Existence and Multiplicity of Traveling Waves in a Lattice Dynamical System

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This work proves the existence and multiplicity results of monotonic traveling wave solutions for some lattice differential equations by using the monotone iteration method. Our results include the model of cellular neural networks (CNN). In addition to the monotonic traveling wave solutions, non-monotonic and oscillating traveling wave solutions in the delay type of CNN are also obtained.   © 2000 Academic Press

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I. INTRODUCTION

This work studies the existence and multiplicity of traveling wave solutions of lattice differential equations. As generally considered, lattice differential equations are infinite systems of ordinary differential equations on a spatial lattice, such as the *D*-dimensional integer lattice $\mathbf{Z}^D$. Lattice differential equations arise from many system models such as in chemical reaction theory [14, 24], biology [2, 3], image processing and pattern recognition [11, 13], and material science [4].

An underlying motivation for studying the lattice differential equations is the large array of a locally coupled first-order nonlinear dynamical system, i.e., *Cellular Neural Networks* (CNN). Proposed by Chua and Yang [12, 13], such an information processing system is occasionally referred to as CY-CNN.

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Indeed, cellular neural networks (CNN) without input terms are of the form

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + z + \sum_{|k| \leq d, |l| \leq d} a_{k,l} f(x_{i+k, j+l}) \qquad (i, j) \in \mathbf{Z}^2 \quad (1.1)$$

or

$$\frac{dx_i}{dt} = -x_i + z + \sum_{|l| \leq d} a_l f(x_{i+l}) \qquad i \in \mathbf{Z}^1. \quad (1.2)$$

Here the nonlinearity f is an output function and a piecewise-linear function in CY-CNN. The quantity z is called a threshold or bias term and the numbers $a_{k,l}$ can be arranged into the $(2d+1) \times (2d+1)$ matrix A which is called a space-invariant template.

The study of traveling wave solutions can proceed as follows. Let $\theta \in \mathbf{R}^1$ be given and consider solutions of (1.1) or (1.2) of the form

$$x_{i,j}(t) = x(i \cos \theta + j \sin \theta - ct) \qquad \text{or} \quad x_i(t) = x(i - ct) \quad (1.3)$$

for some unknown function $x: \mathbf{R}^1 \to \mathbf{R}^1$ and some unknown real number c. A solution of the form (1.3) of system (1.1) or (1.2) is called a traveling wave solution of (1.1) (or 1.2). By denoting $s = i \cos \theta + j \sin \theta - ct$ (or $s = i - ct$), x and c satisfy the equation of the form

$$-cx'(s) = G(x(s + r_0), x(s + r_1), ..., x(s + r_N)), \quad (1.4)$$

where $r_0 = 0$, r_i are real numbers for $i = 1$ to N.

If Eq. (1.4) depends on the past and future, i.e., if

$$r_{\min} \equiv \min\{r_i\}_{i=0}^N < 0 < r_{\max} \equiv \max\{r_i\}_{i=0}^N, \quad (1.5)$$

then (1.5) is called a mixed type. If $r_{\min} = 0$ or $r_{\max} = 0$, then (1.4) is called an advance or delay type, respectively.

Previous studies [5, 34] have numerically observed traveling wave solutions and mathematically proven them [1, 6, 26–28] in the case of the discrete reaction–diffusion equation. Chow, Mallet-Paret, and Shen [9] studied the existence and stability of traveling wave solutions in lattice dynamical systems. In a related study Mallet-Paret [28] confirmed the existence and uniqueness of a traveling wave which connects the two stable states in bistable systems. Previous investigations of [28] and [35] provide the basis for this study. Indeed, define $\Phi(x)$ by

$$\Phi(x) = G(x, x, ..., x), \quad (1.6)$$

and assume $x^0 < x^+$ such that

$$\Phi(x^0) = \Phi(x^+) = 0 \quad \text{and} \quad \Phi(x) > 0 \qquad \text{for} \quad x \in (x^0, x^+). \qquad (1.7)$$

By assuming that G is quasi-monotone, i.e., $G(u_0, u_1, ..., u_N)$ is strictly increasing in u_j for $1 \leqslant j \leqslant N$, Wu and Zou [35] verified that a family of monotone traveling wave solutions of (1.4) satisfies the boundary conditions

$$\lim_{s \to -\infty} x(s) = x^0 \qquad \text{and} \qquad \lim_{s \to \infty} x(s) = x^+. \qquad (1.8)$$

A monotonic iteration scheme is employed in [35]. Under certain conditions on G, Wu and Zou constructed upper and lower solutions of (1.4), thereby satisfying the boundary conditions in (1.8).

In this work, we first generalize the results in [35]. Indeed, we denote the characteristic equation of (1.4) at $\bar{x}$ by

$$\Delta(\sigma, c, \bar{x}) = -c\sigma - \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(\bar{x}) \, e^{\sigma r_j}. \qquad (1.9)$$

The assumptions needed for this mixed type problem are:

(G.1)   Assume that $\sum_{j=0}^{N}(\partial G/\partial u_j)(x^0) > 0$.

(G.2)   Assume that G is quasi-monotonic for $u_j, j \geqslant 1$, in $[x^0, x^+]^{N+1}$, i.e.,

$$\frac{\partial G}{\partial u_j}(u) > 0, \qquad \text{for} \quad u \in [x^0, x^+]^{N+1}, \quad j \geqslant 1.$$

(G.3)   Assume that

$$G(u) \leqslant \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(x^0)(u_j - x^0) \qquad \text{for} \quad u \in [x^0, x^+]^{N+1}.$$

The first main results are

THEOREM 1.1.   *By assuming that $r_{\min} < 0 < r_{\max}$ with* (G.1), (G.2), *and* (G.3) *being held, $c^* \leqslant 0$ exists such that for any $c < c^*$ Eq.* (1.4) *has a nondecreasing solution satisfying the boundary conditions* (1.8). *Herein, c^* satisfies*

$$\Delta(\sigma^*, c^*, x^0) = 0 \qquad and \qquad \Delta'(\sigma^*, c^*, x^0) = 0 \qquad (1.10)$$

for some $\sigma^* > 0$, where "'" denotes the partial derivative of $\Delta(\sigma, c, x^0)$ with respect to σ. Moreover, (σ^*, c^*) is a unique solution of (1.10).

With Condition (G.1) we can construct lower solutions. Condition (G.2) ensures the validity of the monotone iteration scheme. Condition (G.3), a global sublinearity of G at x^0, allows us to construct an upper solution. Notably, the condition (G.3) is much weaker than that in [35]. Condition (G.3) holds in many models, such as in the reaction–diffusion equation and CNN.

In Theorem 1.1, c^* is the critical velocity which verifies the existence of a monotonic traveling wave connecting x^0 and x^+. Furthermore, in the delay case, our results indicate that (G.3) is redundant. Indeed, the following results are obtained.

THEOREM 1.2. *Assume* (G.1), (G.2), *and that* $r_{\max} = 0$. *Then, for any* $c \leqslant 0$, *a non-decreasing solution of* (1.4) *satisfies the boundary conditions* (1.8).

All general results of Theorems 1.1 and 1.2 can be applied to CNN, enabling us to obtain monotone traveling waves. However, owing to the simplicity of the piecewise-linear nonlinearity of CY-CNN, the solutions can be obtained explicitly in the case of the delay or advance type. In addition to monotone traveling waves, non-monotonic waves can also be obtained in the case when G is quasi-monotone. Furthermore, overshoot non-monotonic waves can be obtained in the case when G is not quasi-monotone. Previous investigations have not rigorously proved these non-monotonic waves.

The rest of this paper is organized as follows. Section 2 introduces a novel monotone iteration scheme to construct upper and lower solutions of (1.4). In Section 3 we prove the main theorems by using the monotone iteration scheme. Section 4 applies the results in Section 3 to examine the CNN problem and also obtains non-monotonic solutions when G is either quasi-monotone or not quasi-monotone.

II. MONOTONE ITERATION SCHEME

In this section, we consider the differential equation (1.4) with

$$G(u) \equiv G(u_0, u_1, ..., u_N) \colon \mathbf{R}^{N+1} \to \mathbf{R}^1$$

being a C^2-function, $c < 0$, and r_i in $\mathbf{R}^1$ for $i = 0$ to N. In general, the smoothness of G can be relaxed, say $G \in C^1$, except at a finite set. Hereafter, we assume (1.7) and that $r_0 = 0$.

These conditions (1.7) occur quite frequently in many models, and the two zeros of Φ correspond to the homogeneous steady states of (1.4). For simplicity, we also denote $x^0 = (x^0, ..., x^0) \in \mathbf{R}^{N+1}$, etc., when it does not cause any confusion.

This section largely focuses on obtaining monotonic traveling wave solutions of (1.4). The method employed herein to study Eqs. (1.4) and (1.8) is the well-known monotone iteration method. Importantly, the characteristic equation of (1.4), which occurs with the linearization of (1.4) about some trivial solutions, e.g. x^0 and x^+, must be considered.

Clearly, a pair of upper and lower solutions can be constructed according to the roots of the characteristic equation of (1.4).

Herein, we denote the characteristic functions about x^0 and x^+ by $\Delta(\sigma, c, x^0)$ and $\Delta(\sigma, c, x^+)$ respectively, which are defined by

$$\Delta(\sigma, c, x^0) = -c\sigma - \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(x^0)\, e^{\sigma r_j} \qquad (2.1)$$

and

$$\Delta(\sigma, c, x^+) = -c\sigma - \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(x^+)\, e^{\sigma r_j}. \qquad (2.2)$$

Proving the existence of a traveling wave requires that G satisfies the assumptions (G.1), (G.2), and (G.3).

We recall the definition of upper and lower solutions of (1.4).

DEFINITION 2.1. A continuous function $U: \mathbf{R}^1 \to \mathbf{R}^1$ is called an upper solution of (1.4) if it is differentiable almost everywhere and satisfies

$$-cU'(s) \geqslant G(U(s+r_0), ..., U(s+r_N)). \qquad (2.3)$$

Similarly, the lower solution $L(s)$ satisfies

$$-cL'(s) \leqslant G(L(s+r_0), ..., L(s+r_N)). \qquad (2.4)$$

To construct the upper and lower solutions of (1.4), we need some properties of characteristic function $\Delta(\sigma, c, x^0)$. Indeed, by differentiating with respect to σ, we have

$$\frac{\partial \Delta(\sigma, c, x^0)}{\partial \sigma} = -c - \sum_{j=1}^{N} \frac{\partial G}{\partial u_j}(x^0)\, e^{\sigma r_j} r_j \qquad (2.5)$$

and

$$\frac{\partial^2 \Delta(\sigma, c, x^0)}{\partial \sigma^2} = -\sum_{j=1}^{N} \frac{\partial G}{\partial u_j}(x^0)\, e^{\sigma r_j} r_j^2. \qquad (2.6)$$

According to (2.5) and (2.6), we have the following result.

LEMMA 2.2. *By assuming that* (G.1) *and* (G.2) *hold,* $c^* \leqslant 0$ *exists such that for any* $c < c^*$, $\sigma^0(c) > 0$ *and* $\varepsilon_0(c) > 0$ *satisfy*

$$\Delta(\sigma^0, c, x^0) = 0$$

and

$$\Delta(\sigma^0 + \varepsilon, c, x^0) > 0 \qquad for \quad 0 < \varepsilon < \varepsilon_0.$$

Proof. According to (G.2) and (2.6), $\Delta(\sigma, c, x^0)$ is a concave function of σ. Hence, (G.1) implies that $c^* \leqslant 0$ exists such that for any $c < c^*$, $\sigma^0(c) > 0$ and $\varepsilon_0(c) > 0$ satisfy the results. Therefore, the proof is complete.

In the following proposition, the construction of upper and lower solutions in mixed type resembles that in [35]. The construction of the lower solution in the delay case is new.

PROPOSITION 2.3. (i) *Under the assumptions of Theorem* 1.1, *for the given positive numbers* ζ, h, *and* ε, *define functions*

$$U(s) = \begin{cases} x^+ & if \quad s \geqslant 0, \\ x^0 + (x^+ - x^0) e^{\sigma^0 s} & if \quad s \leqslant 0, \end{cases} \tag{2.7}$$

and

$$L(s) = \begin{cases} x^0 & if \quad s \geqslant s_0, \\ x^0 + \zeta(1 - he^{\varepsilon s}) e^{\sigma^0 s} & if \quad s \leqslant s_0, \end{cases} \tag{2.8}$$

where $s_0 < 0$ *is such that* $he^{\varepsilon s_0} = 1$. *Then* $U(s)$ *is an upper solution of* (1.4), *and positive numbers* h_0, ζ_0, *and* ε_0 *in* $\mathbf{R}^1$ *exist such that if* $h > h_0 > 1$, $0 < \zeta < \zeta_0$, *and* $0 < \varepsilon < \varepsilon_0$, $L(s)$ *is a lower solution of* (1.4).

(ii) *Under the assumptions of Theorem* 1.2, *for given positive numbers* ζ, h, *and* ε, *we define the function*

$$\tilde{L}(s) = \begin{cases} x^0 + \zeta(1 - he^{\varepsilon s_1}) e^{\sigma^0 s_1} & if \quad s \geqslant s_1, \\ x^0 + \zeta(1 - he^{\varepsilon s}) e^{\sigma^0 s} & if \quad s \leqslant s_1, \end{cases} \tag{2.9}$$

with $s_1 = (1/\varepsilon) \ln(\sigma^0/h(\sigma^0 + \varepsilon)) < 0$. *Then, positive numbers* $\tilde{h}_0$, $\tilde{\zeta}_0$, *and* $\tilde{\varepsilon}_0$ *exist such that if* $h > \tilde{h}_0 > 1$, $0 < \zeta < \tilde{\zeta}_0$, *and* $0 < \varepsilon < \tilde{\varepsilon}_0$, *then* $\tilde{L}(s)$ *is a lower solution of* (1.4).

Proof. To demonstrate that $U(s)$ is an upper solution, note that if $s \geqslant 0$ then $U'(s) = 0$, and by (G.2) we have

$$G(U(s + r_0), ..., U(s + r_N)) \leqslant \Phi(x^+) = 0.$$

Hence,

$$-cU'(s) \geqslant G(U(s+r_0), ..., U(s+r_N)).$$

If $s \leqslant 0$, according to the definition of U we have

$$U'(s) = \sigma^0(x^+ - x^0)\, e^{\sigma^0 s}.$$

Now, applying (G.3), we have

$$
\begin{aligned}
G(U(s+r_0), ..., U(s+r_N)) &\leqslant \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(x^0)(U(s+r_j) - x^0), \\
&\leqslant \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(x^0)(x^+ - x^0)\, e^{\sigma^0(s+r_j)}.
\end{aligned}
\tag{2.10}
$$

Since

$$\Delta(\sigma^0, c, x^0) = -c\sigma^0 - \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(x^0)\, e^{\sigma^0 r_j},$$

(2.10) implies

$$-cU'(s) \geqslant G(U(s+r_0), ..., U(s+r_N)),$$

for $s \leqslant 0$. Hence, $U(s)$ is an upper solution of (1.4).

Next, we prove L is a lower solution. If $s \geqslant s_0$, we have $L'(s) = 0$ and (G.2) implies that

$$G(L(s+r_0), ..., L(s+r_N)) \geqslant G(x^0, ..., x^0) = 0.$$

Hence, for $s \geqslant s_0$,

$$-cL'(s) \leqslant G(L(s+r_0), ..., L(s+r_N)).$$

If $s \leqslant s_0$, then from the definition of L we have

$$L'(s) = \zeta(\sigma^0 - h(\sigma^0 + \varepsilon)\, e^{\varepsilon s})\, e^{\sigma^0 s}.$$

Now, applying Taylor's expansion of G about x^0, if ζ is small then we can write

$$
G(u_0, ..., u_N) = \Phi(x^0) + \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(x^0)(u_j - x^0) + Q(u - x^0)
\tag{2.11}
$$

for u in $[x^0, x^+]^{N+1}$ and

$$|Q(u - x^0)| \leqslant K_0\, |u - x^0|^2 \qquad \text{for some} \quad K_0 \geqslant 0. \tag{2.12}$$

Thus, by (2.11) and direct computation, we have

$$
\begin{aligned}
G(L(s+r_0), &..., L(s+r_n)) + cL'(s) \\
&= \zeta e^{(\sigma^0+\varepsilon)s} h\, \Delta(\sigma^0+\varepsilon, c, x^0) + Q(L(s+r_0) - x^0, ..., L(s+r_N) - x^0).
\end{aligned}
\tag{2.13}
$$

From (2.12), the constant $K > 0$ exists such that

$$
|Q(L(s+r_0) - x^0, ..., L(s+r_N) - x^0)| \leqslant K\zeta^2 e^{2\sigma^0 s}.
$$

Since (G.1) holds and by Lemma 2.2 we know that there exists an $\varepsilon_0 > 0$ such that

$$
\Delta(\sigma^0+\varepsilon, c, x^0) > 0 \qquad \text{for} \quad 0 < \varepsilon < \varepsilon_0,
$$

there exists an $h_0 > 1$ such that if $h > h_0$, the right-hand side of (2.13) is positive, i.e.,

$$
-cL'(s) \leqslant G(L(s+r_0), ..., L(s+r_N)).
$$

for $s \leqslant s_0$. Hence, $L(s)$ is a lower solution of (1.4).

Finally, we show that $\tilde{L}(s)$ is a lower solution when $r_{\max} = 0$. First, we choose positive numbers ζ, h, and ε such that $L(s)$ is a lower solution and define $\tilde{L}(s)$ as in (2.9). Let $L(s_1)$ be the maximum of $L(s)$ in $\mathbf{R}^1$, i.e.,

$$
s_1 = \frac{1}{\varepsilon} \ln \frac{\sigma^0}{h(\sigma^0+\varepsilon)} < 0,
$$

then $\tilde{L}'(s) = 0$ for $s \geqslant s_1$. From (G.1), we have

$$
\frac{\partial G}{\partial u_0}(x^0) + \sum_{j=1}^{n} \frac{\partial G}{\partial u_j}(x^0)\, e^{\sigma^0 r_j} > 0,
$$

and this implies

$$
\begin{aligned}
G(\tilde{L}(s+r_0), &..., \tilde{L}(s+r_N)) \\
&= \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(x^0)(\tilde{L}(s+r_j) - x^0) + Q(L(s+r_0) - x^0, ..., L(s+r_N) - x^0) \\
&\geqslant \tilde{K}\left(\frac{\partial G}{\partial u_0}(x^0) + \sum_{j=1}^{N} \frac{\partial G}{\partial u_j}(x^0)\, e^{\sigma^0 r_j} \right) \\
&> 0,
\end{aligned}
$$

for some positive constant $\tilde{K}$. Hence,

$$-c\tilde{L}'(s) \leqslant G(\tilde{L}(s + r_0), ..., \tilde{L}(s + r_N)),$$

for $s \geqslant s_1$. If $s < s_1$, then by an argument similar to that used in proving that $L(s)$ is a lower solution we can also obtain

$$-c\tilde{L}'(s) \leqslant G(\tilde{L}(s + r_0), ..., \tilde{L}(s + r_N)),$$

for $s \leqslant s_1$. By combining these results, $\tilde{L}(s)$ is a lower solution of (1.4).

The proof is complete.

After construction of upper and lower solutions of (1.4), using the quasi-monotonicity of G, we present a novel monotone iteration scheme to obtain the non-decreasing solutions of (1.4) and (1.8).

From (G.2), a $\mu > 0$ exists such that the function $H(u_0, ..., u_N): \mathbf{R}^{N+1} \to \mathbf{R}^1$ defined by

$$H(u_0, ..., u_N) = -\frac{1}{c} G(u_0, ..., u_N) + \mu u_0 \tag{2.14}$$

is monotonic in $u_j \in [x^0, x^+]$ for each $j \geqslant 0$. Thus we rewrite (1.4) as

$$x'(s) = H(x(s + r_0), ..., x(s + r_N)) - \mu x(s). \tag{2.15}$$

Then $x(s)$ is easily verified to be a solution of (2.15) if and only if $x(s)$ satisfies

$$x(s) = e^{-\mu s} \int_{-\infty}^{s} e^{\mu t} H(x(t + r_0), ..., x(t + r_N)) \, dt. \tag{2.16}$$

If we define the operator T by

$$(T\varphi)(s) = e^{-\mu s} \int_{-\infty}^{s} e^{\mu t} H(x(t + r_0), ..., x(t + r_N)) \, dt, \tag{2.17}$$

then by (2.16) the fixed point of T satisfies (2.15), and vice versa.

In the following, we apply the monotonic iteration method to find the fixed point of T. Clearly, $\varphi(s)$ is an upper (lower) solution of (1.4) if and only if

$$\varphi(s) \geqslant (\leqslant) (T\varphi)(s). \tag{2.18}$$

Denote the set by

$$\tilde{\Gamma} = \{\varphi \,|\, \varphi: \mathbf{R}^1 \to [x^0, x^+] \text{ and continuous}\}$$

and the set of profiles by

$$\Gamma = \{ \varphi \in \tilde{\Gamma} \mid \varphi \text{ is non-decreasing and satisfies } (1.8) \},$$

then T has the following properties on Γ.

LEMMA 2.4. *Assume that* (G.2) *holds, then*

(i)   *If* $\varphi(s), \tilde{\varphi}(s) \in \tilde{\Gamma}$ *and* $\varphi(s) \leqslant \tilde{\varphi}(s)$ *for all s in* $\mathbf{R}^1$*, then*

$$(T\varphi)(s) \leqslant (T\tilde{\varphi})(s) \qquad \text{for all s in } \mathbf{R}^1.$$

(ii)   *If* φ *is an upper* (*or lower*) *solution of* (1.4), *then* $(T\varphi)(s)$ *is also an upper* (*or lower*) *solution of* (1.4).

(iii)   *if* $\varphi \in \Gamma$ *then* $(T\varphi)(s) \in \Gamma$*, too.*

Proof.   Since H is non-decreasing, (i) follows. Next, assume that φ is an upper solution of (1.4). By (2.18) we have $(T\varphi)(s) \leqslant \varphi(s)$ for all s in $\mathbf{R}^1$. By (i), we obtain

$$T(T\varphi)(s) \leqslant (T\varphi)(s) \qquad \text{for all s in } \mathbf{R}^1.$$

Hence, $(T\varphi)(s)$ is also an upper solution of (1.4), and (ii) follows.

To prove (iii), note that H is non-decreasing. Hence, $\varphi \in \Gamma$ obviously implies that $T\varphi$ is also non-decreasing. To demonstrate that $T\varphi$ satisfies (1.8), note that

$$H(x^0) = \mu x^0 \qquad \text{and} \qquad H(x^+) = \mu x^+.$$

Now, according to L'Hospital's rule, it is easy to verify that

$$\lim_{s \to -\infty} (T\varphi)(s) = x^0 \qquad \text{and} \qquad \lim_{s \to \infty} (T\varphi)(s) = x^+.$$

Hence, $(T\varphi)(s)$ lies in Γ. The proof is complete.

III. PROOF OF THE MAIN THEOREMS

Proof of Theorem 1.1.   By assuming that (G.1), (G.2), and (G.3) hold, then by Proposition 2.3 U and L are the upper and lower solutions of (1.4), respectively. For any positive integer n, define $U_n(s)$ and $L_n(s)$ by

$$U_n(s) = (T^n U)(s) \qquad \text{and} \qquad L_n(s) = (T^n L)(s), \tag{3.1}$$

with $U_0 = U$ and $L_0 = L$. Then using (2.18) and Lemma 2.4, we have

$$x^0 \leqslant \cdots \leqslant U_n(s) \leqslant \cdots \leqslant U_1(s) \leqslant U(s) \leqslant x^+.$$

According to Lebesgue's dominated convergence theorem, the limiting function $U_*(s)$ defined by

$$U_*(s) = \lim_{n \to \infty} U_n(s)$$

exists and is a fixed point of T. Moreover, $U_*(s)$ is non-decreasing and satisfies (1.4). Therefore, it must be verified that $U_*(s)$ satisfies the boundary conditions (1.8). However, $L(s)$, constructed in (2.8), is a non-trivial lower solution. Since $U \geqslant L$ in $\mathbf{R}^1$, it is also easy to verify that $U_n \geqslant L$ for all n, hence $U_* \geqslant L$. Since U_n is non-decreasing and satisfies (1.8), U_* lies in Γ and is a non-decreasing solution of (1.4) and (1.8).

It remains to show that c^* satisfies (1.10). Since

$$\Delta(0, c, x^0) < 0 \qquad \text{and} \qquad \Delta''(\sigma, c, x^0) < 0,$$

it is clear that there are a unique $c^* < 0$ and $\sigma^* > 0$ that satisfy (1.10). Indeed, c^* satisfies

$$c^* = -\sum_{j=1}^N \frac{\partial G}{\partial u_j}(x^0) \, e^{\sigma^* r_j} r_j \tag{3.2}$$

with

$$\sigma^* = \inf \left\{ \sigma > 0 \, \middle| \, \sum_{j=1}^N \frac{\partial G}{\partial u_j}(x^0) \, e^{\sigma r_j}(\sigma r_j - 1) > \frac{\partial G}{\partial u_0}(x^0) \right\}. \tag{3.3}$$

The proof is complete.

Remark 3.1. According to Theorem 1.1, the critical velocity c^* exists, thereby ensuring a monotone traveling wave solution connecting x^0 and x^+ for any $c \in (-\infty, c^*)$. When $c = c^*$, it is not easy to construct the lower solution as (2.8) to show the existence of a solution of (1.4) and (1.8). However, we believe that such a solution exists. For example, in [38], Zinner *et al.* studied the discrete Fisher equation and obtained the traveling wave solutions when $c \leqslant c^*$.

For another example, consider one-dimensional cellular neural networks by

$$\frac{dx_i}{dt} = -x_i + af(x_i) + \beta f(x_{i+1}) \tag{3.4}$$

with $f(x) = (|x+1| - |x-1|)/2$. Define $L(s)$ by

$$L(s) = \begin{cases} x^0 + \delta & \text{for} \quad s \geqslant 0, \\ x^0 + \delta e^{\sigma^* s} & \text{for} \quad s \leqslant 0, \end{cases} \tag{3.5}$$

then $L(s)$ is a lower solution of (3.4) when $a + \beta - 1 > 0$ and δ is positive and small enough. Hence, we have a traveling wave solution of (3.4) and (1.8) when $c = c^*$. In addition, the global structure of the traveling wave solutions of (3.4) is completely classified in [19]. Of relevant interest is whether or not a traveling wave of (1.4) exists which may be non-monotone for $c > c^*$.

Proof of Theorem 1.2.   Since $r_{\max} = 0$, by (2.5) and (2.6), we have that $\Delta(\sigma, c, x^0)$ is a concave function in σ and $\Delta'(\sigma, c, x^0) > 0$ for any $c < 0$. Hence (G.1) holds for any $c < 0$. Now from Proposition 2.3(ii), we know that $\tilde{L}(s)$ is a lower solution of (1.4). If we denote $\tilde{L}_n(s)$ by

$$\tilde{L}_n(s) = (T^n \tilde{L})(s). \tag{3.6}$$

with $\tilde{L}_0 = \tilde{L}$ and apply Lemma 2.4, we obtain

$$x^0 \leqslant \tilde{L}_0(s) \leqslant \tilde{L}_1(s) \leqslant \cdots \leqslant \tilde{L}_n(s) \leqslant \cdots \leqslant x^+.$$

By the Lebesgue dominated convergence theorem again, the limiting function $\tilde{L}_*(s)$ defined by

$$\tilde{L}_*(s) = \lim_{n \to \infty} \tilde{L}_n(s)$$

is the fixed point of T. It remains to be shown that $\tilde{L}_*(s)$ satisfies the boundary conditions. Clearly, $\tilde{L}_*(s)$ is non-decreasing due to the monotonicity of T. This is because we do not have a non-trivial upper solution as a barrier function to separate $\tilde{L}_*(s)$ from a trivial solution x^+; to overcome this difficulty, we need to show $\tilde{L}_*(s) \to x^0$ as $s \to -\infty$. This can be achieved inductively on $\tilde{L}_n(s)$.

By (3.6), we have

$$\tilde{L}_{n+1} = e^{-\mu s} \int_{\infty}^{s} e^{\mu t} H(\tilde{L}_n(t + r_0), ..., \tilde{L}_n(t + r_N)) \, dt.$$

We begin with the study of $\tilde{L}_1(s)$ as $s \to -\infty$. By (2.11), $\tilde{L}_1$ can be written as

$$\tilde{L}_1(s) = e^{-\mu s} \int_{-\infty}^{s} e^{\mu t} \left[-\frac{1}{c} \sum_{j=0}^{N} \frac{\partial G}{\partial u_j}(x^0)(\tilde{L}(t + r_j) - x^0) + \mu \tilde{L}(t) \right.$$

$$\left. + Q(\tilde{L}(t + r_0) - x^0, ..., \tilde{L}(t + r_N) - x^0) \right] dt.$$

As s tends to $-\infty$, we have

$$\tilde{L}_1(s) = x^0 + \zeta e^{\sigma^0 s} - \zeta h \left(1 - \frac{\Delta(\sigma^0 + \varepsilon, c, x^0)}{\mu + \sigma^0 + \varepsilon} \right) e^{(\sigma^0 + \varepsilon)s}$$

$$+ e^{-\mu s} \int_{-\infty}^{s} e^{\mu t} Q(\tilde{L}(t + r_0) - x^0, ..., \tilde{L}(t + r_N) - x^0)] \, dt. \quad (3.7)$$

However, (2.12) implies that a positive constant $\bar{K}$ exists such that

$$e^{-\mu s} \int_{-\infty}^{s} e^{\mu t} Q(\tilde{L}(t + r_0) - x^0, ..., \tilde{L}(t + r_N) - x^0) \, dt$$

$$\leqslant \frac{\bar{K}}{\mu + 2\sigma^0} e^{2\sigma^0 s}. \quad (3.8)$$

Define ξ and ρ as

$$\xi = 1 - \frac{\Delta(\sigma^0 + \varepsilon, c, x^0)}{\mu + \sigma^0 + \varepsilon} \qquad \text{and} \qquad \rho = \frac{\bar{K}}{\mu + 2\sigma^0}.$$

In addition, by combining (3.7) with (3.8), we have $0 < \xi < 1$ and $\rho < \frac{1}{2}$, for μ large enough. Hence, $\tilde{L}_1(s)$ can be written as

$$\tilde{L}_1(s) = x^0 + \zeta(1 - \xi h e^{\varepsilon s}) e^{\sigma^0 s} + r_1(s),$$

where

$$r_1(s) = e^{-\mu s} \int_{-\infty}^{s} e^{\mu t} Q(\tilde{L}(t + r_0) - x^0, ..., \tilde{L}(t + r_N) - x^0) \, dt$$

and

$$|r_1(s)| \leqslant \rho e^{2\sigma^0 s}.$$

Let $\zeta \leqslant \rho$, and by induction $\tilde{L}_n(s)$ can be written as

$$\tilde{L}_n(s) = x^0 + \zeta(1 - \xi^n h e^{\varepsilon s}) e^{\sigma^0 s} + r_n(s) \quad (3.9)$$

and

$$|r_n(s)| \leqslant \rho e^{2\sigma^0 s}. \quad (3.10)$$

Hence, $\tilde{L}_n$ and $\tilde{L}_*$ tend to x^0 as s tends to $-\infty$. Thus $\tilde{L}_*$ is not the trivial solution x^+. Since $\tilde{L}_*$ is monotonously increasing, according to (1.7), we have

$$\lim_{s \to \infty} \tilde{L}_*(s) = x^+.$$

The proof is complete.

Remark 3.2. By using a comparison theorem obtained in [28], we can prove that the monotone solution obtained in Theorems 1.1 and 1.2 is unique for $c \in (-\infty, c^*)$. We only sketch the proof in the following and omit the details.

It is not difficult to prove that if $x(s)$ is a monotonic solution of (1.4) and (1.8) then we have $x(s) = x^0 + O(e^{\sigma^0 s})$, as $s \to -\infty$. On the other hand, if $\sum_{j=0}^{N}(\partial G/\partial u_j)(x^+) < 0$ then (1.4) satisfies the hyperbolicity at x^+; see [27]. Hence, as in [27], we have $x(s) = x^0 + O(e^{\sigma^+ s})$, as $s \to \infty$. Here $\sigma^+ < 0$ and satisfies $\Delta(\sigma^+, c, x^+) = 0$. By an argument similar to that used in proving Proposition 6.5 of [28], the uniqueness result follows.

IV. APPLICATIONS TO CNN

In this section, we initially apply the above results to obtain a monotonic traveling wave solution in CNN. For a CY-CNN with delay or advance type, we demonstrate that the solutions can be obtained explicitly. In addition to the non-decreasing traveling waves, we obtain non-monotonic traveling waves. The various results obtained for CY-CNN allow us to study the general case of (1.4) even when G is not quasi-monotonic.

For simplicity we only study the one-dimensional CNN; the higher dimensional cases can be treated analogously. Consider

$$\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}) \tag{4.1}$$

where z, α, a, and β are constants. Here f is a non-decreasing continuous function which is differentiable except for finite points. A typical case is

$$f(x) = f_0(x) \equiv \tfrac{1}{2}(|x+1| - |x-1|). \tag{4.2}$$

In this case, it is called CY-CNN. Assuming

$$x_i(t) = x(i - ct) = \varphi(s) \qquad \text{for} \quad i \in \mathbf{Z^1}, \tag{4.3}$$

where $s = i - ct$ and $\varphi(s)$ is in $C^1(\mathbf{R}^1, \mathbf{R}^1)$, then $\varphi(s)$ satisfies

$$-c\varphi'(s) = -\varphi(s) + z + \alpha f(\varphi(s-1)) + af(\varphi(s)) + \beta f(\varphi(s+1)), \qquad (4.4)$$

and the boundary conditions are

$$\lim_{s \to -\infty} \varphi(s) = x^0 \qquad \text{and} \qquad \lim_{s \to \infty} \varphi(s) = x^+. \qquad (4.5)$$

Now,

$$\Phi(x) = -x + z + (\alpha + a + \beta) f(x). \qquad (4.6)$$

Assume that $x^0 < x^+$ are the two zeros of $\Phi(x)$ such that $\Phi(x) > 0$ for $x \in (x^0, x^+)$. For CY-CNN we have

$$x^0 = \frac{-z}{-1 + a + \alpha + \beta} \qquad \text{and} \qquad x^+ = z + a + \alpha + \beta,$$

whenever $a + \alpha + \beta - 1 \neq 0$.

Applying Theorem 1.1, we obtain

THEOREM 4.1. *Suppose α, a, and β are real numbers and f is a continuous function differentiable except for finite points. If $f'(x) \geqslant 0$, $f'(x^0)$ exists and satisfies the conditions*

    (i) $-1 + (\alpha + a + \beta) f'(x^0) > 0$,

    (ii) $\alpha > 0$ and $\beta > 0$,

    (iii) $f(x) \leqslant f(x^0) + (\alpha + a + \beta) f'(x^0)(x - x^0)$, *for x in $[x^0, x^+]$,*

then $c^* < 0$ exists such that for each $c < c^*$ there is a non-decreasing solution satisfying (4.4) and (4.5). Moreover, c^* satisfies

$$-c^* = f'(x^0)(-\alpha e^{-\sigma^*} + \beta e^{\sigma^*}) \qquad and$$
$$-c^* \sigma^* = -1 + f'(x^0)(a + \alpha e^{-\sigma^*} + \beta e^{\sigma^*})$$

for some $\sigma^* > 0$.

Proof. It is easy to verify that the assumptions (i), (ii), and (iii) imply (G.1), (G.2), and (G.3) with

$$\Delta(\sigma, c, x^0) = -c\sigma + 1 - f'(x^0)(a + \alpha e^{-\sigma} + \beta e^{\sigma}).$$

The following result immediately occurs. Therefore, the proof is complete.

In the following theorem, we observe whether the assumption (G.2) fails. A traveling wave solution may exist which satisfies (4.5) which overshoots

the steady state x^+. For simplicity, we consider the delay type of CY-CNN, i.e., $\beta = 0$. Now, the monotonic traveling wave can be solved explicitly.

THEOREM 4.2 (Delay case of CY-CNN). *Assume that $\beta = 0$ and $f = f_0$ in (4.2). If $-1 + \alpha + a > 0$, then:*

(i) *If $\alpha > 0$, then for any $c < c^*$, monotonic traveling wave solutions of (4.1) and (4.5) exist.*

(ii) *If $a > 1$, $\alpha < 0$ and we define $c^* = (\ln(-\alpha/a))^{-1}$, then for any $c \leqslant c^*$, monotonic traveling wave solutions of (4.1) and (4.5) exist.*

(iii) *If $a > 1$, $\alpha < 0$, and $c > c^*$, then a solution ϕ of (4.1) and (4.5) exists which has a single maximum. In this case, ϕ is not monotonic.*

Furthermore, in any case the solution $\phi(s)$ can be expressed as

$$\phi(s) = \begin{cases} x^0 + (1 - x^0) e^{\sigma^0 s} & \text{for } s \leqslant 0, \\ x^0 + le^{\sigma^0 s} + al(1 - e^{-\delta s}) - \dfrac{al\delta}{\sigma^0 + \delta} (e^{\sigma^0 s} - e^{-\delta s}) & \text{for } s \in [0, 1], \\ x^+ + e^{-\delta(s-1)}(\phi(1) - x^+) & \text{for } s \in [1, \infty), \end{cases} \tag{4.7}$$

where $l = 1 - x^0$, $\delta = -1/c$, $x^0 = -z/(-1 + a + \alpha)$, $x^+ = z + a + \alpha$, and $\Delta(\sigma^0, c, x^0) = 0$.

Proof. Since f is piecewise linear, the problems (4.4) and (4.5) can be decomposed into the equations

$$-c\phi'(s) = \begin{cases} -\phi(s) + z + \alpha\phi(s-1) + a\phi(s) & \text{if } s \in (\infty, 1], \\ -\phi(s) + z + \alpha\phi(s-1) + a\phi(s) & \text{if } s \in [-1, 0], \\ -\phi(s) + z + \alpha\phi(s-1) + a & \text{if } s \in [0, 1], \\ -\phi(s) + z + \alpha + a & \text{if } s \in [0, \infty). \end{cases} \tag{4.8}$$

Herein, assume that $\phi(0) = 1$, $\phi(-\infty) = x^0$, and $\phi(\infty) = x^+$. Now, (4.8) can be solved and the solution is given in (4.7). Since the proof is elementary but lengthy, the detail is omitted. Therefore, the proof is complete.

Remark 4.3. In the case (iii) of Theorem 4.2, the assumption (G.2) does not hold and the solution is now nonmonotonic, as shown in Fig. 1.

Remark 4.4. According to Theorem 4.2, a bifurcation diagram can be drawn which exhibits how the monotonic traveling wave changes into a non-monotonic traveling wave when α changes from positive to negative.
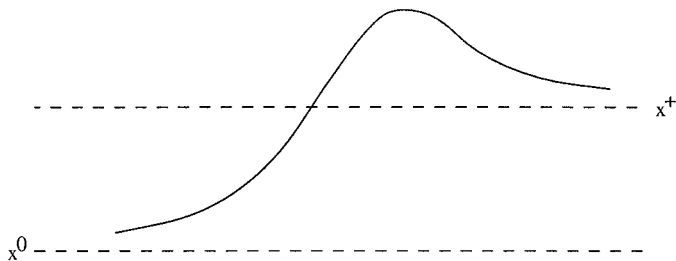
FIGURE 1

Indeed, if we assume that $a > 1$ and $c < 0$ will be given fixed numbers, we define $\alpha^*(a, c)$ and $c^*(a)$ by

$$\alpha^*(a, c) = -ae^{1/c} \qquad \text{and} \qquad c^*(a) = \ln\frac{a-1}{a}.$$

If $c > c^*(a)$ then the bifucation pictures with respect to α are given as follows.

In Case (ii) of Fig. 2, the traveling wave solution $\phi(s)$ is equal to x^+ for s greater than some s^*.

Finally, the oscillating traveling wave solution of CY-CNN is considered as follows

THEOREM 4.5.   *While considering* (4.4) *and* (4.5) *with* $\beta = 0$, *a solution* $\phi_{\text{osc}}(s)$ *exists which is given by*

$$\phi_{\text{osc}}(s) = \begin{cases} x^0 + le^{\lambda s}\cos(vs) + me^{\lambda s}\sin(vs) & \text{if} \quad s \in (-\infty, 0], \\ x^0 + le^{\lambda s}\cos(vs) + me^{\lambda s}\sin(vs) + \tilde{\phi}_{\text{osc}}(s) & \text{if} \quad s \in [0, 1], \end{cases}$$
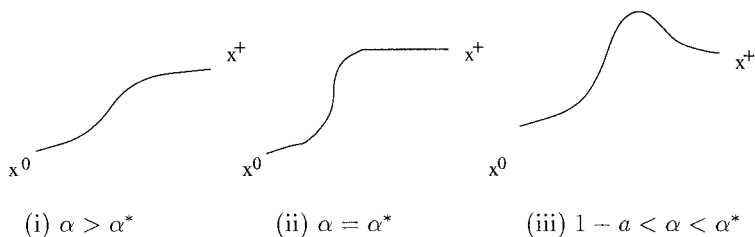
$$(4.9)$$

(i) $\alpha > \alpha^*$ \qquad (ii) $\alpha = \alpha^*$ \qquad (iii) $1 - a < \alpha < \alpha^*$

FIG. 2. Bifurcation diagram.

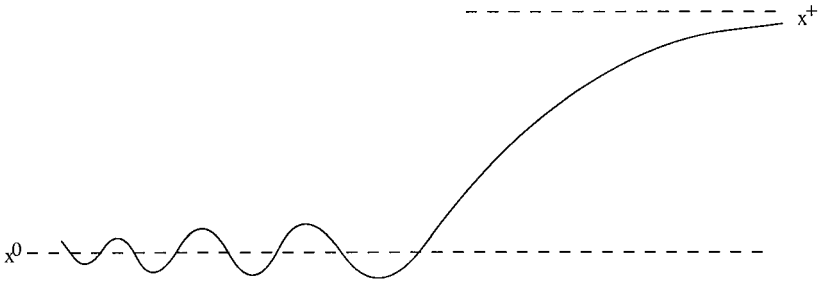**FIG. 3.** Oscillating wave.

where $\lambda > 0$, l and m are non-zero real numbers, and $\delta = -1/c$,

$$\tilde{\phi}_{\text{osc}}(s) = al - ale^{-\delta s} - al\delta e^{-\delta s}g(s) - am\delta e^{-\delta s}v(s),$$

$$g(s) = \frac{(\lambda + \delta)(e^{(\lambda + \delta) s}\cos(vs) - 1) + ve^{(\lambda + \delta) s}\sin(vs)}{v^2 + (\lambda + \delta)^2},$$

$$v(s) = \frac{(\lambda + \delta) e^{(\lambda + \delta) s}\sin(vs) - v(e^{(\lambda + \delta) s}\cos(vs) - 1)}{v^2 + (\lambda + \delta)^2},$$

and $\Delta(\lambda + vi, c, x^0) = 0$.

Proof. The proof of the theorem is the same as that used in proving Theorem 4.2 by solving (4.8) with suitable l and m. Since the proof is elementary but lengthy, the details are omitted here.

EXAMPLE 4.6. Within a certain parameter range of a, α, z, and appropriate choices of l and m, we can prove that ϕ_{osc} satisfies $\lim_{s \to \infty} \phi_{\text{osc}}(s) = x^+$. Here is an example with the aid of numerical computation: If we choose $a = 200$, $\alpha = 144.14\pi$, $z = 0$, $v = 1.1\pi$, $c = -17.3277$, $l = 1$, and $m = 0.2$, then the oscillating wave $\phi_{\text{osc}}(s)$ is given in Fig. 3.

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