JOURNAL OF ALGEBRA 58, 208–216 (1979)

Seminormality and Projective Modules over Polynomial Rings

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Communicated by J. Dieudonné

Received July 5, 1978

INTRODUCTION

This paper is concerned with the problem of determining those integral domains D which have the property that finitely generated projective modules over $D[X_1, ..., X_n]$ are extended from D. As Quillen and Suslin have shown, Dedekind domains have this property. In fact, Lequain and Simis show in [10] that any Prüfer domain has the property. For geometric purposes this beautiful result is no improvement on Quillen-Suslin because a Noetherian Prüfer domain is a Dedekind domain. On the other hand, we shall show in this paper that the techniques of Lequain–Simis can be applied in the one-dimensional case to obtain geometric results.

Obviously, a first step in attacking the general problem would be to determine those domains D such that rank one projectives over $D[X_1, ..., X_n]$ are extended —that is, those domains D such that $\mathrm{Pic}(D) = \mathrm{Pic}(D[X_1, ..., X_n])$. This problem was effectively treated by Traverso [14] for the case of a Noetherian domain having affine normalization. A clever argument of Gilmer and Heitmann reduces the general case to the case handled by Traverso and so we are able in Theorem 1 to give a characterization of those domains D such that $\mathrm{Pic}(D) = \mathrm{Pic}(D[X_1, ..., X_n])$ for any positive integer n. In fact, our characterization of Traverso's *seminormality* is an internal one which arose in the 1962 paper of Bass [2]. Precisely, $\mathrm{Pic}(D) = \mathrm{Pic}(D[X_1, ..., X_n])$ if and only if whenever α belongs to the quotient field of D with $\alpha^2, \alpha^3 \in D$, then $\alpha \in D$. We then prove our main result, Theorem 2, which says that if D has Prüfer normalization plus the property that $\mathrm{Spec}(D_P)$ is finite for each prime ideal P of D, then finitely generated projective $(D[X_1, ..., X_n])$-modules are extended if and only if D is seminormal. This completely solves the problem for one-dimensional Noetherian domains and so for coordinate rings of irreducible algebraic curves. For completeness, we prove again a result of Salmon [13] characterizing the irreducible seminormal affine plane curves, thereby enabling us to illustrate our results for the well known curves $y^2 = x^3$ and $y^2 = x^2 + x^3$.

Throughout this paper $\bar{D}$ will denote the integral closure of the integral

0021-8693/79/050208–09$02.00/0

domain D. If D is a subring of a ring R, integral over D, ${}_R^+D$ will indicate the seminormalization of D in R. Recall from [14] that ${}_R^+D$ is the largest subring of R which contains D and has the property that for each prime ideal P of D, i) there is a unique prime ideal ${}^+P$ of ${}_R^+D$ lying over P, and (ii) the canonical homomorphism of residue class fields $k(P) \to k({}^+P)$ is an isomorphism. D is said to be seminormal if $D = {}_D^+D$.

We shall have occasion to use the statement "$\mathrm{Pic}(D) = \mathrm{Pic}(D[X_1,\ldots, X_n])$". By this we mean that the natural monomorphism $\mathrm{Pic}(D) \to \mathrm{Pic}(D[X_1,\ldots, X_n])$ is an isomorphism, i.e. that rank one projective $D[X_1,\ldots, X_n]$-modules are extended from D.

Main Results

Bass presented in [2, Prop. 2.1(6)] an argument of Schanuel which we now give in a slightly modified form. The argument will show that if D is an integral domain with quotient field K and $\alpha \in K$ is an element such that $\alpha \notin D$, but α^2, $\alpha^3 \in D$, then there is a projective rank one module over the polynomial ring $D[X]$ which is not extended. In fact, consider the fractional ideals $I = (\alpha^2, 1 + \alpha X)$ and $J = (\alpha^2, 1 - \alpha X)$ of $D[X]$. Then $IJ = (\alpha^4, \alpha^2 + \alpha^3 X, \alpha^2 - \alpha^3 X, 1 - \alpha^2 X^2) \subseteq D[X]$. Now $X^4\alpha^4 + (1 + \alpha^2X^2)(1 - \alpha^2X^2) = 1$, so $IJ = D[X]$ and I and J are invertible, with $J = I^{-1}$. We claim that I is not extended from D. For suppose it were. By localizing D at a prime ideal P such that $\alpha \notin D_P$, we may assume that D is quasi-local. Then I would be principal, say $I = (f)$, for some $f \in I$. Since $IK[X] = K[X], f \in K$. Then $f - f\alpha X = f(1-\alpha X) \in D[X]$, so $f \in D$. This gives $I = (f) \subseteq D[X]$, contradicting $\alpha \notin D$.

We say that a domain with quotient field K is (2, 3)-*closed* if every element $\alpha \in K$ such that α^2, $\alpha^3 \in D$ is an element of D. The argument of Schanuel above, then, shows that the property of being (2, 3)-closed is a necessary condition for a domain D to have the property that rank one $D[X]$-projectives are extended. What seems astonishing to us is that, although weaker than normality, it is also sufficient.

THEOREM 1. *Let D be an integral domain with $\{X_i\}_{i=1}^{\infty}$ a family of indeterminates over D. The following conditions are equivalent*:

(1) $\mathrm{Pic}(D) = \mathrm{Pic}(D[X_1,\ldots, X_n])$ *for each positive integer n.*

(2) $\mathrm{Pic}(D) = \mathrm{Pic}(D[X_1])$.

(3) *D is seminormal.*

(4) *For each $\alpha \in \bar{D}\backslash D$, the conductor of D in $D[\alpha]$ is a radical ideal of $D[\alpha]$.*

(5) *D is* (2, 3)-closed.

Ideas for the proof come from a number of sources, among them Bass [2], Gilmer–Heitmann [7], Hamann [8], and Traverso [14]. In particular, the proof that condition (4) implies condition (1) is due to Gilmer and Heitmann and appears in article [7].

Proof. (1) ⇒ (2). Clear.

(2) ⇒ (5). This follows from Schanuel's argument given above.

(5) ⇒ (4). Suppose (4) does not hold. Then there is an element $\alpha \in \bar{D}$ such that the conductor C of D in $D[\alpha]$ is not a radical ideal in $D[\alpha]$. Then $\sqrt{C} \not\subseteq D$, since C is the largest common ideal of D and $D[\alpha]$. Choose $\beta \in \sqrt{C}\backslash D$. Say $\beta^n \in C$. Since C is an ideal of $D[\alpha]$, $\beta^m \in C$ for all $m \geqslant n$. Let n_0 be the least integer such that $\beta^m \in D$ for all $m \geqslant n_0$. Then $n_0 > 1$ and $\gamma = \beta^{n_0-1} \notin D$. But $\gamma^2, \gamma^3 \in D$, contradicting (5).

(4) ⇒ (3). From the characterization of seminormalization given in the introduction, it follows that for $D \subseteq R \subseteq \bar{D}$, ${}_R^+D = R \cap {}_{\bar{D}}^+D$. (Thus D is seminormal if and only if D is seminormal in $D[\alpha]$ for every $\alpha \in \bar{D}$.) Now suppose D is not seminormal and let $\alpha \in {}_{\bar{D}}^+D\backslash D$. Let $R = D[\alpha]$. Then ${}_R^+D = R$, so for each prime ideal P of D there is a unique prime ideal Q of R lying over P and furthermore, the map $k(P) \rightarrow k(Q)$ is an isomorphism. We claim that in this situation, the conductor C of D in R is not a radical ideal of R, contradicting (4). For $C \neq 0$ since R is a finite D-module, and $C \neq D$ since $R \neq D$. Let P be a prime ideal of D minimal with respect to containing C, and let Q be the prime ideal of R lying over P. Then CD_P is the conductor of D_P in R_P and since $CD_P \neq D_P$, $D_P \neq R_P$. Using the Cohen–Seidenberg theorems we see that $R_P = R_Q$. If C is a radical ideal of R, then it is one in D, whence $CD_P = PD_P$. But QR_Q is the only prime of R_P containing PD_P, so $CD_P = QR_Q$. Thus $k(P) = D_P/PD_P$, while $k(Q) = R_P/PD_P$. From the fact that $k(P) \rightarrow K(Q)$ is an isomorphism we deduce that $D_P = R_P$, which is false.

(3) ⇒ (5). Let α be an element of the quotient field of D such that $\alpha^2, \alpha^3 \in D$. Since D is seminormal, D is seminormal in $D[\alpha]$, i.e. ${}_{D[\alpha]}^+D = D$. By [14, Lemma 1.3], the conductor C of D in $D[\alpha]$ is a radical ideal of $D[\alpha]$. Clearly $\alpha \in C \subseteq D$.

An interesting consequence of Theorem 1 is that if D is seminormal, so is $D[X_1]$. This is a K-theoretic result which we shall use in the proof of Theorem 2. We shall also need the following lemma:

LEMMA 1. *Let D be an integral domain.*

(a) *If D is seminormal, then D_S is seminormal for any multiplicatively closed subset S of D.*

(b) *If $D_{\mathcal{M}}$ is seminormal for each maximal ideal $\mathcal{M}$ of D, then D is seminormal.*

Proof. Both assertions can be easily verified using condition (5) of Theorem 1.

THEOREM 2. *Let D be an integral domain with $\{X_i\}_{i=1}^{\infty}$ a family of indeterminates over D. Suppose that the following two conditions hold*:

(i) *$\bar{D}$ is a Prüfer domain, and*

(ii) *$\operatorname{Spec}(D_P)$ is a finite set for each prime ideal P of D.*

Then finitely generated projective $D[X_1, ..., X_n]$-modules are extended for each positive integer n if and only if D is seminormal.

Proof. ($\Rightarrow$): This implication is obvious from Theorem 1.

($\Leftarrow$): We shall use the Lequain–Simis induction theorem [10, Theorem A]. Thus, let $\mathscr{C}$ be the class of integral domains satisfying (i), (ii), and (iii), where (iii) is the condition that the domain D is seminormal. We have four properties of the class $\mathscr{C}$ to verify:

(c.0) Each non-maximal prime of any domain in $\mathscr{C}$ has finite height. This property is clear by (ii).

(c.1) If $D \in \mathscr{C}$, then $D_P \in \mathscr{C}$ for each prime ideal P of D. Condition (ii) obviously holds in D_P and condition (iii) holds by Lemma 1. As for condition (i), if $S = D\backslash P$, it is well known that $\bar{D}_S = \overline{(D_P)}$. Thus $\overline{(D_P)}$ is a Prüfer domain.

(c.3) If $D \in \mathscr{C}$ and D is quasi-local, then every finitely generated projective module over $D[X_1]$ is free. To see this, by condition (ii), Spec D is a finite set. Thus, by [10, Lemma 1] and [1, Ch.IV, Cor. 2.7], every finitely generated projective module over $D[X_1]$ is the direct sum of a free $D[X_1]$-module plus an ideal of $D[X_1]$. Since D is seminormal and quasi-local, the ideal is principal by Theorem 1.

(c.2) If $D \in \mathscr{C}$, then $(D[X_1])_{P[X_1]} \in \mathscr{C}$ for each prime ideal P of D. Here is where the work comes in. If D is seminormal, then $D[X_1]$ is seminormal and hence $(D[X_1])_{P[X_1]}$ is seminormal by Lemma 1. This disposes of condition (iii).

To verify condition (i) for $(D[X_1])_{P[X_1]}$, we must show that $\overline{((D[X_1]_{P[X_1]})}$ is a Prüfer domain. Let $S = D[X_1]\backslash P[X_1]$. Then $\overline{(D[X_1]_{P[X_1]})} = \overline{(D[X_1])}_S = \bar{D}[X_1]_S$. If Q is a prime ideal of $\bar{D}[X_1]$ with $Q \cap S = \varnothing$, then $Q \cap D[X_1] \subseteq P[X_1]$. By the Cohen–Seidenberg theorems applied to the integral extension $D[X_1] \subseteq \bar{D}[X_1]$, there is a prime ideal Q' of $\bar{D}[X_1]$ such that $Q' \supseteq Q$ and $Q' \cap D[X_1] = P[X_1]$. Thus every maximal ideal of $\bar{D}[X_1]_S$ is of the form Q_S, where $Q \cap D[X_1] = P[X_1]$.

Now suppose $Q \cap D[X_1] = P[X_1]$ and let $Q_0 = Q \cap \bar{D}$. Then $Q_0 \cap D = P$, so $Q_0[X_1] \cap D[X_1] = P[X_1]$. Since $Q_0[X_1] \subseteq Q$, we must have $Q = Q_0[X_1]$. It follows that every maximal ideal of $\bar{D}[X_1]_S$ is of the form $Q_0[X_1]_S$, where Q_0 is a prime ideal of $\bar{D}$ and $Q_0 \cap D = P$. We now see that the localization of $\bar{D}[X_1]_S$ at a maximal ideal $Q_0[X_1]_S$ is $\bar{D}[X_1]_{Q_0[X_1]}$ which is a valuation ring, since $\bar{D}$ is a Prüfer domain [6, Prop. 18.7]. This shows that $\overline{(D[X_1]_{P[X_1]})}$ is a Prüfer domain.

To verify condition (ii) for $D[X_1]_{P[X_1]}$, we must show that it has finitely many prime ideals. In order to make the argument easier to follow, we isolate three more or less general facts about domains whose integral closure is a Prüfer domain:

CLAIM 1. *Suppose R is a domain with $\bar{R}$ a Prüfer domain, and let P be a prime ideal of R. Then $\overline{R/P}$ is a Prüfer domain.*

Proof of Claim 1. Let Q be a prime ideal of $\bar{R}$ lying over P. Then $\bar{R}/Q$ is integral over R/P and is a Prüfer domain. Let L be the quotient field of R/P. Since $\bar{R}/Q$ is integrally closed and integral over R/P, $\overline{R/P} = L \cap (\bar{R}/Q)$. As $\bar{R}/Q$ is integral over $\overline{R/P}$, $\overline{R/P}$ is a Prüfer domain by [6, Theorem 22.4].

CLAIM 2. *Let R be a domain with $\bar{R}$ a Prüfer domain, and let P be a prime ideal of R. Then for each valuation overring V of R, there is at most one prime ideal of V lying over P.*

Proof of Claim 2. Let $Q_1 \subsetneq Q_2$ be prime ideals of V lying over P. Since $\bar{R} \subseteq V$ and $\bar{R}$ is a Prüfer domain, $V = \bar{R}_Q$ for some prime ideal Q of $\bar{R}$. Thus $Q_1 \cap \bar{R} \subsetneq Q_2 \cap \bar{R}$. But then we contradict the Cohen–Seidenberg theorems upon contracting to R.

CLAIM 3. *Let R be a domain with $\bar{R}$ a Prüfer domain, and let P be a prime ideal of R. If Q is a non-zero prime ideal of $R[X_1]$ with $Q \subseteq P[X_1]$, then $Q \cap R \neq 0$.*

Proof of Claim 3. We proceed exactly as in the proof of (6) ⇒ (3) in [6, Theorem 19.15], using claim 2 and omitting the assumption that R is integrally closed. Thus, assume there is a non-zero prime ideal Q of $R[X_1]$ with $Q \subseteq P[X_1]$ and such that $Q \cap R = 0$. Then R is a subring of the domain $R[X_1]/Q = R[x]$, where x denotes the image of X_1 modulo Q. Since $Q \neq 0$, x is algebraic over R.

Let M be any of the prime ideals of $R[X_1]$ properly containing $P[X_1]$ such that $M \cap R = P$. Set $P_0 = P[X_1]/Q$ and $M_0 = M/Q$. Then P_0 and M_0 are distinct prime ideals of $R[x]$ lying over P such that $P_0 \subseteq M_0$. There is a valuation overring W of $R[x]$ containing prime ideals P_1 and M_1 lying over P_0 and M_0, respectively. Let K be the quotient field of R, and let $V = K \cap W$. Then V is a valuation overring of R and since $R[x]$ is algebraic over R, $P_1 \cap K$ and $M_1 \cap K$ are distinct prime ideals of V [6, Theorem 19.16]. But $P_1 \cap K$ and $M_1 \cap K$ both contract to P in R, contradicting claim 2.

Now let Q be a prime ideal of $D[X_1]$ such that $Q \subseteq P[X_1]$. Let $Q_0 = Q \cap D$. Set $R = D/Q_0$, $P' = P/Q_0$, and $Q' = Q/Q_0[X_1]$. By claim 1, $\bar{R}$ is a Prüfer domain. Now Q' is a prime ideal of $R[X_1]$ such that $Q' \subseteq P'[X_1]$. Since $Q' \cap R = 0$, it follows from claim 3 that $Q' = 0$, and hence that $Q = Q_0[X_1]$. Thus every prime ideal of $D[X_1]_{P[X_1]}$ is the extension of a prime ideal of D_P, and therefore

$D[X_1]_{P[X_1]}$ has a finite number of prime ideals. This completes the proof of Theorem 2.

Remark 1. It would be nice to remove hypotheses (i) and (ii) from Theorem 2, but an examination of the proof shows that a key feature is the interplay between (i) and (ii) in order to verify the hypotheses of Theorem A of [10]. A thorough inspection of the proof of Theorem 2 shows that our assumption (ii) really emanates from [1, Ch.IV. Cor. 2.7]. Thus, one might omit (ii) if one could prove an analogue of [1, Ch.IV, Cor. 2.7] free of dimension restrictions. On the other hand, (ii) could be replaced by the assumption that D is finite dimensional if it were true that whenever R is a finite dimensional quasi-local domain whose integral closure is a Prüfer domain, Spec R is finite. That this need not be the case was shown to us by W. Heinzer. To wit, let K be an algebraically closed field with X and Y indeterminates. Let V be a discrete rank two valuation domain of the form $K + N$ on $K(X, Y)$. Denote by L the algebraic closure of $K(X, Y)$ and by D^* the integral closure of V in L. If $\mathscr{M}$ is the intersection of the maximal ideals of D^*, then $D = K + \mathscr{M}$ is the desired example. The reader desiring further details should consult [9, p. 6].

Remark 2. By means of Theorem 2 we can construct for each positive integer m a non-normal domain D_m of dimension m such that finitely generated $D_m[X_1, ..., X_n]$-modules are extended for each positive integer n. In fact, we can arrange to have $\overline{(D_m)}$ not finitely generated over D_m. Specifically, let L be a field containing a subfield K over which L is infinite algebraic and choose an m-dimensional valuation ring of the form $L + \mathscr{M}$. That $D_m = K + \mathscr{M}$ is the desired example follows from Theorem 2 and the second part of Proposition 1 proved in the sequel.

Obviously, hypothesis (ii) of Theorem 2 holds when the domain is one-dimensional. In fact, if D is a one-dimensional Noetherian domain, D satisfies hypotheses (i) and (ii). This makes the following theorem immediate:

THEOREM 3. *Let D be a one-dimensional domain. If $\bar{D}$ is a Prüfer domain, then finitely generated projective $D[X_1, ..., X_n]$-modules are extended for any positive integer n if and only if D is seminormal. In particular, if D is Noetherian, then finitely generated projective $D[X_1, ..., X_n]$-modules are extended for any positive integer n if and only if D is seminormal.*

The two main results of [5] were Theorems 4.7 and 5.4. It is clear from Theorem 1 and Endo's Theorem 4.1 that seminormality and Endo's weak normality coincide. Consequently, Theorem 3 extends Endo's results to polynomial rings in an arbitrary number of variables over any one-dimensional Noetherian domain.

A natural way in which one-dimensional Noetherian domains arise is as the coordinate rings of irreducible affine curves. We now address ourselves to determining those plane curves whose coordinate rings are seminormal. Toward this end we prove a result which will give a condition for a one-dimensional quasilocal domain to be seminormal.

PROPOSITION 1. *Let $(D, \mathscr{M})$ be a quasi-local domain with C the conductor of D in $\bar{D}$. Then*

(1) *If D is seminormal, then C is a radical ideal of $\bar{D}$. Furthermore, if D is one-dimensional, then $C = 0$ or $C = \mathscr{M}$.*

(2) *If $\mathscr{M}$ is a radical ideal of $\bar{D}$, then D is seminormal.*

Proof. (1) If D is seminormal, then C is a radical ideal of $\bar{D}$ by [14, Lemma 1.3]. If moreover, D is one-dimensional with $C \neq 0$ and if $\mathscr{M}'$ is a maximal ideal of $\bar{D}$, then $\mathscr{M}' \supseteq \mathscr{M} \supseteq C$. Thus, since C is a radical ideal of $\bar{D}$ and since D is one-dimensional, $C = \cap\{\mathscr{M}' \mid \mathscr{M}'$ is a maximal ideal of $\bar{D}\}$. It follows that $C = C \cap D = (\cap\{\mathscr{M} \mid \mathscr{M}'$ maximal in $\bar{D}\} \cap D = \mathscr{M}$.

(2) Suppose that $\mathscr{M}$ is a radical ideal of $\bar{D}$ and let $\alpha \in \bar{D}$ with $\alpha^2, \alpha^3 \in D$. If $\alpha^2 \notin \mathscr{M}$, then $\alpha = \alpha^{-2}\alpha^3 \in D$. If $\alpha^2 \in \mathscr{M}$, then since $\mathscr{M}$ is a radical ideal of $\bar{D}$, $\alpha \in \mathscr{M} \subseteq D$.

COROLLARY 1. *If $(D, \mathscr{M})$ is one-dimensional quasi-local domain with $\bar{D}$ a finite D-module, then D is seminormal if and only if $\mathscr{M}$ is a radical ideal of $\bar{D}$.*

Remark 3. Let L be a field algebraic over a proper subfield K and let t be transcendental over L. If V is a one-dimensional valuation domain of the form $L(t) + \mathscr{M}$, then the domain $R = L + \mathscr{M}$ is a one-dimensional quasi-local integrally closed domain which is not a valuation domain. Moreover, R is the integral closure of the domain $D = K + \mathscr{M}$ and by Corollary 1, D is seminormal. Thus, even in the one-dimensional case, seminormality does not imply Prüfer normalization. Note that our results do not apply to the domain D.

We come now to the promised characterization of "seminormal" plane curves.

PROPOSITION 2. (cf. Salmon [13]). *The coordinate ring of an irreducible affine plane curve over an algebraically closed field K is seminormal if and only if each of its singularities is an ordinary double point.*

Proof. We sketch a proof shown to us by Bill Heinzer. If the curve has a singularity, we can take it to be at the origin.

Thus, let D be the local ring of an ordinary n-fold point at the origin, say $D = ([X, Y])_{(X,Y)}/(f)$, where f has leading form $\prod_{i=1}^{n}(X - a_iY)$ with the a_i's distinct. Then in $K[[X, Y]]$, the completion of $(K[X, Y])_{(X,Y)}$, f factors as a product of n distinct power series each of order one. Thus, $\bar{D}$ has n distinct maximal ideals $\bar{\mathscr{M}}_1, \ldots, \bar{\mathscr{M}}_n$ [11, p. 139]. Moreover, the conductor C factors as $C = \bar{\mathscr{M}}_1^{n-1} \cdots \bar{\mathscr{M}}_n^{n-1}$. To see this, note that since the conductor lifts to the completion, it suffices to prove it in $K[[X, Y]]/(f)$. But this is a subring of a direct sum of n copies of $K[[T]]$ for an indeterminate T. Hence, by Corollary 1, D is seminormal if and only if $n = 2$, that is if and only if the origin is an ordinary double point.

If the origin is a non-ordinary singular point ,then D has the form $(K[X, Y])_{(X,Y)}/(f)$ where f has leading form $\prod_{i=1}^{m}(X - a_iY)^{e_i}$ and $e_j > 1$ for some j. An argument similar to the one above shows that the conductor of D in $\bar{D}$ is not a radical ideal of $\bar{D}$ and so D is not seminormal.

This completes the proof.

We remark that Bombieri [3] has proved in more or less the same fashion the general case of Proposition 2. More precisely, he has shown that an algebraic curve is seminormal if and only if its singularities are multiple points of multiplicity n with distinct tangents and tangent space of dimension n. Nevertheless, since our subsequent applications will be to affine plane curves, we felt that a proof in that case should be included for completeness. For a more comprehensive discussion of the results of Bombieri and Salmon, see E. Davis' paper [4].

The following theorem is a geometric corollary to the above work. By assuming Bombieri's result, we can give the theorem its full strength.

THEOREM 4. *Let C be an irreducible affine curve over an algebraically closed field K. Vector bundles over* $C \times \mathbb{A}_k^n$ *are extended from vector bundles over C for any positive integer n if and only if every singularity of C is an ordinary multiple point at which the tangent space has dimension equal to the multiplicity.*

Theorem 4 renders a wealth of examples available to us. We focus on two. The curve $y^2 = x^3$ has a cusp at the origin and no other singularities. It is thus the simplest example of a non-seminormal plane curve. This undoubtedly accounts for its recurrence when an example of a projective $D[X]$-module not extended from D has been sought. The curve $y^2 = x^2 + x^3$ has an ordinary double point at the origin as its only singularity. Its coordinate ring is therefore an example of a non-normal noetherian domain D for which finitely generated $D[X_1, ..., X_n]$-modules are extended.

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