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ORIGINAL ARTICLE

An auxiliary ordinary differential equation and the exp-function method

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Abstract In this paper, the new idea of finding the exact solutions of the nonlinear evolution equations is introduced. The idea is that the exact solutions of the auxiliary ordinary differential equation are derived by using exp-function method, and then the exact solutions of the nonlinear evolution equations are derived with aid of the auxiliary ordinary differential equation. As examples, the classical KdV equation, Boussinesq equation, (3 + 1)-dimensional Jimbo–Miwa equation and Benjamin–Bona–Mahony equation are discussed and the exact solutions are derived.

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Introduction

In this paper, the equation is considered as

$$Au'' + Bu + Cu^2 = 0, \tag{1}$$

where A, B, C are arbitrary constants. Eq. (1) is one of the most important auxiliary equations, because many nonlinear evolution equations can be converted to Eq. (1) using the traveling wave reduction (it is shown in "The applications of the auxiliary ordinary different equation (1)" section).

Recently, He and Wu (2006) proposed a straightforward and concise method called "exp-function method" to explore the exact solutions of modified KdV equation, the Dodd–Bullough–Mikhailov equation. This method has been paid attention by many researchers. Up to now, the exp-function method has been applied to find the solutions of a class of the nonlinear evolution equations, such as the nonlinear Schrödinger equations with cubic and power law nonlinearity (Khani et al., 2007), combined KdV–mKdV equation (Ebaid, 2007), KdV equation with variable coefficients (Zhang, 2007a), the discrete (2 + 1)-dimensional Toda lattice equation (Zhu, 2008), and the Maccari's system (Zhang, 2007b). Thus it is easy to see that the exp-function method is a very powerful method and can be used to study the exact solutions of the high-dimensional system, the discrete system, and the system with variable coefficients.

According to the introduction above, the method presented in this paper is described as follows: exact solutions of the auxiliary ordinary differential Eq. (1) are firstly derived using exp-function method. The exact solutions of a class of nonlinear evolution equations which can be converted to Eq. (1) using the traveling wave reduction are presented with the aid of the auxiliary ordinary differential Eq. (1).

This paper is organized as follows: in the next section, the exact solutions of the auxiliary ordinary differential Eq. (1) are derived by exp-function method. A class of the nonlinear evolution equations, such as the classical KdV equation, Boussinesq equation, (3 + 1)-dimensional Jimbo–Miwa equation and Benjamin–Bona–Mahony equation, which can be converted to auxiliary ordinary differential Eq. (1) using the traveling wave reduction are introduced in "The applications of the auxiliary ordinary different equation (1)" section, and the exact solutions can be obtained with the aid of the auxiliary ordinary different Eq. (1). Some conclusions and discussions are given in the final section.

The exact solution of auxiliary ordinary differential Eq. (1)

According to the exp-function method, we suppose that the exact solutions of Eq. (1) is in the form as

$$u(\eta) = \frac{\sum_{n=-d}^{c} a_n \exp(n\eta)}{\sum_{m=-q}^{p} b_m \exp(m\eta)} = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{2}$$

Thus we have

$$u'' = \frac{c_3 \exp[(c+3p)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots}, \tag{3}$$

$$u^2 = \frac{c_1 \exp[(2c+2p)\eta] + \cdots}{c_2 \exp[4p\eta] + \cdots}, \tag{4}$$

where $c_i (i = 1, \cdots, 4)$ are constants determined later.

Considering the balancing between u'' and u^2 in Eqs. (3) and (4) yields

$$c = p.$$

From Eq. (2), we have

$$u'' = \frac{\cdots + d_3 \exp[-(3q+d)\eta]}{\cdots + d_4 \exp[-4q\eta]}, \tag{5}$$

$$u^2 = \frac{\cdots + d_1 \exp[-(2q+2d)\eta]}{\cdots + d_2 \exp[-4q\eta]}, \tag{6}$$

where $d_i (i = 1, \cdots, 4)$ are constants determined later.

Considering the balancing between u'' and u^2 in Eqs. (5) and (6) yields

$$q = d.$$

Case 1 $c = p = 1,\ d = q = 1$

The Eq. (2) is converted as

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}, \tag{7}$$

Substituting (7) into (1) and using *Mathematica,* the left hand side of Eq. (1) is converted into a polynomial in $e^{n\eta}$. Setting the coefficients of $e^{n\eta}$ to zero yields a set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}$.

With the help of *Mathematica*, we can obtain the solutions of the algebraic equations above as

Case 1.1 $a_{-1} = 0, a_1 = 0, a_0 = -\frac{3Bb_0}{C}, b_0 = b_0, b_{-1} = \frac{b_0^2}{4}, A = -B,$

Substituting the results above in to Eq. (7) yields the exact solutions of Eq. (1) as

$$u(\eta) = -\frac{3Bb_0}{C(e^{\eta} + b_0 + \frac{1}{4}b_0^2 e^{-\eta})}, \tag{8}$$

where $A = -B$.

In (8), setting $b_0 = 2$ yields

$$u(\eta) = -\frac{6B}{C[\cosh(\eta) + 1]}, \tag{9}$$

Case 1.2 $a_{-1} = -\frac{Bb_0^2}{4C}, a_1 = -\frac{B}{C}, a_0 = \frac{2Bb_0}{C}, b_0 = b_0, b_{-1} = \frac{b_0^2}{4}, B^2 = AC$

Similar to *Case 1.1*, the exact solutions of Eq. (1) is obtained as

$$u(\eta) = -\frac{B(4e^{\eta} - 8b_0 + b_0^2 e^{-\eta})}{C(4e^{\eta} + 4b_0 + b_0^2 e^{-\eta})}. \quad (10)$$

where $B^2 = AC$.

Case 2 $c = p = 2, d = q = 1$

The Eq. (1) is in the form as

$$u(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \quad (11)$$

Case 2.1 $B = -1$,

Case 2.1.1

Similar to *Case 1*, we obtain

$$a_{-1} = 0, a_0 = 0, a_1 = a_1, a_2 = 0, b_1 = \frac{2a_1}{3}, b_0 = \frac{a_1^2 C^2}{36}, B = -1.$$

Substituting the results above into (11) yields

$$u(\eta) = \frac{a_1 e^{\eta}}{e^{2\eta} + \frac{2a_1}{3} e^{\eta} + \frac{a_1^2 C^2}{36}}. \quad (12)$$

Case 2.1.2

Similar to *Case 1*, we obtain

$$a_2 = 0, a_1 = a_1, a_0 = a_0, a_{-1} = 0, b_1 = \frac{2Ca_1^2 + 6a_0}{6a_1}, b_0 = -\frac{24Ca_0 + a_1^2 C}{72}, b_{-1} = \frac{a_0 a_1 C^2}{36},$$

Substituting the results above into (11) yields

$$u(\eta) = \frac{a_1 e^{\eta} + a_0}{e^{2\eta} + \frac{2Ca_1^2 + 6a_0}{6a_1} e^{\eta} - \frac{24Ca_0 + a_1^2 C}{72} + \frac{a_0 a_1 C^2}{36} e^{-\eta}}. \quad (13)$$

Case 2.1.3

Similar to *Case 1*, we obtain

$$a_{-1} = \frac{2a_2^3 b_1^3 - 3a_2^2 a_1 b_1^2 + a_1^3}{108a_2^2}, a_0 = -\frac{5a_2^2 b_1^2 - 3a_1^2 - 2b_1 a_1 a_2}{12a_2}, a_1 = a_1, a_2 = a_2,$$

$$b_1 = b_1, b_0 = \frac{3a_2^2 b_1^2 - 2b_1 a_1 a_2 - a_1^2}{12a_2^2}, b_{-1} = \frac{2a_2^3 b_1^3 - 3a_2^2 a_1 b_1^2 + a_1^3}{108a_2^3},$$

Substituting the results above into (11) yields

$$u(\eta) = \frac{a_2 e^{2\eta} + a_1 e^{\eta} - \frac{5a_2^2 b_1^2 - 3a_1^2 - 2b_1 a_1 a_2}{12a_2} + \frac{2a_2^3 b_1^3 - 3a_2^2 a_1 b_1^2 + a_1^3}{108a_2^2} e^{-\eta}}{e^{2\eta} + b_1 e^{\eta} + \frac{3a_2^2 b_1^2 - a_1^2 - 2b_1 a_1 a_2}{12a_2^2} + \frac{2a_2^3 b_1^3 - 3a_2^2 a_1 b_1^2 + a_1^3}{108a_2^3} e^{-\eta}}. \quad (14)$$

where $a_2 = \frac{1}{C}$.

Case 2.2 $B > 0$, We can obtain the exact solutions of Eq. (1), here we omit for simplicity.

Case 3 $c = p = 2, d = q = 2$

The Eq. (2) is in the form as

$$u(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{\exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)}, \tag{15}$$

In Eq. (15), setting $b_{-1} = b_1 = 0$ for simplicity, Eq. (15) is converted to

$$u(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{\exp(2\eta) + b_0 + b_{-2} \exp(-2\eta)}. \tag{16}$$

Case 3.1

Similar to *Case 1*, we obtain

$$a_2 = 0, a_1 = 0, a_0 = a_0, a_{-1} = 0, a_{-2} = 0, b_0 = -\frac{2Ca_0}{6B}, b_{-2} = \frac{a_0^2 C^2}{36B^2}.$$

Substituting the results above into (16) yields

$$u(\eta) = \frac{a_0}{e^{2\eta} - \frac{2Ca_0}{6B} + \frac{a_0^2 C^2}{36B^2} e^{-2\eta}}, \tag{17}$$

Case 3.2

$$a_2 = 0, a_1 = a_1, a_0 = \frac{Ca_1^2}{3B}, a_{-1} = \frac{a_1^3 C^2}{36B^2}, a_{-2} = 0, b_{-2} = \frac{C^2 a_1^4}{5184B^2}, b_0 = -\frac{C^2 a_1^2}{18B^2}.$$

Thus the exact solutions of Eq. (1) can be obtained as

$$u(\eta) = \frac{a_1 e^{\eta} + \frac{Ca_1^2}{3B} + \frac{a_1^3 C^2}{36B^2} e^{-\eta}}{e^{2\eta} - \frac{C^2 a_1^2}{18B^2} \frac{C^2 a_1^4}{5184B^2} e^{-2\eta}}. \tag{18}$$

Case 3.3

$$a_2 = a_2, a_1 = 0, a_0 = -2a_2 b_0, a_{-1} = 0, a_{-2} = \frac{a_2 b_0^2}{4}, b_0 = b_0, b_{-2} = \frac{b_0^2}{4}, B = -Ca_2.$$

Thus the exact solutions of Eq. (1) can be obtained as

$$u(\eta) = \frac{a_2 e^{2\eta} - 2a_2 b_0 + \frac{1}{4} a_2 b_0^2 e^{-2\eta}}{e^{2\eta} + b_0 + \frac{1}{4} b_0^2 e^{-2\eta}}. \tag{19}$$

where $a_2 = -\frac{B}{C}$.

Case 3.4

$$a_2 = a_2, a_1 = a_1, a_0 = \frac{5a_1^2}{18a_2}, a_{-1} = \frac{a_1^3}{36a_2^2}, a_{-2} = \frac{a_1^2}{1296a_2^3}, b_0 = -\frac{a_1^2}{18a_2^2}, b_2 = \frac{a_1^4}{1296a_2^4}.$$

The exact solutions of Eq. (1)are in forms as

$$u(\eta) = \frac{a_1 e^{\eta} + a_2 e^{2\eta} + \frac{5a_1^2}{18a_2} + \frac{a_1^3}{36a_2^2} e^{-\eta} + \frac{a_1^4}{1296a_2^3} e^{-2\eta}}{e^{2\eta} - \frac{a_1^2}{18a_2^2} + \frac{a_1^4}{1296a_2^4} e^{-2\eta}}. \tag{20}$$

where $a_2 = -\frac{B}{C}$.

Remark 1: We can obtain other type solutions of Eq. (1) such as we set $f = c = 4, d = g = 4$.

The applications of the auxiliary ordinary different equation (1)

In this section, the classical KdV equation, Boussinesq equation, (3+1)-dimensional Jimbo–Miwa equation and Benjamin–Bona–Mahony equation are considered again and the exact solutions are derived with the aid of the auxiliary ordinary different Eq. (1).

KdV equation

We firstly consider KdV equation (Mei and Zhang, 2005; Siraj-ul-Islam et al., 2008; Wang et al., 2008; Zhang, 2009) as

$$u_t + uu_x + \beta u_{xxx} = 0, \tag{21}$$

where β is constant.

Supposing the exact solutions of Eq. (21) are in the form as

$$u(x,t) = u(\xi), \xi = k(x - ct) + \xi_0, \tag{22}$$

where ξ_0 is constant.

Substituting (22) into (21) yields

$$-cu' + uu' + \beta k^2 u''' = 0. \tag{23}$$

Integrating Eq. (23) once and setting integral constant to zero, Eq. (23) is converted to Eq. (1), where

$$A = \beta k^2, B = \frac{1}{2}, C = -c. \tag{24}$$

Boussinesq equation

The Boussinesq equation (Abassy et al., 2007; Inc, 2008; Javidi and Jalilian, 2008) considered as

$$u_{tt} - c_0^2 u_{xx} - \alpha u_{xxxx} - \beta(u^2)_{xx} = 0. \tag{25}$$

Supposing the exact solutions of Eq. (25) are in the form as

$$u(x,t) = u(\eta) = u(kx + \omega t), \tag{26}$$

where k, ω are constants.

Similar to the "KdV equation" section, by using Eq. (26), Eq. (25) is converted to Eq. (1),where

$$A = \alpha k^4, B = c_0^2 k^2 - \omega^2, C = \beta.$$

(3+1)-Dimensional Jimbo–Miwa equation

(3+1)-Dimensional Jimbo–Miwa equation (Liu and Jiang, 2004; Ma and Lee, 2009; Ma et al., 2009; Turgut and İsmail, 2008) is considered as

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0. \tag{27}$$

Supposing the exact solutions of Eq. (27) are in the form as

$$u(x,y,z,t) = u(\eta), \eta = kx + my + rz + \omega t, \tag{28}$$

where k, m, r, ω are constants.

Similar to the "KdV equation" section, by using Eq. (28), Eq. (27) is converted to Eq. (1),where

$$A = k^3 m, B = 2m\omega - 3kr, C = 3k^2 m.$$

Benjamin–Bona–Mahony equation

In this section, we consider Benjamin–Bona–Mahony equation (El-Wakil et al., 2008; Lai et al., 2009) as

$$u_t - u_{xxt} + u_x + \left(\frac{u^2}{2}\right)_x = 0, \tag{29}$$

Supposing the exact solutions of Eq. (29) are in the form as

$$u(x,t) = u(\eta), \eta = kx - \omega t, \tag{30}$$

where k, ω are constants.

Similar to the "KdV equation" section, by using Eq. (30), Eq. (29) is converted to Eq. (1),where

$$A = k^2\omega, B = k - \omega, C = \frac{1}{2}k.$$

Remark 2: We can obtain the solutions of Eqs. 21, 25, 27, and 29 with the aid of the auxiliary ordinary different Eq. (1). Here we omit for simplicity.

Remark 3: There are other partial differential equations which can be converted to the auxiliary ordinary different Eq. (1) with the aid of the traveling wave reduction.

Conclusions and discussions

The auxiliary equation method is very important in finding the exact solutions of nonlinear evolution equations, and the auxiliary ordinary different Eq. (1) is one of most important auxiliary equations because many nonlinear evolution equations, such as the classical KdV equation, Boussinesq equation, $(3+1)$-dimensional Jimbo–Miwa equation and Benjamin–Bona–Mahony equation and other PEDs, can be converted to this equation using the traveling wave reduction. In this paper, we apply exp-function method to derive the exact solutions of the auxiliary ordinary different Eq. (1). The exact solutions of the classical KdV equation, Boussinesq equation, $(3+1)$-dimensional Jimbo–Miwa equation and Benjamin–Bona–Mahony equation are derived with the aid of the auxiliary ordinary different Eq. (1). The idea introduced in this paper can be applied to other nonlinear evolution equations.

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