) assumes every finite complex number infinitely often. E. Mues [3] then proved that \( \phi \) assumes zero infinitely often in the case \( n = 4 \); also, he gave an example to show that for every \( c \neq 0 \), there exists a nonconstant meromorphic function \( f \) which satisfies \( f' + f^4 \neq c \). Afterward N. Steinmetz [4] proved that if \( f' + f^4 \neq c \), then \( f \) satisfies a Riccati differential equation \( w' = 2p^2(w^2 - p^2), \) \( p^2 = c \neq 0 \). Corresponding to normal family criteria, J. K. Langley [5] and Li [6] proved respectively that if \( F \) is a family of meromorphic functions on a domain \( D \) and for every \( f \in F, \) \( f' + f^n \neq c, \) \( n \geq 5 \), then \( F \) is normal on \( D \). Pang Xuecheng [8] and W. Schwick [9] then extended the preceding result. Compare also the discussion in Schiff [10].

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Here we shall improve the above results by proving the following:

**Theorem 1.** Let \( f \) be a meromorphic function in the complex plane, let \( \phi = f^n + f^{(k)} + a_1 f^{(k-1)} + \cdots + a_k f + a_{k+1} \), where \( n = k + 3 \) and \( f^{(k)} + a_1 f^{(k-1)} + \cdots + a_k f + a_{k+1} \neq 0 \), and let \( a_1, \ldots, a_{k+1} \) be \( k + 1 \) complex numbers. If \( N(r, 1/\phi) = S(r, f) \), then \( f \) satisfies the Riccati differential equation \( w' = p_1 w^2 + p_2 w + p_3 \), where \( p_1^{n-2} = H/2k! \), \( p_2 = -6a_1/(3n-2)k, \) \( p_3 = H/n, \) and \( H \) is a constant.

**Theorem 2.** Let \( f \) and \( \phi \) be assumed as in Theorem 1. If \( N(r, 1/\phi) = S(r, f) \) and one of the following conditions is satisfied,

1. \( n \) is an odd number and \( a_1 = 0 \),
2. \( n \) is an even number and \( a_1 = a_2 = 0 \),

then \( f \) is a constant.

Concerning the normality criterion, we have

**Theorem 3.** Let \( F \) be a family of meromorphic functions on a domain \( D \). Suppose that for every \( f \in F \), \( f^n + f^{(k)} + a_1 f^{(k-1)} + \cdots + a_k f + a_{k+1} \neq 0 \), where \( n \geq k + 3 \), and \( a_1(z), \ldots, a_k(z), a(z) \) are \( k + 1 \) holomorphic functions on \( D \). Then \( F \) is normal.

### 2. Some Lemmas and Proofs of Theorems

**Lemma 1.** Let \( f \) be a transcendental meromorphic function and \( n \) be a positive integer. If \( f^n P(f) = Q(f) \), and \( \deg Q \leq n \), where \( P \) and \( Q \) are two differential polynomials in \( f \) with coefficient functions satisfying \( m(r, a) = S(r, f) \), then \( m(r, P) = S(r, f) \).

Lemma 1 is a modified version of Clunie’s theorem [11].

**Lemma 2 [4].** Let \( f \) be a meromorphic function in \( |z| < \infty \), which satisfies \( m(r, f) = S(r, f) \). Suppose that \( N(f, f) = N(r, f) - \overline{N(r, f)} = S(r, f) \) and in some neighbourhood of the simple poles \( z_v \) of \( f \)

\[
f(z_v) = \frac{a(z_v)}{z - z_v} + b(z_v) + o(z - z_v),
\]

where \( o(z) \) and \( b(z) \) are small functions of \( f \), i.e., \( T(r, a(z)) = o(S(r, f)) \). Then \( f \) satisfies the Riccati differential equation \( w' = A_0(z)w^2 + A_1(z)w + A_2(z) \), where \( A_0, A_1, A_2 \) are small functions of \( f \).

Lemma 2 is also true if \( m(r, 1/f) = N(f, 1/f) = S(r, f) \) and for all simple zeros \( z_v \) of \( f, z \to z_v, f(z) = a(z_v)(z - z_v) + b(z_v)(z - z_v)^2 + O((z - z_v)^3) \). (See [4, Lemma 2].)
Proof of Theorem 1. We set \( \varphi = \phi' / \phi \); then

\[
n f^{n-1} f' + f^{(k+1)} + a_1 f^{(k)} + \cdots + a_{k-1} f'' + a_k f' = \varphi \left[ f^n + f^{(k)} + a_1 f^{(k-1)} + \cdots + a_{k-1} f' + a_k f + a_k f + a_{k+1} \right].
\]

Rewriting (1) in the form

\[
H f^{n-1} = \varphi \left[ f^{(k)} + a_1 f^{(k-1)} + \cdots + a_k f + a_{k+1} \right] - f^{(k+1)} - a_1 f^{(k)} - \cdots - a_k f',
\]

where

\[
H = nf' - \varphi f,
\]

we have \( H \neq 0 \). Otherwise by integration, we easily have \( (1 - C) f^n + f^{(k)} + \cdots + a_1 f^{(k-1)} + \cdots + a_k f + a_{k+1} = 0 \). But this is impossible according to assumptions \( n = k + 3 \) and

\[
f^{(k)} + a_1 f^{(k-1)} + \cdots + a_k f + a_{k+1} \neq 0.
\]

By Lemma 1, \( m(r, H) = S(r, f) \). By (2) we see that if \( z_0 \) is a pole of \( f \), then \( z_0 \) cannot be the pole of \( H \) because \( n = k + 3 \). Only the zeros of \( \varphi \) can cause the poles of \( H \), so \( N(r, H) = N(r, 1/\varphi) = S(r, f) \). Hence \( T(r, H) = S(r, f) \).

Again by (2), (3),

\[
m(r, f) + m \left( \frac{1}{r-f} \right) = S(r, f).
\]

From (2), it is easy to see that \( N_1(r, f) = T(r, f) + S(r, f) \), where \( N_1(r, f) \) denotes the \( N \) function with respect to the simple poles of \( f \). If \( z_0 \) is a simple pole of \( f \) and \( f(z) = b_{-1} / (z - z_0) + b_0 + \cdots \) in a neighbourhood of \( z_0 \), then

\[
\varphi = \frac{-n}{z - z_0} + \frac{nb_0}{b_{-1}} + O(z - z_0).
\]
By computation of (2), the left side of (2) is
\[
\frac{H_0 b_{-1}^{n-1}}{(z - z_0)^{n-1}} + \frac{H_1 b_{-1}^{n-1} + H_0(n-1)b_0 b_{-1}^{n-2}}{(z - z_0)^{(n-2)}} + \text{terms of higher degree of } (z - z_0)
\]
and the right side of (2) is
\[
\frac{(-1)^{k+1}(nk! - (k + 1)!)b_{-1}}{(z - z_0)^{k+2}} + \frac{(-1)^k a_1 b_{-1}(k - 1)!(n - k) + (-1)^k nk! b_0}{(z - z_0)^{k+1}} + \text{terms of higher degree of } (z - z_0),
\]
where \( H_0 = H(z_0) \) and \( H_1 = H'(z_0) \).
By comparing their coefficients,
\[
H_0 b_{-1}^{n-2} = (-1)^{k+1}2k!
\]
and
\[
\frac{b_0}{b_{-1}} = -\frac{2}{3n - 2} \left( \frac{H_1}{H_0} \right) - \frac{3a_1}{(3n - 2)k}.
\]
So by Lemma 2,
\[
\varphi' = \frac{1}{n} \varphi^2 + A_1 \varphi + A_2,
\]
where
\[
A_1 = \frac{4}{3n - 2} \left( \frac{H'}{H} \right) + \frac{6a_1}{(3n - 2)k}
\]
and \( A_2 \) is a small function.
Rewriting (3) in the form
\[
f' = \frac{H}{n} + \frac{1}{n} \varphi f
\]
and differentiating $f$ and $\varphi$, we have

$$f'' = \frac{2}{n^2} \varphi^2 f + \frac{A_1}{n} \varphi f + \frac{A_2}{n^2} f + \frac{\varphi H}{n} + \frac{H'}{n}$$

$$f''' = \frac{6}{n^3} \varphi^3 f + \frac{6A_1}{n^2} \varphi^2 f + \left( \frac{5A_2}{n^2} + \frac{A_1}{n} + \frac{A_1}{n} \right) \varphi f + \left( \frac{A_1 A_2}{n} + \frac{A_2^2}{n} \right) \varphi^2 f + \left( \frac{3H}{n^2} \varphi^2 + \text{terms of lower degree of } \varphi \right)$$

$$f^{(k)} = \frac{k!}{n^k} \varphi^k f + \frac{B_k}{n^{k-1}} A_1 \varphi^{k-1} f$$

$$+ \left( \frac{C_k}{n^{k-1}} A_2 + \frac{D_k}{n^{k-2}} A_1^2 + \frac{E_k}{n^{k-2}} A_1^2 \right) \varphi^{k-1} f$$

$$+ \left( \text{terms of lower degree of } \varphi \right) f + \frac{k!}{2n^k} H \varphi^{k+1}$$

where $B_1 = C_1 = D_1 = E_1 = 0$ and for $k \geq 2$

$$B_k = (k - 1)((k - 1)! - B_{k-1}); \quad C_k = (k - 1)(k - 1)! + (k - 2)C_{k-1},$$

$$D_k = B_{k-1} + (k - 2)C_{k-1}; \quad E_k = (k - 2)(B_{k-1} + E_{k-1}).$$

If we substitute all $f^{(i)}$ in (2), then in the case $k \geq 2$,

$$H f^{n-1} = \left( \frac{2k!}{n^{k+1}} \varphi^{k+1} + \left( \frac{nB_k - B_{k+1}}{n^{k-1}} A_1 + \frac{3a_1(k - 1)!}{n^k} \right) \varphi^k \right.$$

$$\left. + \left( \frac{nC_k - C_{k+1}}{n^k} A_2 + \frac{nD_k - D_{k+1}}{n^k} A_1^2 + \frac{nE_k - E_{k+1}}{n^k} A_1^2 \right. \right.$$ 

$$\left. + \frac{nB_k - B_{k+1}}{n^{k-1}} A_1^2 + \frac{4a_1(k - 2)!}{n^{k-1}} \right) \varphi^{k-1}$$

$$+ \left( \text{terms of lower degree of } \varphi \right) f + \frac{k!}{n^{k+1}} H \varphi^{k+1}$$

$$+ \text{terms of lower degree of } \varphi$$

$$\triangleq P_{k+1}(\varphi) f + P_k(\varphi).$$
Differentiation of (9) gives
\[ H'f^{n-1} + (n - 1)Hf^{n-2}f' = \left( P_{k+1}(\varphi) \right)'f + P_{k+1}(\varphi)f' + (P_k(\varphi))', \tag{10} \]
If we substitute (8) and (9) in (10), then (10) can be written in the form
\[ \frac{n - 1}{n}H^2f^{n-2} = \left( P_{k+1}(\varphi) \right)' + \frac{P_{k+1}(\varphi)}{n} - \frac{H'}{H}P_{k+1}(\varphi) \]
\[ - \frac{n - 1}{n}P_{k+1}(\varphi)\varphi f + \frac{H}{n}P_{k+1}(\varphi) + (P_k(\varphi))' \]
\[ - \frac{H'}{H}P_k(\varphi) - \frac{n - 1}{n}P_k(\varphi)\varphi. \tag{11} \]
The coefficient of \( \varphi^{k+1} \) in
\[ \frac{H}{n}P_{k+1}(\varphi) + (P_k(\varphi))' - \frac{H'}{H}P_k(\varphi) - \frac{n - 1}{n}P_k(\varphi)\varphi \]
\[ = \frac{2k!H}{n^{k+2}} + \frac{kk!H}{n^{k+2}} - \frac{(n - 1)k!H}{n^{k+2}} = 0, \]
so its degree on \( \varphi \) is at most \( k = n - 3 \).
Let \( f(z) = b_{-1}(z - z_0) + b_0 + \cdots, z \to z_0 \); then in the neighbourhood of \( z_0 \), the left side of (11) equals \( ((n - 1)/n)H_0^2b_{-1}^2(z - z_0)^{n-2} \) + terms of higher degree of \( (z - z_0) \), and the right side of (11) equals \( u(z_0)n^k b_{-1}^2(z - z_0)^{n+2} \) + terms of higher degree of \( (z - z_0) \), where \( H_0 = H(z_0) \) and \( u(z) \) is a small function. Hence
\[ \frac{n - 1}{n}H_0^2b_{-1}^2 = u(z_0)n^k b_{-1} \]
and by (6),
\[ b_{-1} = \frac{(n - 1)H_0(-1)^{k+1}2k!}{n^{k+1}u(z_0)} \]
and
\[ b_0 = b_{-1}\left(-\frac{2}{3n-2}\left(\frac{H_1}{H_0}\right) - \frac{3a_1}{(3n-2)k}\right). \]
Again by Lemma 2,
\[ f' = p_1f^2 + p_2f + p_3, \tag{12} \]
where \( T(r, p_i) = S(r, f), i = 1 \) to 3.
It is easy to see that $T(r, f) = T(r, \varphi) + S(r, f)$. Hence by (3) and (12),

\[ \varphi = np_1 f + np_2, \]

and

\[ np_3 = H. \]  \hspace{1cm} (14)

Differentiating (13) gives $\varphi' = np'_1 f + np_1 f' + np'_2$. Combining this with (7), (12), (13), and (14), we have

\[ A_1 = \frac{p'_1}{p_1} - p_2, \]

and

\[ A_2 = np_1 p_3 + np'_2 - np_2 \frac{p'_1}{p_1}. \]  \hspace{1cm} (16)

Rewriting (13) in the form $f = \varphi/np_1 - p_2/p_1$ and substituting it in (9), then by comparing their coefficients on

\[ H \left( \frac{1}{np_1} \right)^{n-1} = \frac{2k!}{n^{k+2} p_1}, \]

and

\[
HC_{n-1}^2 \left( \frac{1}{np_1} \right)^{(n-3)} \left( \frac{p_2}{p_1} \right)^2 
= - \left( \frac{nB_k - B_{k+1}}{n^k} A_1 + \frac{3a_3(k - 1)!}{n^k} \right) \frac{p_2}{p_1} 
+ \left( \frac{nC_k - C_{k+1}}{n^k} A_2 + \frac{nD_k - D_{k+1}}{n^{k-1}} A_1 \right) 
+ \frac{nE_k - E_{k+1}}{n^{k-1}} A_1^2 
+ \left( \frac{nB_{k-1} - B_k}{n^{k-1}} a_1 A_1 + \frac{4a_3(k - 2)!}{n^{k-1}} \right) 
\times \frac{1}{np_1} + \frac{k!}{n^{k+1}} H. \]  \hspace{1cm} (18)
By (14)–(17) and the representation of \( A_1 \), (18) can be written in the form
\[
\left( \frac{k!}{n^{k+1}} + \frac{nC_k - C_{k+1}}{n^{k+1}} \right) H p_1
\]
\[
= U_1 \left( \frac{H'}{H} \right) + U_2 \left( \frac{H'}{H} \right)^2 + U_3 \frac{H'}{H} + U_4,
\]
where \( U_i \) (\( i = 1 \) to 4) are constants.

It is easy to show that \( C_k \geq kk!/4 \) by induction, so \( nC_k - C_{k+1} \geq 0 \).

Again by (17) and (19), \( H \) must be a constant. This proves the case \( k \geq 2 \).

In the case \( k = 1 \), in a neighbourhood of the simple zeros \( z_i \) of \( f \), the right side of (3) is
\[
4b_1 + \left( 8b_2 - b_1 \left( \frac{2b_2 + a_1b_1}{b_1 + a_2} \right) \right) (z - z_i)
\]
\[+ \text{terms of higher degree of} \ (z - z_i), \]
where
\[
b_1 = f'(z_i) \quad \text{and} \quad b_2 = \frac{H'(z_i)(b_1 + a_2) + a_1b_1^2}{6b_1 + 8H'(z_i)}.
\]

A gain by Lemma 2, \( f \) can be represented as \( f' = p_1f^2 + p_2f + p_3 \).

Differentiating \( f' \) and substituting it in \( f'' \), we have
\[
f'' = 2p_1^2f^3 + (3p_1p_2 + p_1')f^2 + (p_1' + 2p_1p_3 + p_2^2)f + p_3' + p_2p_3.
\]

Notice that (13) is also true in the case \( k = 1 \), so (2) can be written in the form
\[
Hf^3 = 2p_1^2f^3 + (5p_1p_2 - p_1' + 3a_1p_1)f^2
\]
\[+ (3a_1p_2 + 4a_2p_1 + 2p_1p_3 + 3p_2^2 - p_2')f
\]
\[+ (3p_2p_3 + 4a_2p_2 - p_3' - a_1p_3).
\]

By comparing its coefficients
\[
H = 2p_1^2 \quad (20)
\]
\[
5p_1p_2 - p_1' + 3a_1p_1 = 0 \quad (21)
\]
\[
3a_1p_2 + 4a_2p_1 + 2p_1p_3 + 3p_2^2 - p_2' = 0 \quad (22)
\]
\[
3p_2p_3 + 4a_2p_2 - p_3' - a_1p_3 = 0. \quad (23)
\]

Rewriting (21) in the form \( p_2 = a_1/5 + p_1/5p_1' \), we have \( T(r, p_2) = 5T(r, p_1) = S(r, p_2) \). Also, by (20), (22), and (14), \( T(r, p_3) = S(r, p_2) \) and \( T(r, p_3) = S(r, p_2) \). Thus all \( p_i \) are constants and we complete the proof of
Theorem 1.

Proof of Theorem 2. If \( n \) is an even number and \( a_1 = a_2 = 0 \), then \( U_s = 0 \) and hence \( H = 0 \). So \( f \) is a constant. If \( n \) is an odd number and \( a_1 = 0 \), then \( p_2 = 0 \) and hence \( f = \text{Ctg}(Az + B) \). Substituting \( f \) in \( \varphi \) gives

\[
\varphi = (\text{Ctg}(Az + B))^n + (\text{Ctg}(Az + B))^{(k)} + a_1(\text{Ctg}(Az + B))^{(k-1)} + \ldots + a_{k+1} = C^n\text{tg}^n(Az + B) + Q_{k+1}(\text{tg}(Az + B)),
\]

where \( Q_{k+1} \) is a polynomial of degree \( k + 1 \).

If \( N(r, 1/\varphi) = S(r, f) \), then \( \varphi = C^n(\text{tg}(Az + B) + i)^n(\text{tg}(Az + b) - i)^n \), which implies that \( (p - 1)i + (q - 1)(-i) = (p - q)i = 0 \). But this is impossible because \( n \) is odd and \( p + q = n \). Hence \( f \) must be a constant and the result is proved. \( \blacksquare \)

Corollary 1. Let \( f \) be a meromorphic function in \( C \) and \( n \) and \( k \) be two positive integers. If \( f \) satisfies one of the following conditions,

1. \( f^4 + f^* \neq 0 \),
2. \( f^n + f^{(k)} \neq a, \ n \geq k + 3, \ k \geq 2 \) (or: \( n \geq k + 4, \ k \geq 1 \), \( a \) is a finite complex number,

then \( f \) must be a constant.

Remark. The proof of Corollary 1 when \( n \geq k + 4 \) is the same as that given in [2].

In order to prove Theorem 3, we need the following.

Lemma 3 [7]. Let \( F \) be a family of meromorphic functions on the unit disc. If \( F \) is not normal, then for every given \( k (-1 < k < 1) \) there exist

1. a real number \( r \), \( 0 < r < 1 \),
2. complex numbers \( z_n, |z_n| < r \),
3. functions \( f_n \in F, n = 1, 2, \ldots \), and
4. positive numbers \( \rho_n \)

which satisfy \( \lim_{n \to \infty} \rho_n = 0 \) and \( \lim_{n \to \infty} \|z_n - r \|/\rho_n = +\infty \), such that \( \rho_n f_n(z_n + \rho_n z) \to g(\xi) \) spherically uniformly on compact subsets of \( C \), where \( g \) is a nonconstant meromorphic function on \( C \).

Proof of Theorem 3. Without loss of generality, we can assume that \( D \) is the unit disc. If \( F \) is not normal, then there exist \( r, z_m, \rho_m \) such that \( g_m(\xi) = \rho_m^{k/(n-1)} f_m(z_m + \rho_m \xi) \) is convergent to \( g(\xi) \) uniformly on compact subsets of \( C \), where \( g(\xi) \) is a nonconstant meromorphic function.
If \( g^\prime(\xi) + g^{(k)}(\xi) \neq 0 \), then by Corollary 1, \( g \) must be a constant and we arrive at a contradiction. Hence there exists \( \xi_0 \) such that
\[
g^n(\xi_0) + g^{(k)}(\xi_0) = 0.
\]
It is easy to see that \( g(\xi_0) \neq \infty \). So there exists \( \delta > 0 \) such that \( g(\xi) \) is holomorphic in \( D_\delta = (|\xi - \xi_0| < 2\delta) \), and for all sufficiently large \( m \), \( g_m^{(n)}(\xi) \) are holomorphic in \( D_\delta = (\xi - \xi_0| < \delta) \). Furthermore \( g_m^{(n)}(\xi) \) are convergent to \( g^{(i)}(\xi) \) uniformly on \( D_\delta \) \((i = 0 \text{ to } k)\).

Now we set
\[
L_m(\xi) = \sum_{i=0}^{k-1} a_{k-i}(z_m + \rho_m \xi) \rho_m^{kn/(n-1)-i} g_m^{(i)}(\xi)
\]  
\[
- \rho_m^{kn/(n-1)} a(z_m + \rho_m \xi).
\]
Then \( L_m(\xi) \) tends to zero uniformly on \( D_\delta \) for \( kn/(n-1) - i > k/(n-1) \) and \(|a_{k-i}(z_m + \rho_m \xi)| \leq M(1+r)/2, a_{k-i}(z) < \infty (i = 0 \text{ to } k-1)\). Therefore \( g_m^{(n)}(\xi) + g_m^{(k)}(\xi) + L_m(\xi) \) tends to \( g^n(\xi) + g^{(k)}(\xi) \) uniformly on \( D_\delta \). But
\[
g_m^{(n)}(\xi) + g_m^{(k)}(\xi) + L_m(\xi) = \rho_m^{kn/(n-1)}
\]
\[
\times \left( f_m^n(z_m + \rho_m \xi) + f_m^{(k)}(z_m + \rho_m \xi)
\right.
\]
\[
\left. + \sum_{i=0}^{k-1} a_{k-i}(z_m + \rho_m \xi) f_m(z_m + \rho_m \xi) - a(z_m + \rho_m \xi) \right) \neq 0,
\]
so by the Hurwitz theorem \( g^n(\xi) + g^{(k)}(\xi) \equiv 0 \) on \( D_\delta \). This implies \( g^n + g^{(k)} = 0 \) on \( C \), but this is also impossible.}