Optimal control of damped Klein–Gordon equations with state constraints

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Abstract

In this paper, we study the maximum principles for optimal control problems governed by the damped Klein–Gordon equations with state constraints. And we prove the existence of the optimal parameter and deduce the necessary conditions on the optimal parameter.

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1. Introduction

We study the optimal control problems for the damped Klein–Gordon equation

\[
\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \delta |y|^{\gamma} y = Bu + f, \tag{1.1}
\]

where \(\alpha, \beta, \gamma, \delta\) are physical constants and \(f\) is a forcing function. This Klein–Gordon equation is known as one of the nonlinear wave equation arising in relativistic quantum mechanics and has been studied extensively [7,10]. We also could find the trace by using the following method in Ahmed [1]. Then we may rewrite the state system (1.1) as

\[
y'' + \alpha y' + \beta Ay + \delta k(y) = Bu + f, \tag{1.2}
\]

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where $\Omega$ is an open bounded set of the $n$-dimensional Euclidean space $R^n$, $H = L^2(\Omega)$, $V = H^1_0(\Omega)$, $U$ a real Hilbert space, $A \in \mathbb{L}(V, V^*)$, $B \in \mathbb{L}(U, H)$, $k : V \to H$ a nonlinear operator and $f \in L^2(0, T; H)$.

Recently, many mathematicians have discussed the optimal control problems governed by linear and nonlinear parabolic differential equations. Barbu made many contributions in this field [3]. Lie and Yong studied the maximal principle for optimal control governed by some nonlinear parabolic equations with two point boundary state constraints [9]. Pavel discussed the necessary conditions for optimal control governed by linear parabolic equations with two-point boundary constraints [12]. Wang also dealt with the necessary conditions of optimality for some optimal control problems governed by some parabolic differential equations involving monotone graphs and discussed the two point boundary state constraints [14].

The present work in this paper is concerned with the maximum principles for optimal control problems governed by the damped Klein–Gordon equations.

The paper is organized as follows. In Section 2, we present some preliminaries that are used in the subsequent section. In Section 3, we derive the Pontryagin’s maximum principle for optimal control of the damped Klein–Gordon equation (1.1).

2. Preliminaries

Let $\Omega$ be an open bounded set of the $n$-dimensional Euclidean space $R^n$ with the piecewise smooth boundary $\Gamma = \partial \Omega$. Let $Q = (0, T) \times \Omega$, $\Sigma = (0, T) \times \Gamma$, $R = (−\infty, \infty)$ and $R^+ = [0, \infty)$.

We consider the following control system for the damped Klein–Gordon equation described by

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \delta |y|^\gamma y = Bu + f \quad \text{in} \ Q, $$

$$y = 0 \quad \text{on} \ \Sigma, $$

$$y(0, x) = y_0(x), \quad \frac{\partial y}{\partial t}(0, x) = y_1(x) \quad \text{in} \ \Omega,$$

where $\alpha, \beta, \gamma, \delta \in R$ are constants and $y_0, y_1, f$ are given functions. We set $H = L^2(\Omega)$, $V = H^1_0(\Omega)$ and endow these spaces with the usual scalar products and norms $(\phi, \psi) = \int_\Omega \phi(x)\psi(x) \, dx$, $|\phi| = (\phi, \phi)^{\frac{1}{2}}$ for $\phi, \psi \in H^1_0(\Omega)$. By $V^*$ and $\langle \cdot, \cdot \rangle$ denote the dual space of $V$ and the duality pairing between $V$ and $V^*$, respectively. Let $A \in \mathbb{L}(V, V^*)$ be defined by the bilinear form $\langle A\phi, \psi \rangle = (\phi, \psi)$. Then $A$ is considered as a self-adjoint operator with domain $D(A) = H^1_0(\Omega) \cap H^2(\Omega)$ and for $\phi \in D(A)$, $A\phi = -\Delta \phi$. It is clear that $D(A) \subset V \subset H \subset V^*$ and each space is dense in the following one and the injections are continuous.

We recall the Sobolev embeddings

$$H^1_0(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q < \infty \quad \text{if} \ n = 1, 2,$$

$$q = 6 \quad \text{if} \ n = 3, $$

$$H^1(\Omega) \hookrightarrow C(\overline{\Omega}) \quad \text{if} \ n = 1, $$

$$H^2(\Omega) \hookrightarrow C(\overline{\Omega}) \quad \text{if} \ n = 2, 3. $$

For the exponent $\gamma$, we assume that
0 ≤ γ < ∞ if n = 1, 2,
0 ≤ γ ≤ 2 if n = 3. (2.7)

We define the nonlinear scalar function $k: R \to R$ by $k(s) = |s|^\gamma s$, $\gamma \in R^+$. Then it is easy to see that $k \in C^1(R)$ and $k'(s) = (\gamma + 1)|s|^\gamma$. Under the assumption (2.7) with (2.4), for any $\phi \in H^1_0(\Omega)$ the convolution $k \circ \phi$ is square integrable, i.e., for the nonlinear operator $k: H^1_0(\Omega) \to L^2(\Omega) : u \mapsto k \circ u$ is well defined. Here we assume that $B \in \mathbb{L}(U, H)$, $f \in L^2(0, T; H)$, where $U$ is a real Hilbert space with norm denoted by $|\cdot|_U$ and the scalar product $(\cdot, \cdot)_U$.

Now problem (2.1)–(2.3) is formulated as the control system described by the abstract second-order differential systems in $H$:

$$
y'' + \alpha y' + \beta Ay + \delta k(y) = Bu + f \quad \text{on } (0, T),$$

$$
y(0) = y_0, \quad y'(0) = y_1. \quad (2.8)$$

where $' = \frac{d}{dt}$ and $'' = \frac{d^2}{dt^2}$. We shall denote by $Y$ the space $W^{1,2}([0, T]; H) \cap L^2(0, T; D(A))$, where $W^{1,2}([0, T]; H)$ is the space of all absolutely continuous functions $y:[0, T] \to H$ such that $y' = \frac{dy}{dt} \in L^2(0, T; H)$. We have that $Y \subset C([0, T]; V)$ [5]. We shall denote by $W(0, T)$ the space $(y: y \in L^2(0, T; V), \ y' \in L^2(0, T; H), \ y'' \in L^2(0, T; V^*))$ endowed with the norm $\|y\|_{W(0, T)} = [\int_0^T \|y\|^2 dt + \int_0^T |y'| dt + \int_0^T \|y''\|_{V^*} dt]^{\frac{1}{2}}$. We have that $W(0, T) \subset C([0, T]; H)$ [6, Chapter 4].

**Definition 2.1.** A function $y$ is said to be a weak solution of (2.8) if $y \in W(0, T)$ and

$$
\{y'', \phi\}_{(V^*, V)} + \alpha \{y', \phi\} + \beta \{y, \phi\} + \delta \{y', \phi\} = \{Bu, \phi\} + \{f, \phi\} \quad \text{for all } \phi \in V,
$$

$$
y(0) = y_0, \quad y'(0) = y_1.
$$

We note that $|y(t)|^\gamma y(t) \in H$ a.e. $t \in [0, T]$ if $y \in W(0, T)$.

**Remark 2.1.** The existence and uniqueness of strong solution of (2.8) follows from Theorem 4.1 in Teman [13, p. 214].

The cost functional we shall study in this paper is as follows:

$$
L(y, u) = \int_0^T \left[ g(t, y(t)) + h(u(t)) \right] dt + d_W(F(y)), \quad (2.9)
$$

where we assume the following:

(H1) $g: [0, T] \times V \to R^+$ is measurable in the first variable, $g(t, 0) \in L^\infty(0, T)$ and for every $r > 0$, there exists $L_r > 0$ independent of $t$ such that

$$
|g(t, y_1) - g(t, y_2)| \leq L_r \|y_1 - y_2\|, \quad \forall t \in [0, T],
$$

$$
\|y_1\| + \|y_2\| \leq r. \quad (2.10)
$$
A function \( h: U \rightarrow \bar{R} \equiv (-\infty, \infty] \) is convex and lower semicontinuous. Moreover, there exist \( c_1 > 0 \) and \( c_2 \in R \) such that

\[
h(u) \geq c_1 |u|^2_U - c_2, \quad \forall u \in U. \tag{2.11}
\]

The optimal control problem \((P)\) we shall study in this paper is as follows:

\[
\inf L(y, u) \quad \text{over all } (y, u) \in Y \times L^2(0,T;U) \tag{P}
\]

subject to

\[
y'' + \alpha y' + \beta Ay + \delta |y|^{\gamma} y = Bu + f \quad \text{a.e. in } (0,T), \forall \alpha \in R, \beta > 0, \delta \geq 0,
\]

\[
y(0) = y_0, \quad y'(0) = y_1, \tag{2.12}
\]

with

\[
F(y) \in W, \tag{2.13}
\]

where \( y_0, y_1 \in V \) and we assume the following:

\((H_2)\) \( F: L^2(0,T;V) \rightarrow X \) is continuously Fréchet differentiable, where \( X \) is a Banach space with the dual \( X^* \) strictly convex, \( W \subset X \) is a closed and convex subset and \( d_W(F(y)) \) is the distance of \( F(y) \) to \( W \) in \( X \).

**Theorem 2.1.** Under the hypotheses \((H_1)\) and \((H_2)\), problem \((P)\) has a unique solution \((y^*, u^*)\).

**Proof.** Since \( F(y) \in W \), it follows that \( d_W(F(y)) = 0 \). Let \( \{y_n, u_n\} \) be a minimizing sequence in problem \((P)\), i.e.,

\[
\inf(P) \leq L(y_n, u_n) \leq \inf(P) + \frac{1}{n}, \quad n = 1, 2, \ldots, \tag{2.14}
\]

\[
y''_n + \alpha y'_n + \beta A y_n + \delta |y_n|^{\gamma} y_n = B u_n + f, \quad \text{a.e. in } (0,T),
\]

\[
y_n(0) = y_0, \quad y'_n(0) = y_1. \tag{2.15}
\]

By \((2.11)\), it follows that \( \{u_n\} \) is bounded in \( L^2(0,T;U) \) and therefore on a subsequence, again denoted \( n \), we have

\[
u_n \rightharpoonup u^* \quad \text{weakly in } L^2(0,T;U). \tag{2.16}
\]

Here and throughout the sequel we shall denote by \( C \) several positive constants independent of \( y \) and \( n \). From \((2.15)\), we have

\[
\left|y'_n(t)\right|^2 + \beta \left\|y_n(t)\right\|^2 \leq \frac{\int_0^t \left|y'_n(s)\right|^2 ds + \frac{2\delta}{\gamma + 2} \left\|y_n(t)\right\|^{\gamma + 2}_{L^{\gamma + 2}(\Omega)}}{\left|y_1\right|^2 + \beta \left\|y_0\right\|^2 + \frac{1}{\varepsilon} \int_0^t \left|Bu_n(s)\right|^2 ds + \varepsilon \int_0^t \left|y'_n(s)\right|^2 ds} + \frac{1}{\eta} \int_0^t \left|f(s)\right|^2 ds + \eta \int_0^t \left|y'_n(s)\right|^2 ds \quad \text{for any } \varepsilon, \eta > 0. \tag{2.17}
\]
Here we used the Hölder inequality and Young inequality. Choosing $\varepsilon, \eta$ so small and using Gronwall's inequality, we obtain

\[ \|y_n(t)\|^2 + |y'_n(t)|^2 + \int_0^t |y''_n(s)|^2 \, ds + \|y_n(t)\|^\gamma_{L\gamma+2}(\Omega) \leq C. \quad (2.18) \]

From (2.15) and (2.18) we see that

\[ \|y''_n\|_{L^2(0,T;V^*)} \leq C. \quad (2.19) \]

Thus by (2.16), (2.18), and (2.19), the Arzela–Ascoli theorem and the Aubin compactness theorem, we conclude that there exist $(y^*, u^*) \in Y \times L^2(0, T; U)$ and subsequences of $\{y_n\}$ and $\{u_n\}$, still denoted by themselves, such that, as $n \to \infty$,

\[ y_n \to y^* \quad \text{strongly in } C([0, T]; H) \cap L^2(0, T; V), \]
\[ y'_n \to y'^* \quad \text{weakly in } L^2(0, T; H), \]
\[ y''_n \to y''^* \quad \text{weakly in } L^2(0, T; V'), \]
\[ Ay_n \to Ay^* \quad \text{weakly in } L^2(0, T; V'), \]
\[ |y_n|^\gamma y_n \to |y^*|^\gamma y^* \quad \text{weakly in } L^{\gamma+2}(0, T; H), \]
\[ u_n \to u^* \quad \text{weakly in } L^2(0, T; H). \quad (2.20) \]

From (2.20)–(2.25) we may pass to the lim $n \to \infty$ in (2.15) to derive that

\[ y''^* + \alpha y'^* + \beta Ay^* + \delta |y^*|^\gamma y^* = Bu^* + f \quad \text{a.e. in } (0, T). \]

By (2.10) and (2.20) we obtain that

\[ \liminf_{n \to \infty} \int_0^T g(t, y_n(t)) \, dt = \int_0^T g(t, y^*(t)) \, dt. \]

Since $h$ is convex and lower semicontinuous, it follows from (2.16) that

\[ \liminf_{n \to \infty} \int_0^T h(u_n) \, dt \geq \int_0^T h(u^*) \, dt. \quad (2.26) \]

From (2.14) and (2.16) we obtain

\[ L(y^*, u^*) = \inf(P). \]

Since the function $k(y) = |y|^\gamma y, \gamma \in \mathbb{R}^+$, is locally Lipschitz, the uniqueness of solution of problem $(P)$ can be proved as in Khalifa and Elgamal [8, pp. 451–475]. This completes the proof. \(\square\)
3. Maximum principle for problem $(P)$

Let $(y^*, u^*)$ be optimal for problem $(P)$. In this section, we shall state and prove the necessary conditions for $(y^*, u^*)$. First we recall approximations $g^\varepsilon$ of $g$ and $h^\varepsilon$ of $h$ as follows:

For the details, we refer the reader to [3, Chapter 1]. Let $P_m : V \rightarrow X_m$ be the projection operator, where $X_m$ is the finite-dimensional space generated by $\{e_i\}^{m}_{i=1}$ and $\{e_i\}^{\infty}_{i=1}$ is an orthonormal basis in $V$ and $\bigwedge_m : R^m \rightarrow X_m$ is defined by $\bigwedge_m (\tau) = \sum^{m}_{i=1} \tau_i e_i$, $\tau = (\tau_1, \ldots, \tau_m)$. Let $g^\varepsilon : [0, T] \times V \rightarrow R^+$ be defined by

$$g^\varepsilon(t, y) = \int_{R^m} g\left(t, P_m y - \varepsilon \bigwedge_m \tau\right) \rho_m(\tau) d\tau,$$

where $m = [\varepsilon^{-1}]$, $\rho_m$ is a mollifier on $R^m$, i.e., $\rho_m(\theta) = 0$ for $\|\theta\|_m > 1$, $\rho_m > 0$, $\int_{R^m} \rho_m(\theta) d\theta = 1$ and $\rho_m(\theta) = \rho_m(-\theta)$ for all $\theta \in R^m$. Let $h^\varepsilon : U \rightarrow R$ be defined by

$$h^\varepsilon(u) = \inf \left\{ \frac{|u - v|^2}{2\varepsilon} + h(v) : v \in U \right\}.$$  \hspace{1cm} (3.2)

Now, for each $\varepsilon > 0$, we define a penalty functional $L_\varepsilon : Y \times L^2(0, T; U) \rightarrow R$ by

$$L_\varepsilon(y, u) = \int^T_0 \left[ g^\varepsilon(t, y) + h^\varepsilon(u) \right] dt + \frac{1}{2} \int^T_0 \| y - y^* \|^2 dt + \frac{1}{2} \int^T_0 |u - u^*|^2 dt$$

$$+ \frac{1}{2\varepsilon} \int^T_0 |y'' + \alpha y' + \beta Ay + \delta |y|^{\gamma} y - Bu - f|^2 dt$$

$$+ \frac{1}{2\varepsilon} \left[ \varepsilon + d_W(F(y)) \right]^2,$$

where $d_W(F(y))$ denotes the distance of $F(y)$ to $W$ in $X$.

Since $y \in Y \subset C([0, T]; V)$, $L_\varepsilon$ is well defined and we may define

$$Y_0 = \{ y \in Y : y(0) = y_0, \; y'(0) = y_1 \}.$$  \hspace{1cm} (3.3)

Consider the approximation problem $(P_\varepsilon)$ as follows:

$$\inf L_\varepsilon(y, u) \quad \text{over all } (y, u) \in Y_0 \times L^2(0, T; U).$$ \hspace{1cm} (P_\varepsilon)

We have the following existence and approximation results for problem $(P_\varepsilon)$.

**Lemma 3.1.** For each $\varepsilon > 0$, problem $(P_\varepsilon)$ has at least one solution.

**Proof.** It is clear that $\inf(P_\varepsilon) > -\infty$. Let $(y_n, u_n) \in Y_0 \times L^2(0, T; U)$ be such that

$$\inf(P_\varepsilon) \leq L_\varepsilon(y_n, u_n) \leq \inf(P_\varepsilon) + \frac{1}{n}, \quad n = 1, 2, \ldots,$$

where $y_n, u_n$ depend on $\varepsilon$. By (3.3) and (3.4), we imply that

$$\|u_n\|_{L^2(0, T; U)} \leq C$$  \hspace{1cm} (3.5)

and
\[ \|y_n\|_{L^2(0,T;V)} \leq C, \quad (3.6) \]

here and throughout the proof of Lemma 3.1, we shall denote by \( C \) several positive constants independent of \( n \). By (3.3) and (3.4) again, there exist \( f_n \in L^2(0,T;H) \), \( n = 1, 2, \ldots \), such that

\[ \|f_n\|_{L^2(0,T;H)} \leq C \quad (3.7) \]

and

\[ y''_n + \alpha y'_n + \beta A y_n + \delta |y_n| \gamma y_n = Bu_n + f_n \quad \text{a.e. in } (0,T), \quad \forall \alpha \in \mathbb{R}, \quad \beta > 0, \quad \delta \geq 0, \]
\[ y_n(0) = y_0, \quad y'_n(0) = y_1. \quad (3.8) \]

Multiplying (3.8) by \( y'_n \), integrating on \((0,t)\), we get that

\[ \|y'_n(t)\|^2 + \beta \|y_n(t)\|^2 + 2\alpha \int_0^t |y'_n(s)|^2 \, ds + \frac{2\delta}{\gamma + 2} \|y_n(t)\|_{L^{\gamma + 2}(\Omega)}^{\gamma + 2} \]
\[ = |y_1|^2 + \beta \|y_0\|^2 + 2 \int_0^t (Bu_n(s), y'_n(s)) \, ds + 2 \int_0^t (f_n(s), y'_n(s)) \, ds \]
\[ \leq |y_1|^2 + \beta \|y_0\|^2 + \frac{1}{\varepsilon} \int_0^t |Bu_n(s)|^2 \, ds + \varepsilon \int_0^t |y'_n(s)|^2 \, ds \]
\[ + \frac{1}{\eta} \int_0^t |f_n(s)|^2 \, ds + \eta \int_0^t |y'_n(s)|^2 \, ds. \]

By Gronwall’s inequality, we obtain

\[ \|y_n(t)\|^2 + |y'_n(t)|^2 + \int_0^t |y'_n(s)|^2 \, ds + \|y_n(t)\|_{L^{\gamma + 2}(\Omega)}^{\gamma + 2} \leq C. \quad (3.9) \]

From (3.8) and (3.9) we obtain that

\[ \|y''_n\|_{L^2(0,T;V^*)} \leq C. \quad (3.10) \]

Thus by (3.5), (3.6), (3.9) and (3.10), the Arzela–Ascoli theorem and the Aubin compactness theorem \([2]\), we conclude that there exist \((\bar{y}, \bar{u}) \in Y \times L^2(0,T;U)\) and subsequences of \(\{y_n\}\) and \(\{u_n\}\), still denoted by themselves, such that, as \( n \to \infty \),

\[ y_n \to \bar{y} \quad \text{strongly in } C([0,T];H) \cap L^2(0,T;V), \quad (3.11) \]
\[ y'_n \to \bar{y}' \quad \text{weakly in } L^2(0,T;H), \quad (3.12) \]
\[ y''_n \to \bar{y}'' \quad \text{weakly in } L^2(0,T;V'), \quad (3.13) \]
\[ Ay_n \to A\bar{y} \quad \text{weakly in } L^2(0,T;V'), \quad (3.14) \]
\[ |y'_n|^{\gamma} y_n \to |\bar{y}'|^{\gamma} \bar{y} \quad \text{weakly in } L^{\gamma + 2}(0,T;H), \quad (3.15) \]
\[ u_n \to \bar{u} \quad \text{weakly in } L^2(0,T;H). \quad (3.16) \]
By (3.11)–(3.16), we infer that
\[
\lim_{n \to \infty} \int_0^T \|y_n - y^*\|^2 \, dt \geq \int_0^T \|\tilde{y} - y^*\|^2 \, dt \tag{3.17}
\]
and
\[
\lim_{n \to \infty} \int_0^T \left| y''_n + \alpha y'_n + \beta A y_n + \delta |y_n|^\gamma y_n - Bu_n - f \right|^2 \, dt \\
\geq \int_0^T \left| \tilde{y}'' + \alpha \tilde{y}' + \beta A \tilde{y} + \delta |\tilde{y}|^\gamma \tilde{y} - B\tilde{u} - f \right|^2 \, dt. \tag{3.18}
\]
From (2.10), (3.1), (3.9) and (3.11) we have that
\[
\int_0^T \left| g^\varepsilon(t, y_n) - g^\varepsilon(t, \tilde{y}) \right| \, dt \leq L \int_0^T \|y_n - \tilde{y}\| \, dt \\
\to 0 \quad \text{as} \quad n \to \infty, \tag{3.19}
\]
where \(L > 0\) is independent of \(n\). Since \(h_\varepsilon\) is convex and lower semicontinuous, it follows from (3.16) that
\[
\lim_{n \to \infty} \int_0^T h_\varepsilon(u_n) \, dt \geq \int_0^T h_\varepsilon(\tilde{u}) \, dt. \tag{3.20}
\]
By (3.11) and (H2), \(F(y_n) \to F(\tilde{y})\) as \(n \to \infty\). Thus we have
\[
\frac{1}{2\varepsilon} \left( \varepsilon + d_W(F(y_n)) \right)^2 \to \frac{1}{2\varepsilon} \left( \varepsilon + d_W(F(\tilde{y})) \right)^2 \quad \text{as} \quad n \to \infty. \tag{3.21}
\]
On the other hand, since \(y_n(0) = y_0\), \(y_n(0) \to \tilde{y}(0)\) strongly in \(V\) and \(y'_n(0) = y_1\), \(y'_n(0) \to \tilde{y}'(0)\) weakly in \(H\) by (3.8) and (3.12) we have that \(\tilde{y}(0) = y_0\), which shows that \(\tilde{y} \in Y_0\). Thus it follows immediately from (3.4) and (3.17)–(3.21) that \((\tilde{y}, \tilde{u})\) is optimal for problem \((P_\varepsilon)\). This completes the proof. \(\square\)

**Lemma 3.2.** Let \((y_\varepsilon, u_\varepsilon)\) be optimal for problem \((P_\varepsilon)\). Then there exists a generalized subsequence of \((y_\varepsilon, u_\varepsilon)\), still denoted by itself, such that
\[
\begin{align*}
u_\varepsilon & \to u^* \quad \text{strongly in} \ L^2(0, T; U), \\
y_\varepsilon & \to y^* \quad \text{strongly in} \ Y \quad \text{as} \quad \varepsilon \to 0.
\end{align*}
\]

**Proof.** Since \((y_\varepsilon, u_\varepsilon)\) is optimal for \((P_\varepsilon)\), it follows from (3.3) that
\[
L_\varepsilon(y_\varepsilon, u_\varepsilon) \leq L_\varepsilon(y^*, u^*) \\
\leq \int_0^T \left[ g^\varepsilon(t, y^*) + h_\varepsilon(u^*) \right] \, dt + \frac{\varepsilon}{2}. \tag{3.22}
\]
By (3.1) and by the same argument as in [3, Chapter 3], we obtain
\[ |g^\varepsilon(t, y^\varepsilon) - g(t, y^*)| \leq L\left(\|y^* - P_my^\varepsilon\| + \varepsilon\right), \tag{3.23} \]
where \( L > 0 \) is a constant independent of \( \varepsilon \). By (3.22), (3.23) and the same argument as in [9], we get that
\[ \lim_{\varepsilon \to 0} L(y^\varepsilon, u^\varepsilon) \leq L\left(y^*, u^*\right). \tag{3.24} \]

On the other hand, it follows from (3.3) and (3.24) that
\[ \|y^\varepsilon\|_{L^2(0,T;V)} + \|u^\varepsilon\|_{L^2(0,T;U)} \leq C, \tag{3.25} \]
\[ \int_0^T \left| y''^\varepsilon + \alpha y'^\varepsilon + \beta Ay^\varepsilon + \delta|y^\varepsilon|^\gamma y^\varepsilon - Bu^\varepsilon - f \right|^2 \, dt \leq 2C, \tag{3.26} \]
and
\[ \left[ \varepsilon + d_W(F(y^\varepsilon)) \right]^2 \leq 2C. \tag{3.27} \]

Here and throughout the proof of Lemma 3.2, we shall denote by \( C \) several positive constants independent of \( \varepsilon \). By (3.3) and (3.24), there exists \( v^\varepsilon \in L^2(0,T;H) \) for each \( \varepsilon > 0 \) such that
\[ y''^\varepsilon + \alpha y'^\varepsilon + \beta Ay^\varepsilon + \delta|y^\varepsilon|^\gamma y^\varepsilon = Bu^\varepsilon + v^\varepsilon + f \quad \text{a.e. in } (0,T), \]
\[ y^\varepsilon(0) = y_0, \quad y'^\varepsilon(0) = y_1. \tag{3.28} \]

By (3.25) and (3.28), using the same argument as in the proof of Lemma 3.1, we obtain that there exist \( \tilde{y} \in Y, \tilde{u} \in L^2(0,T;U) \) and subsequences of \( \{y^\varepsilon\} \) and \( \{u^\varepsilon\} \), still denoted by themselves, such that, as \( \varepsilon \to 0 \),
\[ y^\varepsilon \to \tilde{y} \quad \text{strongly in } C([0,T];H) \cap L^2(0,T;V), \tag{3.29} \]
\[ \|y^\varepsilon\|_{C([0,T];V)} \leq C, \tag{3.30} \]
\[ y'^\varepsilon \to \tilde{y}' \quad \text{weakly in } L^2(0,T;H), \tag{3.31} \]
\[ y''^\varepsilon \to \tilde{y}'' \quad \text{weakly in } L^2(0,T;V'), \tag{3.32} \]
and
\[ u^\varepsilon \to \tilde{u} \quad \text{weakly in } L^2(0,T;U). \tag{3.33} \]

By (3.29) and (3.30), we see that
\[ |y^\varepsilon|^\gamma y^\varepsilon \to |	ilde{y}|^\gamma \tilde{y} \quad \text{weakly in } L^{\gamma+2}(0,T;H) \text{ as } \varepsilon \to 0. \tag{3.34} \]

From (3.29)–(3.34) we may pass to the limit for \( \varepsilon \to 0 \) in (3.28) to derive that
\[ \tilde{y}'' + \alpha \tilde{y}' + \beta A\tilde{y} + \delta|\tilde{y}|^\gamma \tilde{y} = B\tilde{u} + f \quad \text{a.e. in } (0,T), \]
\[ \tilde{y}(0) = y_0, \quad \tilde{y}'(0) = y_1. \tag{3.35} \]

It follows from (3.29) and (H_2) that
\[ F(y_\varepsilon) \to F(\bar{y}) \text{ strongly in } X \text{ as } \varepsilon \to 0. \] (3.36)

However, by (3.27), we have that \( d_W(F(y_\varepsilon)) \to 0 \) as \( \varepsilon \to 0 \), which, combined with (3.36), indicates that

\[ F(\bar{y}) \in W, \] (3.37)

since \( W \) is closed. Thus, by (3.35) and (3.37), we have

\[ L(y^*, u^*) \leq L(\bar{y}, \bar{u}), \] (3.38)

because \((y^*, u^*)\) is optimal for \((P_\varepsilon)\). Now from (2.10), (3.1), (3.29) and using the same argument as in [4, Proposition 2.15], we get that

\[ \|g_\varepsilon(t, y_\varepsilon) - g_\varepsilon(t, \bar{y})\| \leq L \|y_\varepsilon - \bar{y}\|, \] (3.39)

\[ \lim_{\varepsilon \to 0} g_\varepsilon(t, \bar{y}(t)) = g(t, \bar{y}(t)), \quad \forall t \in [0, T], \] (3.40)

\[ \|g_\varepsilon(t, \bar{y}(t)) - g(t, \bar{y}(t))\| \leq L(\|\bar{y} - P_m \bar{y}\| + \varepsilon), \] (3.41)

where \( L > 0 \) is independent of \( \varepsilon \) and \( P_m \) was given (3.1). Thus, by (3.39)–(3.41) and the Lebesgue dominated convergence theorem, we obtain

\[ \int_0^T \int_0^T \left| g_\varepsilon(t, y_\varepsilon) - g(t, \bar{y}) \right| dt \leq \int_0^T \left[ \|g_\varepsilon(t, y_\varepsilon) - g_\varepsilon(t, \bar{y})\| + \|g_\varepsilon(t, \bar{y}) - g(t, \bar{y})\| \right] dt \]

\[ \to 0 \text{ as } \varepsilon \to 0. \] (3.42)

By the same argument as in [3, Chapter 5], we deduce that

\[ \lim_{\varepsilon \to 0} \int_0^T \left[ h_\varepsilon(u_\varepsilon) + \frac{1}{2} \|u_\varepsilon - u^*\|^2 \right] dt \geq \int_0^T \left[ h(\bar{u}) + \frac{1}{2} \|\bar{u} - u^*\|^2 \right] dt. \] (3.43)

Now it follows from (3.38), (3.42) and (3.43) that

\[ \lim_{\varepsilon \to 0} L_\varepsilon(y_\varepsilon, u_\varepsilon) \geq L(\bar{y}, \bar{u}) \geq L(y^*, u^*). \] (3.44)

Thus, from (3.24) and (3.44), we infer that \( \bar{u} = u^*, \bar{y} = y^*, \)

\[ u_\varepsilon \to u^* \text{ strongly in } L^2(0, T; U) \text{ as } \varepsilon \to 0, \] (3.45)

and

\[ y_\varepsilon \to y^* \text{ strongly in } L^2(0, T; U) \text{ as } \varepsilon \to 0. \] (3.46)

Finally, we shall prove that \( y_\varepsilon \to y^* \) strongly in \( Y \) as \( \varepsilon \to 0 \). To this end, we first observe that

\[ (y_\varepsilon - y^*)'' + \alpha(y_\varepsilon - y^*)' + \beta A(y_\varepsilon - y^*) + \delta (|y_\varepsilon|^p y_\varepsilon - |y^*|^p y^*) = B(u_\varepsilon - u^*) + v_\varepsilon \]

a.e. in \((0, T)\),

\[ (y_\varepsilon - y^*)'(0) = 0, \quad (y_\varepsilon - y^*)'(0) = 0. \] (3.47)
Multiplying (3.47) by \((y_\varepsilon - y^*)'\) and integrating on \((0, t)\), we get
\[
\frac{1}{2} |y_\varepsilon'(t) - y^*(t)|^2 + \alpha \int_0^T |y_\varepsilon - y^*|^2 \, dt + \frac{\beta}{2} \|y_\varepsilon - y^*\| \\
\leq \frac{\alpha}{2} \int_0^T |y_\varepsilon' - y^*|^2 \, dt \\
+ C_\varepsilon \left[ \int_0^T \left( \left( |y_\varepsilon'| + |y^*'| \right) |y_\varepsilon - y^*| + |B(u_\varepsilon - u^*)|^2_U + |V_\varepsilon|^2 \right) \, dt \right].
\]

This together with (3.45) and (3.46) yields that
\[
y_\varepsilon' \to (y^*)' \quad \text{strongly in } L^2(0, T; H) \text{ as } \varepsilon \to 0.
\]
Hence
\[
y_\varepsilon \to y^* \quad \text{strongly in } Y \text{ as } \varepsilon \to 0.
\]
This completes the proof. \(\square\)

Now we are in a position to state and prove the necessary conditions for \((y^*, x^*)\). By [6, Theorem 1.2, Chapter 3], the Cauchy problems
\[
\begin{align*}
\varphi'' + \alpha \varphi' + \beta A \varphi + (\gamma + 1) \delta |y|^\gamma \varphi &= f \quad \text{a.e. in } (0, T), \\
\varphi(0) &= x_0, \quad \varphi'(0) = x_1,
\end{align*}
\]
and
\[
\begin{align*}
\psi'' - \alpha \psi' + \beta A \psi + (\gamma + 1) \delta |y|^\gamma \psi &= f \quad \text{a.e. in } (0, T), \\
\psi(T) &= x_0, \quad \psi'(T) = x_1,
\end{align*}
\]
have unique solutions \(\varphi\) and \(\psi\) in \(W(0, T)\) for each \(f \in L^2(0, T; V^*)\), \(x_0 \in V\) and \(x_1 \in H\), respectively. Moreover,
\[
\|\varphi''\|_{L^2(0, T; V^*)} + \|\varphi'*\|^2_{L^2(0, T; V)} + \|\varphi\|^2_{L^2(0, T; V)} \leq C \left( \|f\|^2_{L^2(0, T; V^*)} + \|x_0\|^2 + |x_1|^2 \right)
\]
and
\[
\|\psi''\|_{L^2(0, T; V^*)} + \|\psi'*\|^2_{L^2(0, T; H)} + \|\psi\|^2_{L^2(0, T; V)} \leq C \left( \|f\|^2_{L^2(0, T; V^*)} + \|x_0\|^2 + |x_1|^2 \right).
\]
In order to get the necessary condition for \((y^*, u^*)\), we need one more assumption as follows.

\((H_3)\) The set \(F'(y^*)R_r - W\) has finite codimensionality in \(X\) for some \(r > 0\), where
\[
M(0, r) = \{ v \in L^2(0, T; U) : \|v\|_{L^2(0, T; U)} \leq r \}
\]
and
\[
R_r = \{ z \in Y : z'' + \alpha z' + \beta Az + (\gamma + 1) \delta |y^*|^\gamma z = Bv \text{ a.e. in } (0, T) \}
\]
and \(z(0) = 0\) for some \(v \in M(0, r)\).
For the definition of a set to be finite codimensional in $X$ and for related results, we refer the reader to [10]. Throughout what follows, we shall denote by $\partial g(t, y^*)$ the generalized derivative of $g$ to the second variable at $y^*$ and by $\partial h(u^*)$ the subdifferential of $h$ at $u^*$. For the details, we refer the reader to [3]. We denote by $\langle \cdot, \cdot \rangle_{X^*, X}$ the pairing between $X^*$ and $X$ and by $[F'(y^*)]^*$ and $B^*$ the adjoint operators of $F'(y^*)$ and $B$, respectively.

**Theorem 3.1.** Suppose that (H1)–(H3) holds. Let $(y^*, u^*)$ be optimal for problem (P). Then there exists a triplet $(\lambda_0, \rho, \xi_0) \in \mathbb{R} \times L^2(0, T; V) \cap W(0, T) \times X^*$ with $(\lambda_0, \xi_0) \neq 0$ such that

$$
p'' - \alpha p' + \beta Ap + (\gamma + 1)\delta |y^*| \gamma p + [F'(y^*)]^* \xi_0 \in -\lambda_0 \partial g(t, y^*) \quad \text{a.e. in} \ (0, T),
$$

$$
p'(T) = p(T) = 0,
$$

$$
\langle \xi_0, w - F(y^*) \rangle_{X^*, X} \leq 0, \quad \forall w \in W,
$$

and

$$
B^* p(t) \in \lambda_0 \partial h(u^*(t)) \quad \text{a.e. in} \ (0, T).
$$

Moreover, if $F'(y^*)$ is injective, then $(\lambda_0, \rho) \neq 0$.

**Proof.** Let $Z = \{y \in Y: y(0) = 0, \ y'(0) = 0\}$. For $z \in Z, v \in L^2(0, T; U)$, we set $y^\rho_\varepsilon = y_\varepsilon + \rho z$, $u^\rho_\varepsilon = u_\varepsilon + \rho v$, where $(y_\varepsilon, u_\varepsilon)$ is optimal for problem $(P_\varepsilon)$. It is clear that $y^\rho_\varepsilon \in Y, u^\rho_\varepsilon \in L^2(0, T; U)$,

$$
y^\rho_\varepsilon \to y_\varepsilon \quad \text{strongly in} \ Y \text{ as} \ \rho \to 0,
$$

and

$$
u^\rho_\varepsilon \to u_\varepsilon \quad \text{strongly in} \ L^2(0, T; U) \text{ as} \ \rho \to 0.
$$

One can easily check that

$$
\left( \delta |y^\rho_\varepsilon| \gamma y^\rho_\varepsilon - \delta |y_\varepsilon| \gamma y_\varepsilon \right) / \rho \to (\gamma + 1)\delta |y_\varepsilon| \gamma z \quad \text{as} \ \rho \to 0.
$$

(3.54)

It follows from (3.54) that

$$
\frac{1}{2\varepsilon\rho} \int_0^T \left[ \left( y^\rho_\varepsilon \right)'' + \alpha (y^\rho_\varepsilon)' + \beta Ay^\rho_\varepsilon + \delta |y^\rho_\varepsilon| \gamma y^\rho_\varepsilon - Bu^\rho_\varepsilon - f \right]^2 dt
$$

$$
- \left| y^{\rho_\varepsilon} + \alpha y'_\varepsilon + \beta Ay_\varepsilon + \delta |y_\varepsilon| \gamma y_\varepsilon - Bu_\varepsilon - f \right|^2 dt
$$

$$
\int_0^T q_\varepsilon z'' + \alpha z' + \beta Az + (\gamma + 1)|y_\varepsilon| \gamma z - Bu_\varepsilon \right] dt \quad \text{as} \ \rho \to 0,
$$

(3.55)

where $q_\varepsilon = \frac{1}{\varepsilon} [y''_\varepsilon + \alpha y'_\varepsilon + \beta Ay_\varepsilon + \delta |y_\varepsilon| \gamma y_\varepsilon - Bu_\varepsilon - f]$. Since $\langle Ay, z \rangle = \langle y, z \rangle_V$ for all $y \in D(A), z \in V$, we infer that

$$
\lim_{\rho \to 0} \frac{1}{2\rho} \int_0^T \left[ \left\| y^\rho_\varepsilon - y^* \right\|^2 - \left\| y_\varepsilon - y^* \right\|^2 \right] dt = \int_0^T \left\langle y_\varepsilon - y^*, z \right\rangle_V dt
$$

$$
= \int_0^T \left\langle A(y_\varepsilon - y^*), z \right\rangle dt.
$$

(3.56)
By the same argument as in [3, Chapter 5], we see that
\[
\lim_{\rho \to 0} \frac{1}{\rho} \int_0^T \left[ g^\varepsilon(t, y^\rho_\varepsilon) - g^\varepsilon(t, y_\varepsilon) \right] dt = \int_0^T \langle \nabla g^\varepsilon(t, y_\varepsilon), z \rangle dt \tag{3.57}
\]
and
\[
\lim_{\rho \to 0} \frac{1}{\rho} \int_0^T \left[ \left\{ h^\varepsilon(u^\rho_\varepsilon) - h^\varepsilon(u_\varepsilon) \right\} + \frac{1}{2} \left\{ \| u^\rho_\varepsilon - u^* \|^2 - \| u_\varepsilon - u^* \|^2 \right\} \right] dt
= \int_0^T \langle \nabla h^\varepsilon(u_\varepsilon) + u_\varepsilon - u^*, u \rangle_U dt. \tag{3.58}
\]

By the argument as in [15], we get that
\[
\lim_{\rho \to 0} \left[ (\varepsilon + dW(F(y^\rho_\varepsilon)))^2 - (\varepsilon + dW(F(y_\varepsilon)))^2 \right]
= \frac{\varepsilon + dW(F(y_\varepsilon))}{\varepsilon} \left\{ \xi_\varepsilon, F'(y_\varepsilon)z \right\}_{X^*, X}, \tag{3.59}
\]
where \( \nabla g^\varepsilon(t, y_\varepsilon) \) denotes the gradient of \( g^\varepsilon \) to the second variable at \( y_\varepsilon \) and \( \nabla h^\varepsilon(u_\varepsilon) \) denotes the gradient of \( h^\varepsilon \) at \( u_\varepsilon \), while \( \xi_\varepsilon \in \partial dW(F(y_\varepsilon)) \). Moreover,
\[
\left\{ \begin{array}{ll}
\| \xi_\varepsilon \|_{X^*} = 1 & \text{if } F(y_\varepsilon) \notin W, \\
0 & \text{if } F(y_\varepsilon) \in W,
\end{array} \right. \tag{3.60}
\]
because \( W \) is convex and closed and \( X^* \) is strictly convex [11, Chapter 5].

Since \((L^\rho_\varepsilon(y^{\rho_\varepsilon}_\varepsilon, u^{\rho_\varepsilon}_\varepsilon) - L^\rho_\varepsilon(y_\varepsilon, u_\varepsilon)) / \rho \geq 0 \) for all \( \rho > 0 \), it follows from (3.3) and (3.55)–(3.59) that
\[
0 \leq \lambda_\varepsilon \int_0^T \left\{ \langle \nabla g^\varepsilon(t, y_\varepsilon), z \rangle + \langle \nabla h^\varepsilon(u_\varepsilon), v \rangle_U \\
+ \| y_\varepsilon - y^* \|^2 [A(y_\varepsilon - y^*), z] + [u_\varepsilon - u^*, v]_U \right\} dt
+ \int_0^T \langle p_\varepsilon, z'' + \alpha z' + \beta Az + \delta(y + 1)|y_\varepsilon|y_\varepsilon|z - Bv \rangle dt
+ \int_0^T \left\{ \left[ F'(y_\varepsilon) \right]^* \xi_\varepsilon, z \right\} dt, \quad \forall z \in Z, \ v \in L^2(0, T; U), \tag{3.61}
\]
where
\[
\lambda_\varepsilon = \frac{\varepsilon}{\varepsilon + dW(F(y_\varepsilon))}, \quad p_\varepsilon = \lambda_\varepsilon q_\varepsilon \in L^2(0, T; H). \tag{3.62}
\]

By taking \( z = 0 \) in (3.61), we obtain
\[
B^* p_\varepsilon = \lambda_\varepsilon \nabla h^\varepsilon(u_\varepsilon) + \lambda_\varepsilon (u_\varepsilon - u^*) \quad \text{a.e. in } (0, T), \tag{3.63}
\]
while, by taking \( v = 0 \) in (3.61), we get that

\[
0 = \int_0^T \left( \lambda \nabla g^\varepsilon(t, y_\varepsilon) + \left( F'(y_\varepsilon) \right)^* \xi_\varepsilon + \lambda \left\| y_\varepsilon - y^* \right\|^2 A(y_\varepsilon - y^*) \right) dt + \int_0^T \langle p_\varepsilon, z'' + \alpha z' + \beta Az + \delta(\gamma + 1)|y_\varepsilon|\gamma z \rangle dt, \quad \forall z \in Z. \tag{3.64}
\]

We may regard (3.63) and (3.64) as the necessary conditions for \((y_\varepsilon, u_\varepsilon)\).

Now we are in a position to pass to the limit for \( \varepsilon \to 0 \) in (3.63) and (3.64) to derive (3.53) and (3.51), respectively.

First we deal with (3.63). Note that

\[
\alpha_\varepsilon \equiv \lambda \nabla g^\varepsilon(t, y_\varepsilon) + \left( F'(y_\varepsilon) \right)^* \xi_\varepsilon + \lambda \left\| y_\varepsilon - y^* \right\|^2 A(y_\varepsilon - y^*) \in L^2(0, T; V^*)
\]

and \( \{\alpha_\varepsilon\}_{\varepsilon > 0} \) is bounded in \( L^2(0, T; V^*) \). By (3.48b), we may let \( p_{\varepsilon 1} \in W(0, T) \) be the solution to

\[
p_{\varepsilon 1}'' - \alpha p_{\varepsilon 1}' + \beta A p_{\varepsilon 1} + (\gamma + 1)\delta |y_\varepsilon|\gamma y_\varepsilon = 0 \quad \text{a.e. in } (0, T),
\]

\[
p_{\varepsilon 1}'(T) = p_{\varepsilon 1}(T) = 0. \tag{3.65}
\]

Multiplying (3.65) by \( z \) and integrating on \((0, t)\), we have

\[
\int_0^T \langle p_{\varepsilon 1}, z'' + \alpha z' + \beta Az + (\gamma + 1)\delta |y_\varepsilon|\gamma z \rangle dt = -\int_0^T \langle \alpha_\varepsilon, z \rangle dt.
\]

This together with (3.64) implies that

\[
\int_0^T \langle p_\varepsilon - p_{\varepsilon 1}, z'' + \alpha z' + \beta Az + (\gamma + 1)\delta |y_\varepsilon|\gamma z \rangle dt = 0, \quad \forall z \in Z. \tag{3.66}
\]

By (3.48a), for each \( f \in L^2(0, T; H) \), there exists \( z \in Z \) such that

\[
z'' + \alpha z' + \beta Az + (\gamma + 1)\delta |y_\varepsilon|\gamma z = f \tag{in (0, T)}
\]

in \((0, T)\). Thus it follows from (3.66) that \( p_\varepsilon(t) = p_{\varepsilon 1}(t) \) a.e. in \((0, T)\). So, \( p_\varepsilon \in W(0, T) \) and satisfies

\[
\left\| p_\varepsilon'' \right\|^2_{L^2(0, T; V^*)} + \left\| p_\varepsilon' \right\|^2_{L^2(0, T; H)} + \left\| p_\varepsilon \right\|^2_{L^2(0, T; V)} \leq C. \tag{3.67}
\]

By the Aubin compactness theorem and the trace theorem [7, Theorem 3.1 of Chapter 1], there exist \( p \in W(0, T) \) and a subsequence of \( p_\varepsilon \), still denoted by itself, such that

\[
p_\varepsilon \to p \quad \text{strongly in } L^2(0, T; H),
\]

\[
p_\varepsilon' \to p' \quad \text{weakly in } L^2(0, T; V) \text{ as } \varepsilon \to 0,
\]

\[
p_\varepsilon'' \to p'' \quad \text{weakly in } L^2(0, T; V^*) \text{ as } \varepsilon \to 0,
\]

\[
p_\varepsilon(0) \to p(0) \quad \text{weakly in } V \text{ as } \varepsilon \to 0,
\]

\[
p_\varepsilon'(0) \to p'(0) \quad \text{weakly in } H \text{ as } \varepsilon \to 0. \tag{3.68}
\]
By (3.60) and (3.62), we have
\[ 1 \leq \lambda_\varepsilon + \| \xi_\varepsilon \|_{X^*} \leq 2, \quad \forall \varepsilon > 0. \] (3.69)
Thus there exist generalized subsequences of \( \{ \lambda_\varepsilon \} \) and \( \{ \xi_\varepsilon \} \) such that
\[ \lambda_\varepsilon \to \lambda_0 \quad \text{as} \quad \varepsilon \to 0, \] (3.70)
and
\[ \xi_\varepsilon \to \xi_0 \quad \text{weakly}^* \text{ in} \quad X^* \quad \text{as} \quad \varepsilon \to 0. \] (3.71)

By Lemma 3.2 and (3.68), using the same argument as in [4], we may pass to the limit for \( \varepsilon \to 0 \) in (3.63) to derive (3.53).

Next we deal with (3.64), i.e., pass to the limit for \( \varepsilon \to 0 \) in (3.64).

By Lemma 3.2 and by the same argument as in [3, Chapter 5], we infer that
\[ \nabla g_\varepsilon(t, y_\varepsilon) \to \eta \quad \text{weakly in} \quad L^2(0, T; V^*) \]
and
\[ \eta(t) \in \partial g(t, y^*(t)) \quad \text{a.e. in} \quad (0, T). \] (3.72)

By \((H_2)\), Lemma 3.2 and (3.71), we obtain that
\[ \left[ F'(y_\varepsilon) \right]^* \xi_\varepsilon \to \left[ F'(y^*) \right]^* \xi_0 \quad \text{weakly in} \quad L^2(0, T; V^*) \quad \text{as} \quad \varepsilon \to 0. \] (3.73)

From Lemma 3.2 again, we obtain that
\[ \| y_\varepsilon - y^* \|^2 A(y_\varepsilon - y^*) \to 0 \quad \text{strongly in} \quad L^2(0, T; H). \] (3.74)

Moreover, we see that
\[ |y_\varepsilon|^{\gamma} p_\varepsilon \to |y^*|^{\gamma} p \quad \text{weakly}^* \text{ in} \quad L^2(0, T; V^*) \quad \text{as} \quad \varepsilon \to 0. \] (3.75)

By (3.68), (3.70) and (3.72)--(3.75), we may pass to the limit for \( \varepsilon \to 0 \) in (3.65) to obtain that \( p \in W(0, T) \) and satisfies (3.51).

On the other hand, since \( \xi_\varepsilon \in \partial dW(F(y_\varepsilon)) \), we must have
\[ \langle \xi_\varepsilon, w - F(y_\varepsilon) \rangle_{X^*, X} \leq 0, \quad \forall \, w \in W, \]
where \( \partial dW \) denotes the subdifferential of \( dW \). This implies that
\[ \langle \xi_\varepsilon, w - F(y^*) \rangle_{X^*, X} \leq \langle \xi_\varepsilon, F(y_\varepsilon) - F(y^*) \rangle_{X^*, X}. \] (3.76)

Then, by Lemma 3.2, \((H_2)\) and (3.71), we may pass to the limit for \( \varepsilon \to 0 \) in (3.76) to get (3.52).

We have proved (3.51)--(3.53). Now we are in a position to show that \( (\lambda_0, \xi_0) \neq 0 \). To this end, we suppose that \( \lambda_0 = 0 \). Then, by (3.69) and (3.70), there exist \( \varepsilon_1 > 0 \) and \( \delta > 0 \) such that
\[ 2 \geq \| \xi_\varepsilon \|_{X^*} \geq \delta > 0, \quad \forall \varepsilon < \varepsilon_1. \] (3.77)

It follows from (3.61) and (3.76) that
\[ -\eta_{\varepsilon}(z, v) \leq \langle \xi_\varepsilon, F'(y^*)z - w + F(y^*) \rangle_{X^*, X} \]
\[ + \int_0^T \langle p_\varepsilon, z'' + \alpha z' + \beta Az + (\gamma + 1)\delta |y^*|^{\gamma} z - Bv \rangle dt \] (3.78)
for all \((z, v) \in Z \times L^2(0, T; U)\) and \(w \in W\), where

\[
\eta_\varepsilon(z, v) = \lambda_\varepsilon \left[ \int_0^T \left\{ \langle \nabla g_\varepsilon(t, y_\varepsilon), z \rangle + \langle \nabla h_\varepsilon(u_\varepsilon), v \rangle \right\}_U \right] dt
\]

\[
+ \int_0^T \left\{ \| y_\varepsilon - y^* \|^2 \mathcal{A} (y_\varepsilon - y^*), z \right\}_U dt + \int_0^T \{ u_\varepsilon - u^*, v \}_U dt \right]\]

\[
+ \int_0^T \langle p_\varepsilon, (\gamma + 1)\delta (|y_\varepsilon|^\gamma - |y^*|^\gamma z) \rangle dt
\]

\[
+ \langle \xi_\varepsilon, (F'(y_\varepsilon) - F'(y^*)) z + F(y_\varepsilon) - F(y^*) \rangle_{X^*, X}.
\]

(3.79)

For each \(\varepsilon > 0\) and \(v \in M(0, r)\), where \(M(0, r)\) was given in (3.49) and \(r > 0\) was given in \((H_3)\), let \(z_\varepsilon(v)\) be the solution to (3.48a) with \(f = Bv\) and \(x = 0\). Then \(z_\varepsilon(v) \in Z\) and

\[
\| z''_\varepsilon(v) \|^2_{L^2(0,T;V')} + \| z'_\varepsilon(v) \|^2_{L^2(0,T;H)} + \| z_\varepsilon(v) \|^2_{L^2(0,T;V)} \leq C, \quad \forall v \in M(0, r),
\]

(3.80)

where \(C > 0\) is independent of \(\varepsilon\) and \(v\). Moreover, we see that

\[
\int_0^T \langle p_\varepsilon, (\gamma + 1)\delta (|y_\varepsilon|^\gamma - |y^*|^\gamma z) \rangle dt
\]

\[
\leq (\gamma + 1)\delta \| p_\varepsilon \|_{L^2(0,T;V)} \| z_\varepsilon(v) \|_{L^2(0,T;V)} (|y_\varepsilon| - |y^*|)
\]

\[
\times (|y_\varepsilon|^\gamma - 1 + |y_\varepsilon|^\gamma - 2|y^*| + \cdots + |y^*|^\gamma - 1)
\]

\[
\to 0 \quad \text{as } \varepsilon \to 0.
\]

(3.81)

From (3.79)–(3.81) and Lemma 3.2, one can easily check that

\[
\eta_\varepsilon(z_\varepsilon(v), v) \to 0 \quad \text{as } \varepsilon \to 0 \text{ uniformly in } v \in M(0, r).
\]

(3.82)

Thus it follows from (3.9), (3.50) and (3.78) that

\[
\langle \xi_\varepsilon, F'(y^*) z - w + F(y^*) \rangle_{X^*, X} \geq -\eta_\varepsilon, \quad \forall z \in R_r, \ w \in W,
\]

(3.83)

where \(\eta_\varepsilon \to 0\) as \(\varepsilon \to 0\). By \((H_3)\), \(F'(y^*) R_r - W\) has finite codimensionality in \(X\) and so does \(F'(y^*) R_r - W + F(y^*)\). By [10, Lemma 3.6], we conclude from (3.77), (3.82) and (3.83) that \((\lambda_0, \xi_0) \neq 0\).

Finally, if \(F'(y^*)\) is injective and \((\lambda_0, p) = 0\), then it follows from (3.51) that \([F'(y^*)]^* = 0\), which implies that \(\xi_0 = 0\). This contradiction leads \((\lambda_0, p) \neq 0\). This completes the proof. \(\Box\)

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References