ELSEVIER Stochastic Processes and their Applications 70 (1997) 265–282

stochastic processes and their applications

Long range dependence of point processes, with queueing examples

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Received 23 October 1996; received in revised form 26 March 1997

Abstract

Possible definitions of the long range dependence (LRD) of a stationary point process are discussed. Examples from the standard queueing literature are considered and shown to be amenable to yielding processes with long range count dependence. In particular the effect of the single-server queueing operator, whereby one point process is transformed into another via the mechanism of a simple queue, is examined for possible long range dependence of both the counting and interval properties of the output process. For an infinite server queue, the output is long range count dependent if and only if the input is long range count dependent. © 1997 Elsevier Science B.V.

1. Introduction

Long range dependence (LRD) of a stationary stochastic process $\{X_n: n = 0, \pm 1, \dots\}$ is usually defined in terms of its second order properties (Beran, 1994, Ch. 1): assuming that X_n has finite second moment, $\{X_n\}$ is said to be LRD if the variance of the sample mean of n consecutive observations grows more slowly asymptotically than a sequence of independent identically distributed (i.i.d.) observations, i.e. if

$$\lim_{n\to\infty} \frac{\operatorname{var}(\sum_{i=1}^n X_i)}{n} = \infty. \tag{1.1}$$

A sufficient condition for (1.1) is that $\lim_{n\to\infty} \sum_{i=1}^n \operatorname{cov}(X_0, X_i) = \infty$.

Changing the process to a stationary point process $N(\cdot)$, and first considering for simplicity a stationary point process on the real line, there is an immediate analogue of (1.1) available if we describe the process in terms of the stationary sequence $\{Y_n\}$ of intervals between points of a realization described via its Palm distributions.

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PII S0304-4149(97)00045-8

Definition 1 (*Long range interval dependence*). A stationary point process $N(\cdot)$ on the real line exhibits long range interval dependence (LRiD) when the stationary sequence of intervals $\{Y_n\}$ determined by its Palm measure is LRD in the sense that

$$\limsup_{n\to\infty} \frac{\operatorname{var}(\sum_{i=1}^{n} Y_i)}{n} = \infty. \tag{1.2}$$

Within the context of 'standard' queueing models with a renewal arrival process and i.i.d. service times, we show in Section 2 that when the input process is renewal and therefore definitely not LRiD, we cannot induce LRiD into the output process via any sequence of independent service times.

Stationary point processes are more often described via their counting properties, with $N(A)$ = number of points in the Borel set A. If we were to attempt to mimic (1.1) or (1.2) we should look at introducing a skeleton process like $\{N(nh,(n+1)h]: n = 0, 1, \dots\}$ for some $h > 0$, i.e. the number of points in the half-open intervals $(nh, nh+h]$, and thence at the limit behaviour of $(\operatorname{var} N(0,nh])/n$. Such limit behaviour is the same as that of $V(x)/x$ where $V(x) \equiv \operatorname{var} N(0,x]$ denotes the variance function; this also points to a multi-dimensional extension as at Definition 2′ later.

Definition 2 (*Long range count dependence*). A second order stationary point process $N(\cdot)$ exhibits long range count dependence (LRcD) when its variance function $V(x) \equiv \operatorname{var} N(0,x]$ has

$$\limsup_{x\to\infty} \frac{V(x)}{x} = \infty. \tag{1.3}$$

We note that a renewal process, which can never be LRiD, can exhibit long range count dependence (e.g. Solo, 1995, or Section 4 below). We should not be surprised at this statement: it is simply another manifestation of the fact that count and interval properties of point processes are not identical, even though the probability distributions of stationary point processes with finite intensity are in one-one correspondence with the probability distributions of stationary processes $\{Y_n\}$ whose members are non-negative and have finite first moment (e.g. Section 12.3 of Daley and Vere-Jones, 1988).

Three illustrations of LRD properties of point processes shown in this paper concern the single-server queueing operator which is regarded as transforming one stationary point process, namely the arrival process of a queueing system, into another stationary point process, the output or departure process. We consider stable GI/GI/1 queues, i.e. arrivals occur at the epochs of a stationary renewal process with generic lifetime or inter-arrival time r.v. T and i.i.d. service times with generic r.v. S. We have already referred to work in Section 2 where we show that the renewal property of the inter-arrival times, which means that the input process cannot be LRiD, essentially precludes its introduction into the output process, in the sense that if the generic member D of the stationary inter-departure interval sequence $\{D_n\}$ has finite variance, then the sum $\sum_{i=1}^{n} \operatorname{cov}(D_0, D_i)$ converges to a finite limit as $n \to \infty$. We show in Section 5 that if the input process is LRcD then the output process may also be LRcD, and in Section 6 that when the input process is Poisson (and certainly not

LRcD), the output process can be LRcD if the sequence of service times, regarded as the lifetime r.v.s of a stationary renewal process, would constitute a LRcD point process.

Finally, in Section 7, we consider the infinite server queue with mean service times with finite first moment. For such a system, we show that the output is LRcD if and only if the input is LRcD. This system can be viewed as subjecting each point of a point process to independent translations, in which context there is the well known result (Doob, 1953, p. 405) that when the initial point process is a stationary Poisson process, so too is the randomly translated set of points.

Remark 1.1. Here is one way of extending the concept of long range count dependence to a point process in d-dimensional euclidean space $\mathbb{R}^d$. Call the sequence $\{A_n\}$ of Borel subsets of $\mathbb{R}^d$ a *convex averaging sequence* if (i) each A_n is convex; (ii) $A_n \subseteq A_{n+1}$ for $n = 1, 2, \ldots$; and (iii) $r(A_n) \to \infty$ as $n \to \infty$, where $r(A) = \sup\{r : A$ contains a ball of radius $r\}$ (Daley and Vere-Jones, 1988, Definition 10.2.1).

Definition 2′ (*Long range count dependence*). The stationary point process $N(\cdot)$ in $\mathbb{R}^d$ with finite second moment exhibits long range count dependence when for any convex averaging sequence $\{A_n\}$,

$$\limsup_{n \to \infty} \frac{\operatorname{var} N(A_n)}{\ell(A_n)} = \infty, \tag{1.4}$$

where $\ell(A)$ is the Lebesgue measure of the Borel set A.

Remark 1.2. The underlying motivation for these definitions of long range dependence stems from second order properties over a large range of indices. But when we consider the notion of possible dependence of points of a point process at large distances apart, we could as easily regard the proximity of points of a process to some lattice, as incorporating dependence between the locations of points at large distances apart. In the other direction, a point process of controlled variability, meaning, one for which for example $\operatorname{var} N(0, x]$ is uniformly bounded in x, may well be one for which dependence exists between the occurrence or not of points at large separation.

2. Second moment properties of inter-departure intervals of GI/GI/1

This section refines and complements some work in Daley (1968), referred to as D68, from which results are quoted freely. The essential conclusion to be drawn from our discussion is that when the arrival process of a single-server queue is renewal (and hence, cannot be LRiD), and the service times are i.i.d., the departure process similarly cannot be LRiD.

Start by recalling that the independent pair $\{S_n\}$ and $\{T_n\}$ ($n = 0, \pm 1, \pm 2, \ldots$) of doubly infinite sequences of i.i.d. non-negative r.v.s for which the generic members S and T satisfy $\mathrm{E}S < \mathrm{E}T < \infty$, define a stationary waiting time sequence $\{W_n\}$ whose members have the representation

$$W_n = \sup_{j \geqslant 0} \left\{ \sum_{i=1}^{j} (S_{n-i} - T_{n-i}) \right\}, \tag{2.1}$$

for which $\mathrm{E}W_n < \infty$ if and only if $\mathrm{E}S^2 < \infty$. Further, a stationary sequence $\{D_n\}$ of inter-departure intervals is defined by

$$D_n = T_n + W_{n+1} + S_{n+1} - W_n - S_n = S_{n+1} + (T_n - S_n - W_n)_+, \tag{2.2}$$

where the terms S_{n+1} and $(T_n - S_n - W_n)_+$ are independent. Denote generic members of the stationary waiting time and inter-departure processes by W and D, respectively.

It is immediately evident that the condition $\mathrm{E}S < \mathrm{E}T < \infty$ ensures that $\mathrm{E}D = \mathrm{E}T$, and that the product moments $\mathrm{E}(D_0 D_n) < \infty$ for $n = 1, 2, \ldots$ because

$$\begin{aligned} D_0 D_n &= [S_1 + (T_0 - W_0 - S_0)_+][S_{n+1} + (T_n - W_n - S_n)_+] \\ &\leqslant (S_1 + T_0)(S_{n+1} + T_n), \end{aligned} \tag{2.3}$$

which has finite expectation. Further, $\mathrm{E}D^2 < \infty$ if and only if both $\mathrm{E}S^2$ and $\mathrm{E}T^2$ are finite, and then by (2.2),

$$\operatorname{var} D = \operatorname{var} S + \operatorname{var}(T - S - W)_+. \tag{2.4}$$

In D68, under the stronger condition that $\mathrm{E}S^3 < \infty$ we showed that the result at (2.5) holds.

Theorem 1 (cf. Theorem 2 of D68). *In a stationary GI/GI/1 queue with* $\mathrm{E}S < \mathrm{E}T < \infty$, $\operatorname{cov}(D_0, D_n)$ *is well-defined and finite, while* $\operatorname{var} D < \infty$ *if and only if both* $\mathrm{E}S^2$ *and* $\mathrm{E}T^2$ *are finite, in which case the stationary inter-departure time sequence* $\{D_n\}$ *has*

$$\lim_{n \to \infty} \sum_{j=1}^{j} \operatorname{cov}(D_0, D_j) = \tfrac{1}{2}[\operatorname{var} T - \operatorname{var} D]. \tag{2.5}$$

Irrespective of the finiteness or not of $\mathrm{E}T^2$,

$$\lim_{n \to \infty} \sum_{j=1}^{j} \operatorname{cov}(D_0, D_j) = \mathrm{E}W(\mathrm{E}T - \mathrm{E}S) - \operatorname{var} S. \tag{2.6}$$

Proof. The finiteness properties of $\operatorname{cov}(D_0, D_n)$ and $\operatorname{var} D$ follow from (2.2)–(2.4) and the remarks there.

Assume for the moment that both $\mathrm{E}S^2 < \infty$ and $\mathrm{E}T^2 < \infty$. Recall from the known relation for the mean waiting time (e.g. Eq. (2.2b) of Daley et al., 1992) that $\operatorname{var}(T_0 - W_0 - S_0)_+ = \operatorname{var} S + \operatorname{var} T - 2\mathrm{E}W(\mathrm{E}T - \mathrm{E}S)$, so by (2.4) the right hand sides of (2.5) and (2.6) are equal.

To prove (2.6) under the weaker conditions, first replace the queueing system as described by one with the same inter-arrival time sequence but truncated service time sequence $\{S_n^K\} \equiv \{\min(S_n, K)\}$ for some (large) finite positive K, called the K-system for short; denote the waiting times and inter-departure times of the K-system by $\{W_n^K\}$

and $\{D_n^K\}$. We now show that equation (11) of D68, namely that

$$\sum_{j=1}^{n} \operatorname{cov}(D_0, D_j) = \tfrac{1}{2}[\operatorname{var} T - \operatorname{var} D_0] + \operatorname{cov}(T_0, W_{n+1})$$

$$+ \operatorname{cov}(W_0 + S_0, W_n - W_{n+1})$$

$$= \mathrm{E}W(\mathrm{E}T - \mathrm{E}S) - \operatorname{var} S + \operatorname{cov}(T_0, W_{n+1})$$

$$+ \operatorname{cov}(W_0 + S_0, W_n - W_{n+1}), \tag{2.7}$$

holds under the weaker conditions.

First observe that the K-system r.v.s converge weakly to the r.v.s of the original system as $K \to \infty$. Next, the covariance terms $\operatorname{cov}(D_0^K, D_n^K)$ converge because the products $D_0^K D_n^K$ are bounded by r.v.s with finite expectations much as at (2.3). The terms $\mathrm{E}W^K$ and $\operatorname{var} S^K$ converge by bounded convergence, and bounded convergence also ensures that $\operatorname{cov}(T_0, W_n^K)$ converges. Finally, since $|W_{n+1} - W_n| = |\max(S_n - T_n, -W_n)| \leqslant S_n + T_n$,

$$|(W_0^K + S_0^K)(W_n^K - W_{n+1}^K)| \leqslant (W_0^K + S_0^K)(S_n^K + T_n^K), \tag{2.8}$$

so bounded convergence also implies that the last term of (2.7) for the K-system converges. Thus, (2.7) holds under the weaker condition that $\mathrm{E}S^2 < \infty$.

A similar truncation argument applied to the T_n shows that (2.7) also holds under the weaker condition that $\mathrm{E}T^2$ need not be finite (but, of course, $\mathrm{E}S < \mathrm{E}T < \infty$ and $ES^2 < \infty$).

To demonstrate the convergence of the sum on the left-hand side of (2.6), it suffices to show that the covariance terms on the right-hand side of (2.7) converge to zero as $n \to \infty$. Recall that the transient waiting time sequence defined by arbitrary initial conditions and thereafter by the recurrence relation $W_{n+1} = (W_n + S_n - T_n)_+$ is a Markov chain that converges weakly to a unique weak limit when $\mathrm{E}S < \mathrm{E}T < \infty$. Then the term $\mathrm{E}(T_0 W_n)$ converges to $\mathrm{E}T\,\mathrm{E}W$ by bounded convergence as in D68. The last term at (2.7) equals

$$-\mathrm{E}[(W_0 + S_0)(W_{n+1} - W_n)] = -\mathrm{E}[(W_0 + S_0)\max(S_n - T_n, -W_n)]. \tag{2.9}$$

Recall that with $U_i = S_i - T_i$, W_n is expressible as

$$W_n = \max(0, U_{n-1}, U_{n-1} + U_{n-2}, \ldots, U_{n-1} + \cdots + U_1, U_{n-1} + \cdots + U_1 + U_0 + W_0). \tag{2.10}$$

Let $S' - T' \equiv U' =_d U_n$ be independent of all of the sequence $\{U_i\}$ and W_n. Then from $|W_n - W_{n+1}| \leqslant S_n + T_n$,

$$|\mathrm{E}[(W_0 + S_0)(W_{n+1} - W_n)]| \leqslant \mathrm{E}[(W_0 + S_0)(S_n + T_n)] = \mathrm{E}[(W_0 + S_0)(S' + T')].$$

Using the i.i.d. property of the $\{U_i\}$ (in fact, we are using the exchangeability property of the sequence rather than the full i.i.d. property), W_n is expressible as

$\max(0, U_1, U_1 + U_2, \ldots, U_1 + \cdots + U_n)$, and thus the right hand side of (2.9) equals

$$-\mathrm{E}[(W_0 + S_0)\max(U', -\max(0, U_1, U_1 + U_2, \ldots, U_1 + \cdots + U_{n-1},$$
$$U_1 + \cdots + U_{n-1} + U_0 + W_0)). \tag{2.11}$$

The ergodicity of an i.i.d. sequence like $\{U_i\}$ with $\mathrm{E}U_i < 0$ implies that $U_1 + \cdots + U_n \to -\infty$ a.s. as $n \to \infty$, so the inner maximum term at (2.11) converges pointwise to $\sup_{j \geqslant 0} \sum_{i=1}^{j} U_i \equiv W'$ say, this limit r.v. being independent of W_0, S_0 and U'. Then by bounded convergence we have

$$-\mathrm{E}[(W_0 + S_0)\max(U', -W_n)] \to -\mathrm{E}[(W_0 + S_0)\max(U', -W')]$$
$$= -\mathrm{E}[(W_0 + S_0)(\max(U' + W', 0) - W')]$$
$$= -\mathrm{E}[(W_0 + S_0)](\mathrm{E}(U' + W')_+ - \mathrm{E}W')] = 0. \tag{2.12}$$

(2.6) is proved under the weaker conditions.

Remark 2.1. The proof and result at (2.6) appear to tie the convergence of the infinite sum of covariances there to the finiteness of $\mathrm{E}W$, equivalently, $\mathrm{E}S^2$, though we have not demonstrated whether this is a necessary condition: this is a pointed comment because the finite sums in (2.5) and (2.6) do not require the finiteness of $\mathrm{E}S^2$. What we can show is that with $\mathrm{E}T^2$ finite but $\mathrm{E}S^2 = \infty$, the right-hand side of (2.6) equals $-\infty$; if $\mathrm{E}T^2 = \infty$ as well we have not been able to determine whether this last statement still holds.

Remark 2.2. If the mean waiting time is finite, the infinite sum of the covariances of interdeparture intervals is finite, even if the intervals have infinite variance. Thus, the output process of a stationary GI/GI/1 queue with $\mathrm{E}S^2$ and $\mathrm{E}T^2$ both finite, can never be LRiD.

Remark 2.3. For a single-server queue with stationary inter-arrival sequence $\{T_n\}$, not necessarily a renewal process, and i.i.d. service times $\{S_n\}$, with $0 < \mathrm{E}S < \mathrm{E}T < \infty$, Eqs. (2.1) and (2.2) continue to define stationary waiting time and inter-departure time sequences. Indeed, (2.2) extends to

$$T_0 + \cdots + T_{n-1} + W_n + S_n = W_0 + S_0 + D_0 + \cdots + D_{n-1}. \tag{2.13}$$

When

$$\mathrm{E}T^2 < \infty, \quad \mathrm{E}S^2 < \infty \quad \text{and} \quad \mathrm{E}W^2 < \infty, \tag{2.14}$$

the stationary sequence $\{D_n\}$ has finite second moment, and by taking second moments in (2.13) and using the Cauchy–Schwarz inequality, it is not difficult to prove that $\{D_n\}$ is LRiD if and only if $\{T_n\}$ is LRiD. In other words, the output of a stationary G/GI/1 queue satisfying the conditions (2.14) is LRiD if and only if the arrival process is LRiD.

3. Stationary point processes: the variance function and establishing LRcD

We turn now to questions of LRcD. We start by recalling that the covariance density function $c(\cdot)$, when it exists, is related to the variance function $V(\cdot)$ via

$$V(x) = mx + \int_{-x}^{x} (x - |u|)c(u)\,du \quad (x \geqslant 0), \tag{3.1}$$

where $m = EN(0, 1] \geqslant \lambda$ and λ is the intensity of $N(\cdot)$ (e.g. Section 10.4 of Daley and Vere-Jones, 1988). In the examples we consider here, $N(\cdot)$ is always orderly so $m = \lambda$. When $c(\cdot)$ exists, (3.1) implies that $c(x) = \frac{1}{2}V''(x)$ for $x > 0$.

For the examples of long range count dependence that we give, we use the Laplace–Stieltjes transform (L-ST) $v(\cdot)$ of $V(\cdot)$ defined by

$$v(s) = \int_0^\infty e^{-sx}\,dV(x) \quad (\mathrm{Re}(s) > 0); \tag{3.2}$$

$v(s)$ is well-defined on $\mathrm{Re}(s) > 0$ because $V(x)$ is of bounded variation on finite intervals and at most $O(x^2)$ for large x. It is a standard Abelian property of L-STs that for any $\gamma \geqslant 0$ and any constant C,

$$\limsup_{s\downarrow 0} |s^\gamma v(s) - C| \leqslant \limsup_{x\to\infty} \left| \frac{V(x)}{x^\gamma} \Gamma(1+\gamma) - C \right|, \tag{3.3}$$

where Γ denotes the gamma function (e.g. Widder, 1946, p. 181). Consequently,

$$\limsup_{s\downarrow 0} |sv(s)| = \infty \quad \text{implies} \quad \limsup_{x\to\infty} \frac{V(x)}{x} = \infty, \tag{3.4}$$

i.e. the point process with variance function V is LRcD.

We shall typically use distribution functions (d.f.s) that have regularly varying tails (see Section 4 and the Appendix), and deduce that $v(s) \sim As^{-c}(s\downarrow 0)$ for some positive A and $1 < c < 2$. If we also knew that $V(x)$ is ultimately monotonic then we should be able to use the stronger result of Karamata's Tauberian Theorem (e.g. Bingham et al., 1987, Theorems 1.7.1 and 1.7.6) to conclude essentially that $V(x) \sim A'x^c (x \to \infty)$.

$V(x)$ has the spectral representation

$$V(x) = \int_{\mathbb{R}} \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega} \right)^2 \Gamma_B(d\omega), \tag{3.5}$$

where $\Gamma_B(\cdot)$ is the *Bartlett spectrum*, a non-negative symmetric measure that is finite on bounded sets but not totally finite. For a Poisson process at rate λ, Γ_B equals $\lambda/2\pi$ times Lebesgue measure, while for a deterministic point process with inter-point distance equal to $1/\lambda$, Γ_B has mass λ^2 at each of the points $2\pi\lambda j$ ($j = \pm 1, \pm 2, \ldots$). Eq. (3.5) implies that the L-ST $v(s)$ satisfies

$$v(s) = \int_0^\infty e^{-sx}\,dV(x) = \int_{\mathbb{R}} \frac{2\Gamma_B(d\omega)}{s^2 + \omega^2} = 2s^{-2}\Gamma_B(\{0\}) + \int_{0+}^{\infty} \frac{4\Gamma_B(d\omega)}{s^2 + \omega^2}. \tag{3.6}$$

Example (*Cox process*). The variance function $V(\cdot)$ of a doubly stochastic Poisson or Cox process driven by a second-order stationary process or random measure with mean m and variance function $V_{\mathrm{D}}(\cdot)$, is given by

$$V(x) = V_{\mathrm{D}}(x) + mx \tag{3.7}$$

(see e.g. Exercise 11.2.2 of Daley and Vere-Jones, 1988), so it is LRcD if and only if $\limsup_{x\to\infty} V_{\mathrm{D}}(x)/x = \infty$, i.e. if and only if the driving process is LRD.

Note: The spectral measures G of Daley (1971) and $\Gamma_{\mathrm{B}} \equiv \Gamma$ in equation (11.2.3) of Daley and Vere-Jones (1988) are related by $G(\{0\}) = \Gamma_{\mathrm{B}}(\{0\})$, $G(A) = 2\Gamma_{\mathrm{B}}(A)$ for Borel $A \subset (0, \infty)$.

4. A renewal process can be LRcD

In this section we consider a renewal process for which the d.f. $F(\cdot)$ of the lifetime r.v.s has finite mean $\mu = 1/\lambda$ and a regularly varying tail of order $-c$ where $1 < c < 2$, i.e.

$$1 - F(x) = x^{-c}L(x) \quad (x > 0), \tag{4.1}$$

where $L(x)$ is a function slowly varying at infinity (SV function); see the Appendix. Renewal theorems for $F(\cdot)$, $1 < c < 2$ were obtained by Teugels (1968) who also examined the infinite mean case $0 < c \leqslant 1$ but this will not be considered here. Teugels showed that the renewal function satisfies

$$H(x) - x/\mu \sim \frac{x^{2-c}L(x)}{\mu^2(c-1)(2-c)} \quad (x \to \infty), \tag{4.2}$$

so by (4.4) and Proposition 1.5.8 of Bingham et al. (1987), the variance function satisfies

$$V(x) \sim \frac{2x^{3-c}L(x)}{\mu^3(3-c)(2-c)(c-1)} \quad (x \to \infty). \tag{4.3}$$

Thus, $\lim_{x\to\infty} V(x)/x = \infty$, as required for LRcD at (1.3). References to and applications of Teugels' results are also in Bingham et al. (1987) (see Section 8.6 and references therein) and in Solo (1995) who considered the particular case

$$1 - F(x) \sim d^c x^{-c} \quad (x \to \infty)$$

for some c in $1 < c < 2$; he showed by differentiation of (4.2) that the covariance density satisfies

$$c(x) \sim d^c x^{1-c}/[\mu^3(c-1)],$$

as is consistent with (4.3). Our Theorem 2 below restates Teugels' result (4.3) but gives another proof using the LS-T and (3.4) instead, because we want the intermediate result at (4.7) for use in Sections 5 and 6.

Theorem 2. *A stationary renewal process for which the lifetime d.f. F has a regularly varying tail as at (4.1) for some $1 < c < 2$, is LRcD.*

Proof. Let V denote the variance function of a stationary renewal process whose lifetime d.f. F has mean $1/\lambda$ and $F(0+) = 0$. Then

$$V(x) = \lambda x + 2\int_0^x [H(x) - \lambda x]\,\mathrm{d}x, \tag{4.4}$$

where H is the zero-deleted renewal function, and the L-ST v of V is given by

$$\tfrac{1}{2}(sv(s) - \lambda) = \frac{\varphi(s)}{1 - \varphi(s)} - \frac{\lambda}{s}, \tag{4.5}$$

where $\varphi(\cdot)$ is the L-ST of F. For a d.f. F satisfying (4.1), Lemma A.1 shows that its L-ST φ is expressible

$$\varphi(s) = 1 - s/\lambda + A_c s^c L(1/s)[1 + \mathrm{o}(1)] \quad (s\downarrow 0), \tag{4.6}$$

where $A_c = \Gamma(2 - c)/(c - 1)$; substitution in (4.3) gives

$$\begin{aligned}\tfrac{1}{2}(sv(s) - \lambda) &= \frac{1 - s/\lambda + A_c s^c L(1/s)}{s/\lambda - A_c s^c L(1/s)} - \frac{\lambda}{s}\\ &= \frac{\lambda}{s}[[1 - s/\lambda + \mathrm{o}(s)](1 + \lambda A_c s^{c-1} L(1/s)[1 + \mathrm{o}(1)]) - 1]\\ &= \frac{\lambda}{s}[\lambda A_c s^{c-1} L(1/s)[1 + \mathrm{o}(1)]].\end{aligned}$$

Thus $sv(s) = \mathrm{O}(s^{c-2}L(1/s))\,(s\downarrow 0)$, so $|sv(s)| \to \infty$ as $s\downarrow 0$, which suffices by (3.4) to prove the theorem.

Remark 4.1. We note for later use that the essence of this proof is the demonstration that

$$\frac{s\varphi(s)/\lambda}{1 - \varphi(s)} - 1 = \lambda A_c s^{c-1} L(1/s)[1 + \mathrm{o}(1)] \quad (s\downarrow 0). \tag{4.7}$$

We could have deduced this via properties of the L-ST of the SV property at (4.3).

5. The output of GI/M/1 can be LRcD

Theorem 3. *If the renewal input process of a stationary GI/M/1 queue has generic inter-arrival time T for which*

$$\Pr\{T > x\} = x^{-c}L(x) \quad (x > 0) \tag{5.1}$$

for some c in $1 < c < 2$ and $L(\cdot)$ a SV function, then the output process of the queue is LRcD.

Proof. Daley (1976) showed that the variance function $V(\cdot)$ of the output counting process of a GI/M/1 queue has L-ST $v(s) = \int_0^\infty e^{-sx}\,dV(x)$ given by

$$\tfrac{1}{2}s(sv(s) - \lambda) = \lambda\mu(\delta - \rho) + \frac{\lambda\mu^2(1-\delta)[1-w(s)][\mu\delta(1-\alpha(s)) - s\alpha(s)]}{[s+\mu(1-w(s))][s-\mu(1-\delta)](1-\alpha(s))} \tag{5.2}$$

$$= \lambda\mu(\delta-\rho) + \lambda\mu^2\frac{(1-\delta)[1-w(s)]}{[s+\mu(1-w(s))][s-\mu(1-\delta)]}\left[\mu\delta - \frac{s\alpha(s)}{1-\alpha(s)}\right],$$

where $\alpha(s)$ is the L-ST of the interarrival times, δ is the root in $(0, 1)$ of $\delta = \alpha(\mu[1-\delta])$, μ is the service rate and $w(s)$ is the solution of the equation

$$w(s) = \alpha(s + \mu[1 - w(s)]) \tag{5.3}$$

satisfying $w(s) \to \delta(s\downarrow 0)$. This function $w(s)$ is the particular case $\delta(s, 1)$ of the general solution $z = \delta(s, w)$ of the equation $z = w\alpha(s + \mu[1 - z])$ as in Takacs (1962, p. 113). According to Takacs' Lemma 1 there, $w(s)$ is the unique root satisfying $|w(s)| < 1$ for $\mathrm{Re}(s) \geqslant 0$ because the queue GI/M/1 is assumed to be stable. A power series solution in terms of s is possible (because $\alpha(\cdot)$ is analytic within the right-half plane), so $w'(0+)$ certainly exists, with

$$w'(0+) = \alpha'(\mu[1-\delta])(1 - \mu w'(0+)) = \frac{\alpha'(\mu[1-\delta])}{1 + \mu\alpha'(\mu[1-\delta])}, \tag{5.4}$$

subject of course to $1 \neq -\mu\alpha'(\mu[1-\delta])$, which holds because otherwise the solution is not unique.

Refer to (5.2), and consider first the term

$$\frac{(1-\delta)[1-w(s)]}{(s+\mu[1-w(s)])(s-\mu[1-\delta])}. \tag{5.5}$$

For $s \to 0$ this converges to $-1/\mu^2$. In more detail, because all terms have Taylor series expansions around $s = 0$, the expression equals $-\mu^{-2}(1 + Cs + o(s))$ for some finite constant C. When the inter-arrival time d.f. satisfies (5.1) and has mean $1/\lambda$, it follows that its L-ST $\alpha(s)$ has an expansion as at (4.6), so we can use (4.7) and deduce that the term $[\cdots]$ at (5.2) is given by

$$\mu\delta - \frac{s\alpha(s)}{1-\alpha(s)} = \mu\delta - \lambda(1 + \lambda A_c s^{c-1}L(1/s)[1 + o(1)]). \tag{5.6}$$

Combining these results for the components of (5.2) gives

$$\begin{aligned}\tfrac{1}{2}s(sv(s) - \lambda) &= \lambda\mu(\delta - \rho) \\ &\quad - \lambda(1 + Cs + o(s))[\mu\delta - \lambda(1 + \lambda A_c s^{c-1}L(1/s)[1 + o(1)])] \\ &= \lambda^3 A_c s^{c-1}L(1/s)[1 + o(1)] \quad (s\downarrow 0).\end{aligned}$$

Thus,

$$sv(s) = 2\lambda^3 A_c s^{c-2} L(1/s)[1 + o(1)] \quad (s \downarrow 0). \tag{5.7}$$

Letting $s \to 0$ and invoking (3.4), completes the proof of Theorem 3.

6. The output of M/G/1 can be LRcD

Theorem 4. *If a stationary M/G/1 queue has generic service time S for which*

$$\Pr\{S > x\} = x^{-c} L(x) \quad (x > 0) \tag{6.1}$$

for some c in $1 < c < 2$ and $L(\cdot)$ a SV function, then the output process of the queue is LRcD.

Remark 6.1. Throughout this section L denotes the same SV function.

Proof. Daley (1976) showed that the variance function $V(\cdot)$ of the output counting process of a M/G/1 queue has L-ST v given by

$$\tfrac{1}{2}s(sv(s) - \lambda) = -\lambda^2 + \frac{\lambda s \beta(s)}{1 - \beta(s)}\left[1 - \frac{s\Pi(w(s))}{s + \lambda(1 - w(s))}\right], \tag{6.2}$$

where $\beta(s)$ is the L-ST of the service times, λ is the arrival rate, $w(s)$ is now the solution of the equation

$$w(s) = \beta(s + \lambda[1 - w(s)]) \tag{6.3}$$

satisfying $w(s) \to 1$ $(s \to 0)$ and $\Pi(\cdot)$ is the probability generating function for the stationary distribution of the embedded Markov chain of the queue length just after a departure, so

$$\Pi(z) = \frac{(1-\rho)(1-z)\beta(\lambda[1-z])}{\beta(\lambda[1-z]) - z} = \frac{\pi_0 \zeta \beta(\lambda\zeta)}{\beta(\lambda\zeta) - 1 + \zeta} \quad (|z| < 1) \tag{6.4}$$

(e.g. Takacs, 1962, Eq. (67) p. 72), where $\pi_0 = 1 - \rho$, $\zeta = 1 - z$ and $\rho = \lambda/\mu$, with $1/\mu$ the mean service time.

From (6.1) and Lemma A.1 the L-ST $\beta(s)$ has the expansion

$$\beta(s) = 1 + as + A_c s^c L(1/s)[1 + o(1)], \tag{6.5}$$

where $a = -1/\mu$ and A_c as before. Appealing to (4.7) with β in place of φ, the factor outside $[\cdots]$ in (6.2) becomes

$$\frac{\lambda s \beta(s)}{1 - \beta(s)} = \lambda\mu(1 + \mu A_c s^{c-1} L(1/s)[1 + o(1)]). \tag{6.6}$$

Write the term inside $[\cdots]$ in (6.2) as

$$1 - \frac{s\Pi(w(s))}{s + \lambda[1 - w(s)]} = 1 - \frac{s}{s + \lambda\zeta}\left[\frac{\pi_0 \zeta \beta(\lambda\zeta)}{\beta(\lambda\zeta) - 1 + \zeta}\right], \tag{6.7}$$

where now $\zeta = 1 - w(s)$. Lemma A.2 shows that the function $w(\cdot)$ defined implicitly at (6.3) via the function $\beta(s)$ which has the expansion at (6.5), has the expansion

$$w(s) = 1 + As + BA_c s^c L(1/s)[1 + \mathrm{o}(1)] = 1 + As + B's^c L(1/s)[1 + \mathrm{o}(1)], \tag{6.8}$$

where $A = 1/(\lambda - \mu) = (-1/\mu)/(1 - \rho)$, $B = (1 - \rho)^{-c-1}$ and $B' = BA_c$, so

$$\begin{aligned} \zeta = 1 - w(s) &= -s(A + B's^{c-1}L(1/s)[1 + \mathrm{o}(1)]) \\ &= |A|s(1 - (B'/|A|)s^{c-1}L(1/s)[1 + \mathrm{o}(1)]). \end{aligned} \tag{6.9}$$

Using (6.9) in the iterm outside $[\cdots]$ in the right-hand side of (6.7) gives

$$\begin{aligned} \frac{s}{s + \lambda\zeta} &= \frac{1}{1 - \lambda A - \lambda B's^{c-1}L(1/s)[1 + \mathrm{o}(1)])} \\ &= \frac{1 + \lambda B'(1 - \lambda A)^{-1}s^{c-1}L(1/s)[1 + \mathrm{o}(1)]}{1 - \lambda A} \\ &= (1 - \rho)(1 + \lambda B'(1 - \rho)s^{c-1}L(1/s)[1 + \mathrm{o}(1)]), \end{aligned} \tag{6.10}$$

since $1 - \lambda A = 1 + (\lambda/\mu)/(1 - \rho) = 1/(1 - \rho)$. Next use the expansion of β at (6.5) in the term $[\cdots]$ at (6.7) to give

$$\begin{aligned} \frac{\pi_0 \zeta \beta(\lambda\zeta)}{\beta(\lambda\zeta) - 1 + \zeta} &= \frac{\pi_0 \zeta \beta(\lambda\zeta)}{\zeta + a\lambda\zeta + (\lambda\zeta)^c A_c L(1/\lambda\zeta)[1 + \mathrm{o}(1)]} \\ &= \frac{\pi_0 \beta(\lambda\zeta)}{1 + \lambda a + \lambda A_c (\lambda\zeta)^{c-1} L(1/(\lambda\zeta))[1 + \mathrm{o}(1)]} \\ &= \frac{\pi_0}{1 + \lambda a}[1 - \lambda A_c (1 + \lambda a)^{-1}(\lambda\zeta)^{c-1}L(1/(\lambda\zeta))[1 + \mathrm{o}(1)]] \\ &= 1 - \frac{\lambda A_c (\lambda\zeta)^{c-1} L(1/(\lambda\zeta))[1 + \mathrm{o}(1)]}{1 - \rho}, \end{aligned} \tag{6.11}$$

since $1 + \lambda a = 1 - \rho = \pi_0$. Because L is a SV function and $\lambda\zeta/s \to \lambda|A| = \rho/(1 - \rho)$ for $s \downarrow 0$, (6.11) gives

$$\begin{aligned} \frac{\pi_0 \zeta \beta(\lambda\zeta)}{\beta(\lambda\zeta) - 1 + \zeta} &= 1 - \frac{\lambda A_c (\lambda|A|s)^{c-1} L(1/s)[1 + \mathrm{o}(1)]}{1 - \rho} \\ &= 1 - \frac{\lambda A_c \rho^{c-1} s^{c-1} L(1/s)[1 + \mathrm{o}(1)]}{(1 - \rho)^c}. \end{aligned} \tag{6.12}$$

Substituting (6.10) and (6.12) in (6.7) gives

$$1 - \frac{s\Pi(w(s))}{s + \lambda[1 - w(s)]}$$

$$= 1 - (1 - \rho)(1 + \lambda B'(1 - \rho)s^{c-1}L(1/s)[1 + o(1)])$$

$$\times \left(1 - \frac{\lambda A_c \rho^{c-1} s^{c-1} L(1/s)[1 + o(1)]}{(1 - \rho)^c}\right)$$

$$= \rho + \left(\frac{\lambda A_c \rho^{c-1}}{(1 - \rho)^{c-1}} - (1 - \rho)^2 \lambda B'\right) s^{c-1} L(1/s)[1 + o(1)]$$

$$= \rho - \frac{(1 - \rho^{c-1})\lambda A_c s^{c-1} L(1/s)[1 + o(1)]}{(1 - \rho)^{c-1}}. \tag{6.13}$$

Combining (6.13) with (6.6) as in (6.2), gives

$$\tfrac{1}{2}s(sv(s) - \lambda)$$

$$= -\lambda^2 + \lambda\mu(1 + \mu A_c s^{c-1} L(1/s)[1 + o(1)])$$

$$\times \left[\rho - \frac{(1 - \rho^{c-1})\lambda A_c s^{c-1} L(1/s)[1 + o(1)]}{(1 - \rho)^{c-1}}\right]$$

$$= \lambda^2 \mu A_c \left[1 - \frac{1 - \rho^{c-1}}{(1 - \rho)^{c-1}}\right] s^{c-1} L(1/s)[1 + o(1)]. \tag{6.14}$$

Consequently, since the multiplier on the right-hand side here is non-zero, $sv(s) = O(s^{c-2}L(1/s))$ as in Section 5, and $|s(v(s)| \to \infty$ as $s \downarrow 0$. Theorem 4 is proved.

7. Output of G/GI/∞

Recall that the point process occurring as the output of an infinite server queue is the same as occurs when the points of the input process are subject to independent translation, the translations being the service times. Suppose given an orderly stationary point process with finite variance function V, Bartlett spectrum Γ_B and intensity λ, and let each point be subject to independent translation with a common distribution that has a finite first moment. Let S denote a generic translation r.v., with $E(e^{i\omega S}) = \beta(\omega)$. Then from Daley (1971) (or, exercise 11.2.4 of Daley and Vere-Jones, 1988), the translated point process has variance function V_1 say with Bartlett spectrum Γ_{B1} say given by

$$\Gamma_{B1}(d\omega) = |\beta(\omega)|^2 \Gamma_B(d\omega) + (\lambda/2\pi)(1 - |\beta(\omega)|^2)\, d\omega. \tag{7.1}$$

Then from the spectral representation at (3.5) we have

$$V_1(x) = \int_{\mathbb{R}} \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 |\beta(\omega)|^2 \Gamma_B(\mathrm{d}\omega) + \lambda \int_{\mathbb{R}} \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 \frac{1-|\beta(\omega)|^2}{2\pi}\,\mathrm{d}\omega \tag{7.2}$$

$$\leqslant \int_{\mathbb{R}} \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 \Gamma_B(\mathrm{d}\omega) + \lambda \int_{\mathbb{R}} \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 \frac{\mathrm{d}\omega}{2\pi}$$

$$= V(x) + \lambda x. \tag{7.3}$$

This proves the sufficiency part of the next result.

Theorem 5. *Let the orderly stationary point process N with finite variance function and Bartlett spectrum Γ_B be the input process of a G/GI/∞ queue with generic service time S that has finite first moment, and let N_1 denote the output process. Then N_1 is LRcD if and only if N is LRcD.*

Proof. If $\Gamma_B(\{0\}) > 0$ then $\Gamma_{B1}(\{0\}) > 0$ also, and both N and N_1 are certainly LRcD. Assume below that $\Gamma_B(\{0\}) = 0$. Then it follows from (7.2) that

$$V_1(x) \geqslant \int_{\mathbb{R}} \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 |\beta(\omega)|^2 \Gamma_B(\mathrm{d}\omega) = 2\left[\int_0^\delta + \int_\delta^\infty\right] \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 |\beta(\omega)|^2 \Gamma_B(\mathrm{d}\omega), \tag{7.4}$$

for any positive finite δ. Here the integral on (δ, ∞) is dominated by

$$8 \int_\delta^\infty \frac{\Gamma_B(\mathrm{d}\omega)}{\omega^2},$$

which is finite for such δ and independent of x. Consequently,

$$\limsup_{x\to\infty} \frac{V_1(x)}{x} = \infty \quad \text{if and only if} \quad \limsup_{x\to\infty} \frac{1}{x}\int_0^\delta \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 |\beta(\omega)|^2 \Gamma_B(\mathrm{d}\omega) = \infty. \tag{7.5}$$

Also, applying a similar decomposition to (3.5),

$$\limsup_{x\to\infty} \frac{V(x)}{x} = \infty \quad \text{if and only if} \quad \limsup_{x\to\infty} \frac{1}{x}\int_0^\delta \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 \Gamma_B(\mathrm{d}\omega) = \infty. \tag{7.6}$$

Now $|\beta(\omega)|^2$ is continuous in ω, and equal to 1 at $\omega = 0$, so we can choose δ such that $|\beta(\omega)|^2 > \frac{1}{2}$ for $|\omega| < \delta$. Then for such δ,

$$\int_0^\delta \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 \Gamma_B(\mathrm{d}\omega) \geqslant \int_0^\delta \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 |\beta(\omega)|^2 \Gamma_B(\mathrm{d}\omega) \geqslant \frac{1}{2}\int_0^\delta \left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega}\right)^2 \Gamma_B(\mathrm{d}\omega).$$

Appealing to (7.5) and (7.6) proves the converse of the theorem.

Remark 7.1. The same technique of proof extends to a cluster point process $N(\cdot)$ (or, random measure) with cluster centre process $N_c(\cdot)$ and independent clusters $n_i(\cdot)$ (e.g.

Section 8.2 of Daley and Vere-Jones, 1988), so

$$N(A) = \sum_{\{t_i\} \in N_c(\cdot)} n_i(A - t_i), \tag{7.7}$$

and shows that, when $\mathrm{E}[(n(\mathbb{R}))^2] < \infty$, $N(\cdot)$ is LRcD if and only if $N_c(\cdot)$ is LRcD.

Appendix A. Some results involving slowly varying functions

In this appendix we record two results that involve *slowly varying (SV) functions*; strictly speaking, we use the term SV function to mean a function $L(x)$ that is slowly varying at infinity:

$$\lim_{x \to \infty} L(tx)/L(x) = 1 \quad \text{(every finite } t > 0). \tag{A.1}$$

Then a function $f(x)$ is regularly varying of order c if $f(x) = x^c L(x)$ for some SV function L.

The force of the first lemma is that the Laplace–Stieltjes transform of a d.f. with regularly varying tail of order $-c$ has a power series expansion involving moments of integer order smaller than c and a remainder term that involves a SV function.

Lemma A.1. *Suppose the d.f. F on $(0, \infty)$ has mean $1/\lambda$ and that*

$$1 - F(x) = x^{-c} L(x), \tag{A.2}$$

where $1 < c < 2$ and L is a SV function. Then the L-ST φ of F has the expansion

$$\varphi(s) = \int_0^\infty \mathrm{e}^{-sx} \mathrm{d}F(x) = 1 - \frac{s}{\lambda} + A_c s^c L(1/s)[1 + \mathrm{o}(1)] \quad (s \downarrow 0), \tag{A.3}$$

where $A_c = \Gamma(2 - c)/(c - 1)$.

Proof. Using $\mathrm{d}F(x) = -\mathrm{d}[1 - F(x)]$ and integrating by parts twice gives

$$\varphi(s) = 1 - s\int_0^\infty [1 - F(x)]\,\mathrm{d}x + s^2 \int_0^\infty \mathrm{e}^{-sx}\left[\int_x^\infty [1 - F(u)]\,\mathrm{d}u\right]\mathrm{d}x \quad (\mathrm{Re}(s) > 0). \tag{A.4}$$

We use some standard properties of SV functions to deduce the behaviour of the last term: specifically, we refer to Propositions 1.5.10 and 1.5.8, and Theorem 1.7.1 of Bingham et al. (1987). When $1 - F$ satisfies (A.2), Proposition 1.5.10 implies that

$$\int_y^\infty [1 - F(u)]\,\mathrm{d}u = \int_y^\infty x^{-c} L(x)\,\mathrm{d}x \equiv \frac{y^{-(c-1)} L_1(y)}{c - 1} \sim \frac{y^{-(c-1)} L(y)}{c - 1} \quad (y \to \infty).$$

Then Proposition 1.5.8 implies that

$$U_1(x) \equiv \int_0^x y^{-(c-1)} L_1(y)\,\mathrm{d}y \sim \frac{x^{2-c}L_1(x)}{2-c} \sim \frac{x^{2-c}L(x)}{(2-c)(c-1)} \quad (x \to \infty).$$

Finally, Karamata's Tauberian Theorem (Theorem 1.7.1) implies that

$$\int_0^\infty \mathrm{e}^{-sx}\,\mathrm{d}U_1(x) \sim \frac{s^{-(2-c)}L(1/s)\Gamma(3-c)}{(2-c)(c-1)} = A_c s^{-(2-c)} L(1/s) \quad (s \downarrow 0),$$

so substitution in (A.4) gives for the last term

$$s^2 \int_0^\infty \mathrm{e}^{-sx} \left[\int_x^\infty [1 - F(u)]\,\mathrm{d}u \right] \mathrm{d}x = A_c s^c L(1/s)[1 + \mathrm{o}(1)] \quad (s \downarrow 0),$$

as asserted in (A.2).

The other result we need is similarly a SV function analogue of a result known under more stringent conditions from work of Takacs (e.g. Takacs, 1962, p. 47 and p. 113).

Lemma A.2 (Implicit function of a SV function). *Suppose that the function $f(x)$, defined for $x \geqslant 0$, has the expansion*

$$f(x) = 1 + ax + x^c L(1/x) \quad (x \downarrow 0), \tag{A.5}$$

where $1 < c < 2$ and $L(\cdot)$ is a slowly varying function. If $\lambda a \neq -1$ then any solution $z = z(x)$ of the equation

$$z = f(x + \lambda[1 - z]) \tag{A.6}$$

satisfying

$$z(x) \to 1 = f(0) \quad (x \downarrow 0), \tag{A.7}$$

is of the form, for all sufficiently small $x > 0$,

$$z = 1 + Ax + Bx^c L(1/x)[1 + \mathrm{o}(1)] \quad (x \downarrow 0), \tag{A.8}$$

where $A = a/(1 + \lambda a)$ and $B = 1/(1 + \lambda a)^{c+1}$.

Proof. Suppose that $z \equiv z(x)$ satisfies (A.6) and (A.7). Set $Z = z - 1$, and consider first the case that $a > 0$. Then, provided that $x > \lambda Z$,

$$Z = z - 1 = f(x - \lambda Z) - 1 = (x - \lambda Z)a + (x - \lambda Z)^c L\left(\frac{1}{x - \lambda Z}\right), \tag{A.9}$$

where c and $L(\cdot)$ are as in (A.5). Thus,

$$\text{provided} \quad x \neq \lambda Z \quad (x \downarrow 0,\ x \neq 0), \tag{A.10}$$

we can divide by $x - \lambda Z$ and write

$$\frac{Z}{x - \lambda Z} = a + (x - \lambda Z)^{c-1} L\left(\frac{1}{x - \lambda Z}\right) = a + \mathrm{o}(1) \quad (x \downarrow 0).$$

Hence

$$\frac{x}{Z} = \lambda + \frac{1}{a + \mathrm{o}(1)},$$

i.e. since $\lambda a \neq -1$,

$$Z = \frac{xa}{1 + \lambda a}(1 + \mathrm{o}(1)) = Ax(1 + \mathrm{o}(1)). \tag{A.11}$$

Such a function satisfies condition (A.10), because if for any $x \neq 0$ we have $x = \lambda Z$, then $z = f(0) = 1$, i.e. $Z = 0 = x$, contradicting $x \neq 0$.

The case $a < 0$ yields $Z < 0$ and $x > \lambda Z$ holds a fortiori, and (A.11) holds also.

Rewrite (A.9) as

$$(1 + \lambda a)Z = ax + (x - \lambda Z)^c L\left(\frac{1}{x - \lambda Z}\right). \tag{A.12}$$

Using (A.11) in the last term here yields

$$(x - \lambda Z)^c = x^c(1 - \lambda A[1 + \mathrm{o}(1)])^c = x^c(1 - \lambda A)^c(1 + \mathrm{o}(1)) = \frac{x^c}{(1 + \lambda a)^c}(1 + \mathrm{o}(1))$$

and

$$L\left(\frac{1}{x - \lambda Z}\right) = L\left(\frac{1}{(1 - \lambda A)[1 + \mathrm{o}(1)]x}\right) = L(1/x)(1 + \mathrm{o}(1)) \quad (x \downarrow 0),$$

because $L(\cdot)$ is slowly varying. Combining this with (A.12) yields (A.8) as asserted.

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