ELSEVIER Stochastic Processes and their Applications 69 (1997) 179–193

stochastic processes and their applications

On the spectrum of correlation autoregressive sequences[1]

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Received 21 April 1996; received in revised form 5 January 1997

Abstract

In this paper some properties of the correlation autoregressive (CAR) sequences are studied. A representation for the correlation function of an arbitrary CAR sequence is obtained and the relationship between a CAR equation and the growth of the variance and location of spectral lines is revealed. It is also observed that bounded correlation autoregressive sequences coincide with almost periodically correlated sequences with the spectral measure supported on finitely many lines. As a consequence a characterization of the spectrum of a bounded CAR sequence is provided. © 1997 Elsevier Science B.V.

Keywords: Correlation autoregressive sequence; Almost periodically correlated sequence; Harmonizable process; Spectral representation

AMS classification: 60G10; 60G25

1. Introduction

Recently there has been a growing interest in the study of nonstationary stochastic processes, which are interesting from the theoretical point of view, see Chang and Rao (1966), Cramer (1962), Gardner and Frank (1975), Gladyshev (1961), Hurd (1974), Hurd (1991), Miamee and Salehi (1978), as well as for their numerous applications in science and engineering, see Gardner and Frank (1975), Hardin and Miamee (1990), Miamee and Salehi (1980), Priestly (1988). In most cases the study leads to consideration of different classes of nonstationary processes which in one way or the other extend the class of stationary processes. Periodically correlated processes (Gladyshev, 1961) and almost periodically correlated processes (Hurd, 1991) are two examples of such classes. Hardin and Miamee (1990) introduced a new class of nonstationary stochastic processes, called correlation autoregressive processes. A sequence $x=\{x_n\colon n\in\mathbb{Z}\}$ of complex, zero-meansquare integrable random variables is called

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[1] This research was supported in part by the U.S. Army Research Office Grant No. DAAH 04-96-1-0027 and in part by ONR Grant No. N00014-89-J-1824.

0304-4149/97/$17.00
PII S0304-4149(97)00038-0

correlation autoregressive (CAR, in short) if its correlation function $R_x(m,n)=Ex_m\overline{x_n}$ satisfies a CAR equation

$$R_x(m,n)=\sum_{k=1}^{r} a_k R_x(m+k,n+k), \quad n,m\in\mathbb{Z} \tag{1}$$

for some finite set of scalars a_k, $k=1,\dots,r$.

Below are some examples of CAR sequences.

Example 1. If $x=\{x_n\colon n\in\mathbb{Z}\}$ is a stationary sequence then its correlation function satisfies the equation $R_x(m,n)=R_x(m+1,n+1)$, $m,n\in\mathbb{Z}$. It also satisfies the equation $R_x(m,n)=2R_x(m+1,n+1)-R_x(m+2,n+2)$, $m,n\in\mathbb{Z}$. It shows that a CAR equation of a CAR sequence is not unique.

Example 2. A sequence $x=\{x_n\colon n\in\mathbb{Z}\}$ is called periodically correlated (PC), if there is an integer T such that $R_x(m,n)=R_x(m+T,n+T)$, $m,n\in\mathbb{Z}$. Clearly each PC sequence is CAR.

Example 3. If z_n is a stationary sequence then both $y_n=a^n z_n$, and $w_n=nz_n$, are CAR sequences, for

$$R_y(m,n)=(|a|^2+1)^{-1}R_y(m+1,n+1)+|a|^2(|a|^2+1)^{-1}R_y(m-1,n-1),$$

$$R_w(m,n)=3R_w(m+1,n+1)-3R_w(m+2,n+2)+R_w(m+3,n+3),$$

$m,n\in\mathbb{Z}$. In fact every sequence of the form $x_n=\lambda^n n^k z_n$, where $\lambda\neq 0$ and k is a nonnegative integer is a CAR sequence. This follows from the next example.

Example 4. If $\mathbf{z}=\{(z_n^k)\colon k=1,\dots,N,\ n\in\mathbb{Z}\}$ is an N-dimensional stationary sequence then for all nonnegative integers $k_1,\dots,k_N$, complex nonzero numbers $\lambda_1,\dots,\lambda_N$, and scalars $c_{u,s}$ the sequence x_n defined by

$$x_n=\sum_{s=1}^{N}\sum_{u=0}^{k_s}\lambda_s^n n^u c_{u,s} z_n^s, \quad n\in\mathbb{Z} \tag{2}$$

is CAR.

Proof. Observe that the correlation of x_n can be written as

$$R_x(n+p,n)=\sum_{s,t=1}^{n}(\lambda_s\overline{\lambda_t})^n\left(\sum_{v=0}^{k_s+k_t} n^v C(s,t,v,p)\right),$$

with proper coefficients $C(s,t,v,p)$ that do not depend on n. Let

$$p(z)=A\prod_{s,t=1}^{N}(\lambda_s\overline{\lambda_t}-z)^{k_s+k_t+1},$$

where A is such that the constant term of p is equal 1. From Hildebrand (1968) it follows that $R_x(m,n)$ satisfies the Eq. (1) with the scalars a_k, $k=1,\dots,r$ defined by equation $p(z)=1-\sum_{k=1}^{r}a_k z^k$. $\square$

The question whether or not every CAR sequence is of the form (2) is still open.

In the present work we study only discrete time processes. However, the continuous time CAR processes have been introduced and proved to be useful in applications; cf. Hardin and Miamee (1990) and Dargahi-Noubary and Miamee (1993). For example, in analysis of a helicopter noise an observer records consist of two periodically correlated random noise processes, namely, those generated by the main and tail rotors. Since the periods of these processes are generally incommensurate, the helicopter noise is not periodically correlated. However, once the spectrum of the helicopter noise is analyzed it turns out to be supported on several lines parallel to the line $y=x$. We will see later that the helicopter noise process is actually a CAR process.

Another potential area for the applications of the CAR processes is that of seismic waves. Numerous models ranging from simple stationary white noise to ARMA stationary process have been proposed by different authors for modeling seismic waves. The most popular and successful models are uniformly modulated stationary processes. These models are generally based on the assumption that seismic records y_t are composed of a deterministic envelope function $g(t)$ and a zero-mean stationary stochastic process x_t, i.e. $y_t = g(t)x_t$. Here are some typical envelope functions proven to be useful in fitting some data:

$$g(t) = \theta_1 \exp(\theta_2 t) + \theta_3 \exp(\theta_4 t),$$

$$g(t) = \theta_0 + \theta_1 t + \cdots + \theta_{10} t^{10},$$

$$g(t) = t^{\alpha} \exp(-\beta t).$$

Example 4 shows that all these models are included in the class of CAR processes.

In this paper we discuss some fundamental properties of CAR sequences. In Section 2 a representation for the correlation function of a CAR sequence is provided and the growth of the variance of x_n, as $n \to \pm\infty$, is examined. In Section 3 the spectrum of a CAR sequence is studied. Among other results, it is shown that every bounded CAR sequence is almost periodically correlated.

Throughout the paper $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ will denote the set of real numbers, complex numbers, integers, and nonnegative integers, respectively. $\mathbf{H}$ will stand for a complex Hilbert space (which replaces the space of zero-meansquare integrable complex random variables) and $(\cdot,\cdot)$ will denote the inner product in $\mathbf{H}$. The unit circle $\mathbf{T}$ will be identified with the interval $[0, 2\pi)$ and the torus $\mathbf{T}^2$ with the square $[0, 2\pi)^2$ (the algebraic operation are in the usual modulo 2π sense). $B(\mathbf{T})$ and $B(\mathbf{T}^2)$ will denote the Borel σ-algebras in $\mathbf{T}$ and $\mathbf{T}^2$, respectively. If φ is a function on $\mathbf{T}$ then $\operatorname{supp}(\varphi)$ will denote the closure of the set $\{t\colon \varphi(t) \neq 0\}$. If f is an integrable function on $\mathbf{T}$ ($\mathbf{T}^2$, respectively) then

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nt} f(t)\,\mathrm{d}t, \quad n \in \mathbb{Z}$$

and

$$\hat{f}(m,n) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}(ms+nt)} f(s,t)\,\mathrm{d}s\,\mathrm{d}t, \quad m,n \in \mathbb{Z},$$

respectively, will denote the Fourier transform of f. In the sequel all integrals will be over $[0, 2\pi)$, unless otherwise is stated. The symbols $C^\infty(\mathbf{T})$ ($C^\infty(\mathbf{T}^2)$, respectively) will denote the set of all infinitely many times differentiable functions on $\mathbf{T}$ ($\mathbf{T}^2$, respectively) equipped with the topology of uniform convergence of all derivatives (see Edwards, 1979, 12.1). $C(\mathbf{T})$ and $C(\mathbf{T}^2)$ will stand for the space of all continuous functions on $\mathbf{T}$ and $\mathbf{T}^2$, respectively. If $f, g \in C(\mathbf{T})$ then $f \otimes g$ will stand for the $C(\mathbf{T}^2)$ function defined by $f \otimes g(s,t) = f(s)g(t)$.

2. Representation of the correlation function of a CAR sequence

Recall that, by definition, the correlation function $R_x(m,n) = (x_m, x_n)$ of a CAR sequence $x = \{x_n : n \in \mathbb{Z}\}$ satisfies a CAR equation

$$R_x(m,n) = \sum_{k=1}^{r} a_k R_x(m+k, n+k), \quad n, m \in \mathbb{Z}, \tag{3}$$

where r is a positive integer and a_k, $k = 1, \ldots, r$, are some complex numbers. Example 1 shows that a_k's and r are not uniquely determined by the sequence x. Below we introduce the concept of a miminal CAR equation that is unique.

Definition 1. Let $x = \{x_n : n \in \mathbb{Z}\}$ be a sequence in a Hilbert space $\mathbf{H}$.

1. An admissible polynomial for x is any polynomial $p(z) = 1 - \sum_{k=1}^{r} a_k z^k$ such that the correlation function of x satisfies the equation $R_x(m,n) = \sum_{k=1}^{r} a_k R_x(m+k, n+k)$, $n, m \in \mathbb{Z}$.
2. A minimal admissible polynomial is an admissible polynomial of the lowest degree.

Clearly, x is a CAR sequence if and only if it has at least one admissible polynomial.

Lemma 1. *Each CAR sequence $x = \{x_n : n \in \mathbb{Z}\}$ has only one minimal admissible polynomial and hence a unique minimal CAR equation.*

Proof. Suppose that $p_i(z) = 1 - \sum_{k=1}^{r} a_k^{(i)} z^k$, $i = 1, 2$, are two distinct minimal admissible polynomials of a CAR sequence x. Let $m = \min\{j : a_j^{(1)} \neq a_j^{(2)}\}$. Then $z^{-m}(a_m^{(2)} - a_m^{(1)})^{-1}(p_2(z) - p_1(z))$ is clearly an admissible polynomial with degree lower than n, which leads to a contradiction. □

The unique minimal admissible polynomial of a CAR sequence x will be referred to as *the MAP* of x.

In this section we discuss certain properties of a CAR sequence controlled by its MAP, or more precisely by the roots of its MAP. This includes a representation of the correlation function of a CAR sequence. Most of the proofs are lengthy and of algebraic nature and, hence, for the sake of clarity, they are presented in Appendix.

Theorem 1. (Representation of correlation function). *Let $x = \{x_n : n \in \mathbb{Z}\}$ be a CAR sequence and let $p(z) = 1 - \sum_{k=1}^{r} a_k z^k$, $z \in \mathbb{C}$, be its MAP. Let λ_j, $j = 0, \ldots, q$, be* *distinct roots of p and let $m(\lambda_j)=m_j$, $j=0,\dots,q$, be their multiplicities. Then there are numbers $\alpha_{j,k}(p)$, $p\in\mathbb{Z}$, $j=0,\dots,q$, $k=0,\dots,m_j-1$, such that*

$$R_x(n+p,n)=\sum_{j=0}^{q}\sum_{k=0}^{m_j-1}\lambda_j^n n^k \alpha_{j,k}(p), \tag{4}$$

for all $p,n\in\mathbb{Z}$. Moreover, $\sup_p|\alpha_{j,m_j-1}(p)|>0$, for every $j=0,\dots,q$.

From (4) it is clear that the growth rate of the variance $R_x(n,n)$ of x as $n\to\pm\infty$ is controlled by the roots of the MAP and their multiplicities. This statement is made more precise in the following sequel of theorems.

First we record some properties of the roots of a MAP.

Theorem 2. *Let $p(z)=1-\sum_{k=1}^{r}a_k z^k$ be the MAP of a CAR sequence x, $\lambda_0,\dots,\lambda_q$ denote distinct roots of p and $m(\lambda)$ stand for the multiplicity of a root λ. Furthermore let $\lambda_{\max}=\max\{|\lambda_j|: j=0,\dots,q\}$ and $k_+=\max\{m(\lambda_j)-1: |\lambda_j|=\lambda_{\max}\}$. Similarly, let $\lambda_{\min}=\min\{|\lambda_j|: j=0,\dots,q\}$ and $k_-=\max\{m(\lambda_j)-1: |\lambda_j|=\lambda_{\min}\}$. Then*
1. *All a_k's are real.*
2. *$\lambda=\lambda_{\max}$ is a root of p with multiplicity $m(\lambda_{\max})=k_++1$.*
3. *$\lambda=\lambda_{\min}$ is a root of p with multiplicity $m(\lambda_{\min})=k_-+1$.*

The roots $\lambda_{\max}$ and $\lambda_{\min}$ and their multiplicities govern the growth behavior of the variance of x_n at $\pm\infty$ as we see in Theorem 3. In the next section we observe that the other roots are responsible for the location of spectral lines of a sequence.

Theorem 3. *Let $x=\{x_n: n\in\mathbb{Z}\}$ be a CAR sequence and $p(z)=1-\sum_{k=1}^{r}a_k z^k$ be its MAP. Let $\lambda_{\max}$, k_+, $\lambda_{\min}$ and k_- be as in Theorem 2. Then there are bounded sequences $c_+(n)$ and $c_-(n)$, $n\geq 1$, not converging to zero such that*
1. *$\|x_n\|^2=c_+(n)\lambda_{\max}^n n^{k_+}$, $n\geq 1$, and*
2. *$\|x_{-n}\|^2=c_-(n)\lambda_{\min}^{-n} n^{k_-}$, $n\geq 1$.*

The following are two immediate corollaries from Theorems 2 and 3.

Corollary 1. *Let $x=\{x_n: n\in\mathbb{Z}\}$ be a CAR sequence and $p(z)=1-\sum_{k=1}^{r}a_k z^k$ be its MAP. Let $\lambda_0,\dots,\lambda_q$ denote the roots of p and let $m(\lambda_k)$, $k=0,\dots,q$, be their multiplicities. Then $\|x_n\|^2\leq C|n|^k$, $n\in\mathbb{Z}$, for some $C>0$ and $k\in\mathbb{N}$, if and only if $|\lambda_j|=1$ for all $j=0,\dots,q$. If this is the case then*
1. *$\lambda=1$ is a root of p, i.e. $\sum_{k=1}^{r}a_k=1$,*
2. *$m(1)=\max\{m(\lambda_j): j=0,\dots,q\}\leq k+1$.*

Corollary 2. *A CAR sequence $x=\{x_n: n\in\mathbb{Z}\}$ is bounded if and only if all roots of its MAP are of modulus 1 and multiplicity 1.*

3. Spectrum of a CAR sequence

Roughly speaking, the *spectrum* of an **H**-valued sequence $x=\{x_n: n\in\mathbb{Z}\}$ is the Fourier transform of $R_x(m,-n)$ in whatever sense it exists. If for example a sequence $x=\{x_n: n\in\mathbb{Z}\}\subset\mathbf{H}$ is *harmonizable*, that is if there is an **H**-valued Borel measure μ on **T** such that

$$x_n=\int \exp(int)\mu(dt), \tag{5}$$

for all $n\in\mathbb{Z}$, then the spectrum of x is identified with the bimeasure $B_x(1_{\Delta_1},1_{\Delta_2})=(\mu(\Delta_1),\mu(\Delta_2))$, $\Delta_1,\Delta_2\in B(\mathbf{T})$. If a sequence x is *strongly harmonizable*, that is if there is a Borel measure Γ_x on $\mathbf{T}^2$ such that

$$R_x(m,n)=\int\int e^{i(ms-nt)}\Gamma_x(ds,dt), \tag{6}$$

$m,n\in\mathbb{Z}$, then the spectrum of x is identified with Γ_x.

Recall that a sequence $x=\{x_n: n\in\mathbb{Z}\}$ is *stationary* if its correlation function $R_x(m,n)$ depends only on $m-n$. Each stationary sequence is strongly harmonizable and the measure Γ_x in representation (6) sits on the diagonal $D=\{(s,t)\in\mathbf{T}^2: s=t\}$. Commonly, the spectrum of a stationary sequence is identified with the measure $\Gamma_x\circ\Phi^{-1}$, where $\Phi:(s,s)\to s$ maps D onto **T**. Harmonizable sequences, and hence strongly harmonizable and stationary sequences are bounded.

The notion of the spectrum of a second-order stochastic sequence can be, in a natural way, extended to the class of sequences with polynomial growth of variance as follows. We also include here the definition of the random spectrum, which in the harmonizable case corresponds to the measure μ in the integral (5).

Definition 2. (*cf. Makagon and Mandrekar*, 1990). Let $x=\{x_n: n\in\mathbb{Z}\}$ be a sequence in **H** such that $\|x_n\||\leqslant C|n|^k$ for some $C>0$ and $k\in\mathbb{N}$ and all $n\in\mathbb{Z}$.

1. The mapping $F_x: C^\infty(\mathbf{T})\to\mathbf{H}$ defined by

$$F_x(f)=\sum_n \hat{f}(n)x_n$$

$f\in C^\infty(T)$, is called the *random spectrum* of x.

2. The mapping $D_x: C^\infty(\mathbf{T}^2)\to\mathbb{C}$ defined by the formula

$$D_x(f)=\sum_{m,n}\hat{f}(m,n)R_x(m,-n)$$

$f\in C^\infty(\mathbf{T}^2)$, is called the *spectrum* of x.

Since the Fourier coefficient of functions from C^∞ decrease faster than in polynomial rate, F_x and D_x are well-defined continuous linear operators from $C^\infty(\mathbf{T})$ to **H** and from $C^\infty(\mathbf{T}^2)$ to $\mathbb{C}$, respectively, and hence D_x is a distribution in the Schwartz sense (for definition and other facts about distributions see Edwards, 1979). Also note that

$$(F_x(f),F_x(g))=D_x(f\otimes\bar{g}),\quad f,g\in C^\infty(\mathbf{T}). \tag{7}$$

Lemma 2. *Let $x = \{x_n\colon n \in \mathbb{Z}\}$ be a sequence in* **H** *such that $\|x_n\| \leqslant C|n|^k$ for some $C > 0$ and $k \in \mathbb{N}$ and all $n \in \mathbb{Z}$. Then*

1. *x is CAR if and only if there are scalars a_k, $k = 1, \ldots, r$ such that*

$$D_x\left(\left(1 - \sum_{k=1}^{r} a_k e^{ik(s-t)}\right) h(s,t)\right) = 0, \quad h \in C^{\infty}(\mathbf{T}^2); \tag{8}$$

2. *x is stationary if and only if $D_x((1 - e^{i(s-t)})h(s,t)) = 0$ for all $h \in C^{\infty}(\mathbf{T}^2)$ or equivalently $\|F_x(f(s))\|^2 = \|F_x(e^{is} f(s))\|^2$ for all $f \in C^{\infty}(\mathbf{T})$.*

Proof. Since $R_x(m,n) = D_x(e^{i(ms-nt)})$, x is a CAR sequence if and only if there are scalars a_k, $k = 1, \ldots, r$ such that

$$D_x(e^{i(ms-nt)}) = D_x\left(\sum_{k=1}^{r} a_k e^{ik(s-t)} e^{i(ms-nt)}\right)$$

for every $m, n \in \mathbb{Z}$. Since the Fourier transform determines a distribution uniquely, this holds true if and only if the distribution

$$h \to D_x\left(\left(1 - \sum_{k=1}^{r} a_k e^{ik(s-t)}\right) h(s,t)\right)$$

is identically zero. The argument above also shows that $R_x(m,n) = R_x(m+1, n+1)$ for every $m, n \in \mathbb{Z}$ if and only if $D_x((1 - e^{i(s-t)})h(s,t)) = 0$ for every $h \in C^{\infty}(\mathbf{T}^2)$. □

Recall that the support $\operatorname{supp} D$ of a distribution $D\colon C^{\infty}(\mathbf{T}^2) \to \mathbb{C}$ is the smallest closed subset S of $\mathbf{T}^2$ such that $D(f) = 0$ provided that $\operatorname{supp}(f) \cap S = \emptyset$.

The following theorem shows that the spectrum of a CAR sequence x of polynomial growth is supported on finitely many lines parallel to the diagonal, and that the roots of the MAP of x determine the location of the lines.

Theorem 4. *Let $x = \{x_n\colon n \in \mathbb{Z}\}$ be a CAR sequence such that $\|x_n\| < C|n|^k$, for all $n \in \mathbb{Z}$ and some $C > 0$ and $k \in \mathbb{N}$. Let D_x be the spectrum of x and p be its MAP. Let $\lambda_j = \exp(i\theta_j)$, $j = 0, \ldots, q$, denote distinct roots of the MAP of x (cf. Corollary 1). Then $\operatorname{supp} D_x$ is a subset of $\bigcup_{k=0}^{q} D_{\theta_k}$, where $D_\theta = \{(t + \theta, t)\colon 0 \leqslant t < 2\pi\}$.*

Proof. Since x is a CAR sequence, from (8) it follows that

$$D_x(p(e^{i(s-t)})h(s,t)) = 0 \quad \text{for all } h \in C^{\infty}(\mathbf{T}^2). \tag{9}$$

Let $f \in C^{\infty}(\mathbf{T}^2)$ be such that $\operatorname{supp}(f)$ is disjoint from $S = \bigcup_{k=0}^{q} D_{\theta_k}$. Putting $h(s,t) = f(s,t)/p(e^{i(s-t)})$ into Eq. (9) we obtain that $D_x(f) = 0$. Hence, the support of D_x is contained in S. □

If a CAR sequence x is bounded then certainly the conclusion of Theorem 4 holds true. We will show that in this case D_x is a measure.

We present two proofs of this fact. The first proof, given below, is based on the observation that due to Theorem 1 and Corollary 2 bounded CAR sequences are almost periodically correlated and on an advanced result on the spectrum of almost periodically correlated processes established by Hurd (1991). The second proof, although lengthy, uses elementary distribution theory and is given in Appendix. We feel that the second method is worth to be looked upon for it is more direct and introduces new technique in analysis of APC processes.

Recall that a sequence $x=\{x_n:\ n\in\mathbb{Z}\}$ in a Hilbert space is *almost periodically correlated* (APC) if for every $p\in\mathbb{Z}$ the sequence $R_x(n+p,n)$, $n\in\mathbb{Z}$, is *almost periodic* in n in Bohr sense (that is it is a uniform limit of trigonometric polynomials of the form $\sum_k c_k \exp(\mathrm{i}\xi_k n)$). If x is an APC sequence then the limit

$$a_\theta(p)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}R_x(n+p,n)\mathrm{e}^{-\mathrm{i}n\theta} \tag{10}$$

exists for every $\theta\in[0,2\pi)$. If $a_\theta(p)$ is not identically zero then θ will be referred to as a *cycle frequency* of x. Hurd (1991) showed that if x is APC then x has at most countably many cycle frequencies and for each of them $a_\theta(\cdot)$ is the Fourier transform of a complex measure.

Theorem 5. *Let $x=\{x_n:\ n\in\mathbb{Z}\}$ be a bounded sequence in a complex Hilbert space. The following three conditions are equivalent:*
1. *x is a CAR sequence,*
2. *x is an APC sequence with finitely many cycle frequencies,*
3. *x is strongly harmonizable and its spectral measure is supported by finitely many lines parallel to the diagonal.*

Proof. ($1\Leftrightarrow 2$): Let x be a bounded CAR sequence. From Theorem 1 and Corollary 2 it follows that the correlation function of x is of the form

$$R_x(n+p,n)=\sum_{j=0}^{r-1}\mathrm{e}^{\mathrm{i}n\theta_j}\,\alpha_{j,0}(p). \tag{11}$$

and $\alpha_{j,0}(\cdot)$ is not identically zero for any $j=0,\dots,r-1$. Hence, x is APC and θ_j, $j=0,\dots,r-1$, are its cycle frequencies.

Conversely, suppose that x is APC and it has only finitely many cycle frequencies θ_k, $k=0,\dots,r-1$. By (10)

$$R_x(n+p,n)=\sum_{k=0}^{r-1}\mathrm{e}^{\mathrm{i}n\theta_k}\,a_{\theta_k}(p),\qquad n,p\in\mathbb{Z}. \tag{12}$$

Letting $p(z)=C\prod_{k=1}^{r}(\mathrm{e}^{\mathrm{i}n\theta_k}-z)$, where $C=\overline{\prod_{k=1}^{r}\mathrm{e}^{\mathrm{i}n\theta_k}}$, one can see that p is an admissible polynomial for x (cf. proof of Theorem 1 in the appendix), and hence x is a CAR sequence. The equivalence ($2\Leftrightarrow 3$) is proved in Hurd (1991) in a more general setting. □

In this paper we do not address questions concerning prediction or statistics of CAR sequences. In the case of a bounded CAR sequence, Theorem 5 makes possible to adopt procedures described in Hurd and Leskow (1992) and Leskow (1994) for APC processes to estimate the coefficients $\alpha_{j,0}(p)$ in (11) and the density of the spectral measure D_x restricted to the line D_{θ_j}. A deficiency of this approach is that these procedures assume prior knowledge of cycle frequencies θ_j, $k=0,\dots,r-1$, and at present no consistent estimators of cycle frequencies are known. It seems that it would be more desirable to develop direct estimates of coefficients a_k appearing in the CAR equation (3), and then use Theorem 4 to retrieve cycle frequencies. We feel that more research is to be done in this direction.

Appendix

A.1. Proof of Theorem 1

Recall that every solution to a system of homogenous difference equations

$$\sum_{j=0}^{r} b_j u_{k+j} = 0, \quad b_0, b_r \neq 0, \ \ k \in \mathbb{Z}, \tag{A.1}$$

is of the form

$$u_n = \sum_{j=0}^{q} \sum_{k=0}^{m(\lambda_j)-1} \lambda_j^n n^k \alpha_{j,k}, \qquad n \in \mathbb{Z}, \tag{A.2}$$

where $\lambda_0,\dots,\lambda_q$ are distinct roots of the characteristic polynomial $q(z)=\sum_{j=0}^{r} b_j z^j$ of the Eq. (A.1) and $m(\lambda)$ denotes the multiplicity of a root λ. Note that if in (A.2) $\alpha_{p,m(\lambda_p)-1}=0$ for some $p\in\{0,\dots,q\}$ then the sequence u_n also satisfies an equation of lower degree, namely

$$\sum_{j=0}^{r-1} c_j u_{k+j} = 0, \tag{A.3}$$

where c_j, $j=0,\dots,r-1$, are such that

$$\sum_{j=0}^{r} b_j z^j = \left(\sum_{j=0}^{r-1} c_j z^j\right)(z-\lambda_p).$$

Let now x be a CAR sequence. By assumption the sequence $u_n^p = R_x(n+p,n)$, $n\in\mathbb{Z}$, satisfies the difference equation $u_n^p - \sum_{k=1}^{r} a_k u_{n+k}^p = 0$, $n\in\mathbb{Z}$. Hence, from the discussion above it follows that $R_x(n+p,n)$ has the form (4). If for some j, $\alpha_{j,m_j-1}(p)=0$ for all $p\in\mathbb{Z}$, then for every $p\in\mathbb{Z}$ the sequence $R_x(n+p,n)$ would satisfy the lower degree equation

$$R_x(n+p,n) = \sum_{k=1}^{r-1} c_k R_x(n+p+k, n+k),$$

where $1-\sum_{k=1}^{r-1} c_k z^k = -\lambda_j p(z)/(z-\lambda_j)$, $z\in\mathbb{C}$. This contradicts the minimality of the polynomial $p(z)$. □

A.2. Proof of Theorem 2

We will need the following two lemmas.

Lemma 4.1. (Zygmunt, 1968, pp. 235–238). *Let $\gamma_0,\dots,\gamma_q$ be distinct numbers from $[0,2\pi)$. Then there is a sequence of integers $n_p\to\infty$, such that*

$$\max_{0\leqslant j\leqslant q} |\mathrm{e}^{\mathrm{i}n_p\gamma_j}-1| \overset{p\to\infty}{\longrightarrow} 0.$$

The next lemma is an easy consequence of the above result.

Lemma 4.2. *Let γ_j, $j=0,\dots,q$ be distinct numbers from $[0,2\pi)$ and β_j, $j=0,\dots,q$, be any complex numbers.*
1. *If $\sum_{j=0}^{q}\mathrm{e}^{\mathrm{i}\gamma_j n}\beta_j \overset{n\to\infty}{\longrightarrow} 0$, then $\beta_j=0$, $j=0,\dots,q$.*
2. *If not all β_j's are zero and $\sum_{j=0}^{q}\mathrm{e}^{\mathrm{i}\gamma_j n}\beta_j\geqslant 0$, for all $n\geqslant n_0$, then there is an index j_0, $0\leqslant j_0\leqslant q$, such that $\gamma_{j_0}=0$ and $\beta_{j_0}>0$.*

Proof. 1. Let n_p be as in Lemma 4.1 and let $k\in\mathbb{Z}$. Then

$$0=\lim_{p\to 0}\left(\sum_{j=0}^{q}\mathrm{e}^{\mathrm{i}\gamma_j(n_p+k)}\beta_j\right)=\sum_{j=0}^{q}\mathrm{e}^{\mathrm{i}\gamma_j k}\beta_j=\int_0^{2\pi}\mathrm{e}^{\mathrm{i}tk}\mu(\mathrm{d}t),$$

where $\mu(\mathrm{d}t)=\sum_{j=0}^{q}\beta_j\delta_{\gamma_j}(\mathrm{d}t)$, and δ_a denotes the probabilistic measure concentrated at a point a. Since the Fourier transform determines a complex measure uniquely, $\beta_j=0$, $j=0,\dots,q$.

2. Let $u_n=\sum_{j=0}^{q}\mathrm{e}^{\mathrm{i}\gamma_j n}\beta_j$, $n\geqslant 0$. If all γ_j's are different from zero, then

$$u_0+\cdots+u_n=\sum_{j=0}^{q}\frac{1-\mathrm{e}^{\mathrm{i}\gamma_j(n+1)}}{1-\mathrm{e}^{\mathrm{i}\gamma_j}}\beta_j.$$

Since $u_j\geqslant 0$ for $j\geqslant n_0$, and the right-hand side in the expression above is bounded in n, $\sum_{j=0}^{\infty}u_j<\infty$. Hence, $u_n\to 0$ (as $n\to\infty$), which in view of Part 1 contradicts the assumption that not all β_j's are zero. Therefore, one of the numbers γ_j, say γ_{j_0}, is zero and $\beta_{j_0}\neq 0$. In this case,

$$u_0+\cdots+u_n=\sum_{j=0}^{q}\frac{1-\mathrm{e}^{\mathrm{i}\gamma_j n}}{1-\mathrm{e}^{\mathrm{i}\gamma_j}}\beta_j+(n+1)\beta_{j_0},$$

$n\in\mathbb{N}$, which yields that $\beta_{j_0}=\lim_{n\to\infty}1/(n+1)(u_0+\cdots+u_n)\geqslant 0$. Therefore, $\beta_{j_0}>0$. □

Proof of Theorem 2. 1. Since $\sum_{k=1}^{r}\overline{a_k}R_x(m+k,n+k)=\overline{R_x(n,m)}=R_x(m,n)$, $p^*(z)=1-\sum_{k=1}^{r}\overline{a_k}z^k$ is an admissible polynomial. Therefore, from Lemma 1 it follows that $a_k=\overline{a_k}$, $k=1,\dots,r$.

2. Let $B=\{j\colon |\lambda_j|=\lambda_{\max}\}$, $k_+=\max\{m(\lambda_j)-1\colon j\in B\}$, $B_0=\{j\in B\colon m(\lambda_j)=k_++1\}$, and $B_1=B\backslash B_0$. Then from (4) it follows that

$$\begin{aligned}\frac{R_x(n+p,n)}{\lambda_{\max}^n n^{k_+}} &= \sum_{j\notin B}\sum_{k=0}^{m_j-1}\left(\frac{\lambda_j}{\lambda_{\max}}\right)^n n^{k-k_+}\alpha_{j,k}(p)\\ &+\sum_{j\in B_1}\sum_{k=0}^{m_j-1}\left(\frac{\lambda_j}{\lambda_{\max}}\right)^n n^{k-k_+}\alpha_{j,k}(p)\\ &+\sum_{j\in B_0}\sum_{k=0}^{k_+-1}\left(\frac{\lambda_j}{\lambda_{\max}}\right)^n n^{k-k_+}\alpha_{j,k}(p)+\sum_{j\in B_0}\left(\frac{\lambda_j}{\lambda_{\max}}\right)^n \alpha_{j,k_+}(p),\end{aligned}$$

and hence

$$\frac{R_x(n+p,n)}{\lambda_{\max}^n n^{k_+}}=r_+^p(n)+\sum_{j\in B_0}e^{i\gamma_j n}\alpha_{j,k_+}(p), \tag{A.4}$$

where $r_+^p(n)\to 0$, as $n\to\infty$ and $\gamma_j=\arg(\lambda_j)$, $j\in B_0$. From Theorem 1 it follows that there is an integer p and $j\in B_0$ such that $\alpha_{jk_+}(p)\neq 0$. With this p, in virtue of Lemma 4.2 there is a subsequence $n_q\to\infty$ such that

$$\frac{R_x(n_q+p,n_q)}{\lambda_{\max}^{n_q}n_q^{k_+}}\to\gamma\neq 0.$$

Since $|R_x(n+p,n)|^2\leqslant R_x(n,n)R_x(n+p,n+p)$, we obtain that

$$\begin{aligned}\frac{|R_x(n_q+p,n_q)|^2}{[\lambda_{\max}^{n_q}n_q^{k_+}]^2} &\leqslant \left[r_+^0(n_q)+\sum_{j\in B_0}e^{i\gamma_j n_q}\alpha_{j,k_+}(0)\right]\\ &\times\left[r_+^0(n_q+p)+\sum_{j\in B_0}e^{i\gamma_j n_q}e^{i\gamma_j p}\alpha_{jk_+}(0)\right].\end{aligned}$$

Because $r_+^0(n)\to 0$ as $n\to\infty$, at least one $\alpha_{j,k_+}(0)$ must be nonzero. Taking $p=0$ in the Eq. (A.4) we conclude that

$$\frac{R_x(n,n)}{\lambda_{\max}^n n^{k_+}}=r_+^0(n)+\sum_{j\in B_0}e^{i\gamma_j n}\alpha_{jk_+}(0),\quad n\geqslant 1, \tag{A.5}$$

where not all $\alpha_{j,k_+}(0)=0$, $j\in B_0$. Let $n_q\to\infty$ be a sequence such that $\exp(i\gamma_j n_q)\to 1$ for all $j\in B_0$. Since $r_+^p(n)\to 0$,

$$0\leqslant\frac{R_x(n_q+p,n_q+p)}{\lambda_{\max}^{n_q+p}(n_q+p)^{k_+}}\overset{q\to\infty}{\longrightarrow}\sum_{j\in B_0}e^{i\gamma_j p}\alpha_{j,k_+}(0)\geqslant 0$$

for all $p\in\mathbb{Z}$. From Lemma 4.2 we conclude that one of γ_j's, say γ_{j_0}, $j_0\in B_0$, is equal to 0 and $\alpha_{j_0k_+}(0)>0$, i.e. $\lambda_{j_0}=\lambda_{\max}$ and $m(\lambda_{j_0})=k_++1=\max\{m(\lambda_j)\colon |\lambda_j|=\lambda_{\max}\}$.

3. The proof of Part 3, goes along the same lines as the proof of Part 2. □

A.3. Proof of Theorem 3

1. Let B, B_0, B_1, $r_+^p(n)$ and γ_j be as in the proof of Theorem 2, Part 2. From (A.4) if follows that

$$\|x_n\|^2 = c_+(n)\lambda_{\max}^n n^{k_+}, \tag{A.6}$$

where $c_+(n) = r_+^0(n) + \sum_{j\in B_0} e^{i\gamma_j n}\alpha_{j,k_+}(0)$ does not converge to zero.

The proof of Part 2 follows in the same way from the proof Part 3 of Theorem 2. □

A.4. The second proof of Theorem 5

Let $x=\{x_n\colon n\in\mathbb{Z}\}$ be a bounded CAR sequence, and D_x and F_x be its spectrum and random spectrum, respectively (see Definition 2). Let $p(z)$ be the MAP of x and let λ_j, $j=0,\ldots,r-1$ denote the zeros of p. In view of Corollary 2, we may assume that $\lambda_j = e^{i\theta_j}$, $j=0,\ldots,r-1$, where $\theta_0 = 0 < \theta_1 < \cdots < \theta_{r-1} < 2\pi$. Let $d=\frac{1}{2}\min\{|\theta_k-\theta_j|\colon j\neq k\}$.

With these notation we break the proof into few lemmas.

Lemma 4.3. *If* $\operatorname{supp}(f)\subset\{(s,t)\in\mathbf{T}^2\colon |s-t-\theta_k|<d\}$, *then*

$$D_x(f) = e^{-i\theta_k} D_x(e^{i(s-t)}f(s,t)). \tag{A.7}$$

Proof. Consider the function

$$g(s,t) = \frac{p(e^{i(s-t)})}{(e^{i\theta_k} - e^{i(s-t)})}.$$

Then $f(s,t)/g(s,t) = h(s,t)\in C^\infty(\mathbf{T}^2)$ and from (8) we get $D_x((e^{i\theta_k}-e^{i(s-t)})f(s,t)) = D_x(p(e^{i(s-t)})h(s,t)) = 0$, which proves (A.7). □

Lemma 4.4. *The sequence* $x=\{x_n\colon n\in\mathbb{Z}\}$ *is harmonizable.*

Proof. Let $f_k\in C^\infty(\mathbf{T})$, $k=0,\ldots,N-1$ be functions such that
1. $0\leqslant f_k(t)\leqslant 1$, $k=0,\ldots,N-1$,
2. $\sum_{k=0}^{N-1} f_k(t) = 1$, $t\in[0,2\pi)$,
3. $\operatorname{supp}(f_k\otimes f_k)\subset\{(s,t)\colon |s-t|<d\}$.

Let $F_k(f) = F_x(ff_k)$, $f\in C^\infty(\mathbf{T})$. Since $\lambda_0 = 1$ is a root of p with multiplicity one, the function

$$\eta(s,t) = \frac{p(e^{i(s-t)})}{(1-e^{i(s-t)})}$$

is nonzero in the strip $|s-t|<d$. Moreover, because $\operatorname{supp}(f_k\otimes f_k)\subset\{(s,t)\in\mathbf{T}^2\colon |s-t|<d\}$, the function $f_k(s)\overline{f_k(t)}/\eta(s,t)$ is in $C^\infty(\mathbf{T}^2)$ for every k. Hence, by (8),

$$D_x((1-e^{i(s-t)})f_k(s)\overline{f_k(t)}f(s,t)) = D_x\left(p(e^{i(s-t)})\frac{f_k(s)\overline{f_k(t)}}{\eta(s,t)}f(s,t)\right) = 0 \tag{A.8}$$

for all $f \in C^\infty(\mathbf{T}^2)$. Let $y_n^k = F_k(e^{in\cdot})$, $n \in \mathbb{Z}$, $k = 0,\dots,N-1$. By (7) The spectrum of $y^k = \{y_n^k = F_k(e^{in\cdot})\colon n \in \mathbb{Z}\}$ is given by

$$D_{y^k}(f) = D_x(f_k(s)\overline{f_k(t)}f(s,t)), \quad f \in C^\infty(\mathbf{T}^2)$$

and in view of (A.7) $D_{y^k}(f) = D_{y^k}(e^{i(s-t)}f(s,t))$, for each $k = 0,\dots,N-1$ and $f \in C^\infty(\mathbf{T}^2)$. Therefore, by Lemma 2 Part 2., each sequence $y^k = \{y_n^k\colon n \in \mathbb{Z}\}$, $k = 0,\dots,N-1$, is stationary and hence harmonizable. Since $x_n = \sum_{k=0}^{N-1} y_n^k$, x is also harmonizable, that is there exist an $\mathbf{H}$-valued measure μ such that $x_n = \int e^{int}\mu(dt)$, $n \in \mathbb{Z}$. □

Recall that an r-dimensional stochastic sequence in a Hilbert space $\mathbf{K}$ is a sequence $\boldsymbol{y} = \{\boldsymbol{y}_n\colon n \in \mathbb{Z}\}$, where $\boldsymbol{y}_n = (y_n^k)_{k=0,\dots,r-1}$ and $y_n^k \in \mathbf{K}$. An r-dimensional stochastic sequence $\boldsymbol{y}$ is called stationary if the matrix correlation function $\boldsymbol{R}_y(m,n) = (\boldsymbol{y}_m, \boldsymbol{y}_n) = [(y_m^k, y_n^l)]_{k,l=0,\dots,r-1}$ depends only on $m-n$. The matrix correlation function $\boldsymbol{R}_y$ of an r-dimensional stationary stochastic sequence $\boldsymbol{y}$ admits the representation

$$\boldsymbol{R}_y(m,n) = \int_0^{2\pi} e^{i(m-n)t}\boldsymbol{\Gamma}_y(dt), \quad m,n \in \mathbb{Z} \tag{A.9}$$

where $\boldsymbol{\Gamma}_y$ is a countably additive nonnegative definite matrix-valued measure on $\mathbf{T}$ called the spectral measure of $\boldsymbol{y}$.

Lemma 4.5. *Let* $a = \min\{|\theta_i - \theta_j - \theta_k|\colon \theta_i - \theta_j - \theta_k \neq 0\}$ *and let* $q \in \mathbb{N}$ *be such that* $2\pi \leqslant qa$. *Let* $E_j = [2\pi j/q, 2\pi(j+1)/q)$, $j = 0,\dots,q-1$. *For each* $k = 0,\dots,r-1$, *define a* $\mathbf{K} = \mathbf{H}^q$*-valued sequence* y_n^k *by the formula*

$$y_n^k = \bigoplus_{j=0}^{q-1} \int_0^{2\pi} e^{int} 1_{E_j}(t)\mu(dt - \theta_k) \tag{A.10}$$

$n \in \mathbb{Z}$, *where* μ *is the random spectral measure of* x (*recall that addition in* $[0,2\pi)$ *is modulo* 2π *sense*). *Then* $\boldsymbol{y}_n = (y_n^0,\dots,y_n^{r-1})$, $n \in \mathbb{Z}$, *is an* r*-dimensional stationary sequence in* $\mathbf{K}$.

Proof. Since x is harmonizable,

$$|D_x(f(s)g(t))| = \left|\left(\int f\,d\mu, \int \bar{g}\,d\mu\right)\right| \leqslant \|f\|_2\|g\|_2,$$

where $\|\cdot\|_2$ denotes the L^2-norm with respect to some probability measure (e.g. Graham and Schreiber, 1984). Therefore, the formula (A.7) as well as the fact established in Theorem 4 that $D_x(f) = 0$ provided that $\operatorname{supp}(f) \cap \bigcup_{k=0}^{r-1} D_{\theta_k} = \emptyset$, remain valid for functions f, g which are pointwise limits of uniformly bounded sequences of the form $f_n \otimes g_n$, where $f_n, g_n \in C^\infty(\mathbf{T})$ and $\operatorname{supp}(f_n \otimes g_n)$, $n = 1,2,\dots$, satisfy appropriate constrains. In particular, taking $f(s) = e^{ims}1_{E_j - \theta_k}(s)$ and $g(t) = e^{int}1_{E_l - \theta_l}(t)$, where $0\leqslant k,l\leqslant r-1$, we obtain

$$\left(\int e^{ims}1_{E_j-\theta_k}(s)\mu(ds), \int e^{int}1_{E_j-\theta_l}(t)\mu(dt)\right)$$
$$= D_x(e^{i(ms-nt)}1_{E_j-\theta_k}(s)1_{E_j-\theta_l}(t))$$
$$= \begin{cases} e^{-i\theta_p}D_x(e^{i(s-t)}e^{i(ms-nt)}1_{E_j-\theta_k}(s)1_{E_j-\theta_l}(t)) & \text{if } \theta_l-\theta_k=\theta_p \text{ for some } p, \\ 0 & \text{otherwise} \end{cases}$$
$$= \left(\int e^{i(m+1)s}1_{E_j-\theta_k}(s)\mu(ds), \int e^{i(n+1)t}1_{E_j-\theta_l}(t)\mu(dt)\right).$$

This shows that $(y_m^k, y_n^l)=(y_{m+1}^k, y_{n+1}^l)$, $m,n\in\mathbb{Z}$, and hence $\mathbf{y}$ is stationary. □

Note that the random spectral measure ν_k of $\{y_n^k: n\in\mathbb{Z}\}$ is given by

$$\nu_k(\Delta)=\bigoplus_{j=0}^{q-1}\mu((\Delta\cap E_j)-\theta_k)$$

for all $k=0,\ldots,r-1$ and $\Delta\in B(\mathbf{T})$. Therefore, the spectral measure $\Gamma_y(\Delta)=[\Gamma_{k,l}(\Delta)]_{k,l=0,\ldots,r-1}$ of $\mathbf{y}$ has the form

$$\Gamma_{kl}(\Delta)=(\nu_k(\Delta),\nu_l(\Delta))=\sum_{j=0}^{q-1}(\mu((\Delta\cap E_j)-\theta_k),\mu((\Delta\cap E_j)-\theta_l)).$$

Lemma 4.6. *If we define*
1. $\gamma_k(\Delta)=\Gamma_{0k}(\Delta)$, $\Delta\in B(\mathbf{T})$, $k=0,\ldots,r-1$ *and*
2. $\Gamma(\Delta)=\sum_{k=0}^{r-1}\gamma_k\{s\in\mathbf{T}: (s,s-\theta_k)\in\Delta\}$, $\Delta\in B(\mathbf{T}^2)$,

then Γ *is the spectrum of* x, *that is*

$$D_x(f)=\iint f(s,t)\Gamma(ds,dt), \quad f\in C^\infty(\mathbf{T}^2). \tag{A.11}$$

Proof. Since by Lemma 4.4, D_x is a bimeasure, it is enough to prove (A.11) for functions of the form $f=1_{[u,v)}\otimes 1_{[w,z)}$, where $[u,v)$ and $[w,z)$ are intervals (*arcs*) in $\mathbf{T}$ of length smaller than $a/2$, and $[u,v)$ is contained in only one interval E_k. If $1_{[u,v)}\otimes 1_{[w,z)}$ is zero on each set $D_j=\{(s,t)\in\mathbf{T}^2: s-t=\theta_j\}$, $j=0,\ldots,r-1$, then $D_x(1_{[u,v)}\otimes 1_{[w,z)})=0$, and (A.11) holds true. Otherwise there is exactly one line, say D_j, that intersects $[u,v)\otimes[w,z)$. In the latter case, assuming that $[u,v)\subset E_k$, we have

$$D_x(1_{[u,v)}\otimes 1_{[w,z)})$$
$$=(\mu([u,v)\cap E_k),\mu([w,z)\cap(E_k-\theta_j))$$
$$=(\mu([u,v)\cap E_k),\mu(([w,z)+\theta_j)\cap E_k-\theta_j))$$
$$=(\nu_0[u,v),\nu_j([w,z)+\theta_j))=\Gamma_{0j}([u,v)\cap([w,z)+\theta_j))$$
$$=\gamma_j([u,v)\cap([w,z)+\theta_j))=\int 1_{[u,v)}1_{[w,z)}(s-\theta_j)\gamma_j(ds)$$

$$= \sum_{k=0}^{r-1} \int 1_{[u,v)}(s) 1_{[w,z)}(s - \theta_k) \gamma_k(\mathrm{d}s)$$

$$= \int\int 1_{[u,v)}(t) 1_{[w,z)}(s) \Gamma(\mathrm{d}s, \mathrm{d}t). \qquad \square$$

Note that by the definition the measure Γ sits on lines $D_k = \{(s, s - \theta_k): s \in \mathbf{T}\}$, $k = 0, \ldots, r - 1$, and hence the essential implication 1. $\Rightarrow$ 3. of Theorem 5 is proved.

Acknowledgements

We would like to express our thanks to the referees for their suggestions which significantly improved the presentation.

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