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Equivalences of derived categories for symmetric algebras

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1. Introduction

It is about a decade since Broué made his celebrated conjecture [2] on equivalences of derived categories in block theory: that the module categories of a block algebra A of a finite group algebra and its Brauer correspondent B should have equivalent derived categories if their defect group is abelian. Since then, character-theoretic evidence for the conjecture has accumulated rapidly, but until very recently there have been very few examples where the conjecture has actually been verified. This is because the precise structure of, say, the indecomposable projective modules for A is known only in the simplest cases (although the corresponding structure for B is much easier to determine): this makes it very difficult to carry out explicit calculations to verify an equivalence of derived categories.

Recently, however, Okuyama [6] introduced a method of proving that there is an equivalence of derived categories that needs very little explicit information about A .

In many of the simpler cases where Broué's conjecture is not yet known to be true, there is known to be a 'stable equivalence of Morita type' between A and B : an exact functor between the module categories that is an equivalence of categories 'modulo projective modules'. This is a consequence of an equivalence of derived categories, since the stable module category is a canonical quotient of the derived category. Moreover, recent work of Rouquier [11,12] gives a method

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of constructing such stable equivalences from equivalences of derived categories for smaller groups.

Okuyama's method is a strategy for lifting stable equivalences to equivalences of derived categories. If one can produce an equivalence of derived categories between B and a third algebra C , and if the objects of the derived category $D^b(\text{mod}(B))$ that correspond to the simple C -modules are isomorphic in the stable module category of B (regarded as a quotient category of $D^b(\text{mod}(B))$) to the images of the simple A -modules under a stable equivalence of Morita type, then it follows from a theorem of Linckelmann [5, Theorem 2.1] that A and C are Morita equivalent, and so A and B have equivalent derived categories. Note that to carry out this strategy, one needs to know nothing about A except which objects of the stable module category of B correspond to the simple A -modules.

Okuyama used this method to verify Broué's conjecture for many examples of blocks with defect group $C_3 \times C_3$ [6].

This still leaves the problem of finding a suitable equivalence

$$D^b(\text{mod}(B)) \approx D^b(\text{mod}(C)) \quad (1.1)$$

of derived categories. Okuyama did this by constructing a suitable 'tilting complex' T for B : by the main theorem of [8], T is the object corresponding to the free C -module under an equivalence (1.1), where C is the endomorphism algebra of T . The main theorem of this paper, Theorem 5.1, gives an alternative approach. Rather than characterizing the objects of $D^b(\text{mod}(B))$ that correspond to free modules under equivalences of derived categories, as the definition of a tilting complex does, we characterize the sets of objects that can correspond to the simple modules. Our proof requires only that B should be a symmetric algebra, which is of course the case for a block of a finite group algebra; it is easy to construct counterexamples for general finite-dimensional algebras, but we do not know any counterexamples for self-injective algebras.

Since Linckelmann's theorem focuses on the simple modules, this new characterization of derived equivalence is well suited to applying his theorem as Okuyama did. In Section 7 we give several fairly simple examples. Simple modules have a less complicated structure than projective modules, so it is no surprise that in our examples the objects of $D^b(\text{mod}(B))$ that we construct (corresponding to the simple A -modules) are considerably simpler than the tilting complex (corresponding to a free A -module) would be.

In a recent paper [3], Chuang has used our main theorem to give a proof of Broué's conjecture for the principal block of $SL(2, p^2)$ in characteristic p for an arbitrary prime p . Using different methods, this result has subsequently been extended to $SL(2, p^n)$ for all n by Okuyama [7]. Holloway [4] has also used our main theorem to verify Broué's conjecture for several blocks, including the principal block of the sporadic Hall–Janko group in characteristic five, using computer calculations.

2. Conventions and notation

Throughout most of this paper, k will be an algebraically closed field. This is not really essential, but for the sake of clarity we prefer to avoid the (entirely routine) added complication involved in dealing with a field that is not algebraically closed. In Section 8 we shall give details of the minor modifications that need to be made to deal with a general field.

By a ‘module’ for a ring, we shall mean a *left* module unless we specify otherwise. If Λ is a ring, then $\text{Mod}(\Lambda)$ will be the category of all (left) Λ -modules. If Λ is a finite-dimensional k -algebra, then $\text{mod}(\Lambda)$ will be the category of finitely generated Λ -modules and $\text{proj}(\Lambda)$ will be the category of finitely generated projective Λ -modules.

If Λ is a finite-dimensional self-injective k -algebra (i.e., the injective and projective Λ -modules coincide), then $\text{stmod}(\Lambda)$ will be the stable module category, which is the quotient of $\text{mod}(\Lambda)$ by the ideal of maps that factor through projective modules. This is a triangulated category with shift functor the inverse Ω^{-1} of the Heller translate. The space of maps from M to N in $\text{stmod}(\Lambda)$ will be denoted by $\underline{\text{Hom}}_{\Lambda}(M, N)$.

Our ‘complexes’ will all be *cochain* complexes, so the differentials will have degree +1. If

$$X := \dots \longrightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \longrightarrow \dots$$

is a cochain complex, then $X[m]$ will be X ‘shifted m places to the left,’ i.e., $X[m]^i = X^{i+m}$ and $d_{X[m]}^i = (-1)^m d_X^{i+m}$.

If \mathcal{A} is an additive category, then $K(\mathcal{A})$ will be the chain homotopy category of cochain complexes over \mathcal{A} , $K^-(\mathcal{A})$ will be the full subcategory consisting of complexes X that are ‘bounded above’ (i.e., $X^i = 0$ for $i \gg 0$) and $K^b(\mathcal{A})$ will be the full subcategory of ‘bounded’ complexes (i.e., complexes X with $X^i = 0$ for all but finitely many i).

If \mathcal{A} is an abelian category, then $D(\mathcal{A})$ will be the derived category of cochain complexes over \mathcal{A} , and $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$ will be the full subcategories of complexes that are respectively bounded above and bounded.

We shall regard an abelian category \mathcal{A} as a full subcategory of its derived category $D(\mathcal{A})$ in the usual way, identifying an object X of \mathcal{A} with the complex whose only non-zero term is X in degree zero.

3. Preliminaries on symmetric algebras

A finite-dimensional k -algebra Λ is said to be *symmetric* if there is a *symmetrizing form* on Λ : i.e., a linear map $\lambda : \Lambda \rightarrow k$ such that

$$\forall x, y \in \Lambda, \quad \lambda(xy) = \lambda(yx),$$

and such that the kernel of λ contains no non-zero left or right ideal of Λ .

The principal example is the group algebra kG of a finite group, when λ can be taken to be

$$\lambda(g) = \begin{cases} 1 & \text{if } g = 1, \\ 0 & \text{if } g \in G - \{1\}, \end{cases}$$

extended linearly to the whole of kG . Any block algebra of kG is also a symmetric algebra, using the restriction of the same map λ to the block.

The following theorem, giving characterizations of symmetric algebras in terms of the module category, is well-known, but for the readers' convenience we include a proof.

Theorem 3.1. *Let Λ be a finite-dimensional k -algebra. The following conditions are equivalent.*

- (a) Λ is symmetric.
- (b) Λ and its dual $\Lambda^\vee = \text{Hom}_k(\Lambda, k)$ are isomorphic as Λ -bimodules.
- (c) $\text{Hom}_k(?, k)$ and $\text{Hom}_\Lambda(?, \Lambda)$ are isomorphic as functors from the category of left Λ -modules to the category of right Λ -modules.
- (c') $\text{Hom}_k(?, k)$ and $\text{Hom}_\Lambda(?, \Lambda)$ are isomorphic as functors from the category of right Λ -modules to the category of left Λ -modules.
- (d) For finitely generated projective left Λ -modules P and finitely generated left Λ -modules M , there is an isomorphism of k -vector spaces

$$\text{Hom}_\Lambda(P, M) \cong \text{Hom}_\Lambda(M, P)^\vee,$$

functorial in both P and M .

- (d') For finitely generated projective right Λ -modules P and finitely generated right Λ -modules M , there is an isomorphism of k -vector spaces

$$\text{Hom}_\Lambda(P, M) \cong \text{Hom}_\Lambda(M, P)^\vee,$$

functorial in both P and M .

Proof. Since (a) and (b) are left-right symmetric, it is sufficient to prove that (a) and (b) are equivalent and that (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (b).

First we shall show that (a) implies (b). Let $\lambda : \Lambda \rightarrow k$ be a symmetrizing form on Λ . Consider the map

$$\theta : \Lambda \rightarrow \text{Hom}_k(\Lambda, k)$$

defined by $\theta(x) = x.\lambda$ for $x \in \Lambda$: i.e., $\theta(x) : \Lambda \rightarrow k$ is the map

$$y \mapsto \lambda(yx) = \lambda(xy).$$

Since $\theta(xz)$ is the map

$$y \mapsto \lambda(yxz) = \lambda(zyx) = (x.\lambda.z)(y),$$

θ is a Λ -bimodule homomorphism.

Now let $x \in \Lambda$, and suppose $\theta(x) = 0$. Then $\lambda(yx) = 0$ for all $y \in \Lambda$, and so the left ideal Λx is contained in the kernel of λ . Since λ is a symmetrizing form, this implies $x = 0$. Hence θ is injective, and is therefore an isomorphism, since Λ and Λ^\vee are vector spaces of the same dimension.

To show (b) implies (a), suppose $\theta : \Lambda \rightarrow \Lambda^\vee$ is a Λ -bimodule isomorphism. Let $\lambda = \theta(1)$. Then for $x, y \in \Lambda$,

$$\begin{aligned} \lambda(xy) &= \theta(1)(xy) = (y\theta(1))(x) = \theta(y)(x) = (\theta(1)y)(x) = \theta(1)(yx) \\ &= \lambda(yx). \end{aligned}$$

Also, suppose the left ideal Λx is contained in the kernel of λ . Then for every $y \in \Lambda$,

$$0 = \lambda(yx) = (x\theta(1))(y) = \theta(x)(y),$$

and so $\theta(x) = 0$, and so $x = 0$, since θ is an isomorphism. Similarly, no non-zero right ideal of Λ is contained in the kernel of λ , and so λ is a symmetrizing form on Λ .

Next we show that (b) implies (c). Suppose that Λ and Λ^\vee are isomorphic as Λ -bimodules. Let M be a left Λ -module. There is a chain of natural isomorphisms of right Λ -modules

$$\begin{aligned} \text{Hom}_\Lambda(M, \Lambda) &\cong \text{Hom}_\Lambda(M, \text{Hom}_k(\Lambda, k)) \cong \text{Hom}_k(\Lambda \otimes_\Lambda M, k) \\ &\cong \text{Hom}_k(M, k). \end{aligned}$$

Next assume that (c) is true, and let us deduce (d). Let M and P be finitely generated left Λ -modules. Then $\text{Hom}_\Lambda(M, P) \cong \text{Hom}_k(P^\vee \otimes_\Lambda M, k)$, and since these are all finite-dimensional vector spaces, $\text{Hom}_\Lambda(M, P)^\vee \cong P^\vee \otimes_\Lambda M$, which, since (c) is true, is in turn isomorphic to $\text{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda M$. There is a natural map

$$\text{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda M \longrightarrow \text{Hom}_\Lambda(P, M), \tag{3.1}$$

sending $\alpha \otimes m$ (where $\alpha \in \text{Hom}_\Lambda(P, \Lambda)$ and $m \in M$) to the map $P \rightarrow M$ sending $p \in P$ to $\alpha(p)m$. For $P = \Lambda$, it is easy to check that (3.1) is an isomorphism between vector spaces naturally isomorphic to M . By naturality, (3.1) is also an isomorphism for P any direct summand of a finite direct sum of copies of Λ ; i.e., for any finitely generated projective module P .

Finally, the fact that (d) implies (b) follows by taking $M = P = {}_\Lambda \Lambda$. The natural isomorphism

$$\Lambda \cong \text{Hom}_\Lambda(\Lambda, \Lambda)$$

is an isomorphism of Λ -bimodules, where the bimodule structure on the right-hand side is induced by the *right* action of Λ by multiplication on the two

arguments. But by (d) the right-hand side is naturally isomorphic to its dual, and by naturality this isomorphism is an isomorphism of bimodules. Hence (b). \square

We shall most often be using condition (d).

Here is a corollary about maps in the derived category.

Corollary 3.2. *Let Λ be a finite-dimensional symmetric k -algebra. Let P^* be a bounded complex of finitely generated projective left Λ -modules, and let M^* be a complex of finitely generated left Λ -modules. Then $\text{Hom}_{D(\text{Mod}(\Lambda))}(P^*, M^*)$ and $\text{Hom}_{D(\text{Mod}(\Lambda))}(M^*, P^*)$ are naturally dual.*

Proof. Since P^* is a bounded complex of projective and (since Λ is symmetric) injective modules, calculating homomorphisms from or to P^* in the derived category is equivalent to doing so in the chain homotopy category $K(\text{Mod}(\Lambda))$.

Recall that if X^* and Y^* are complexes of Λ -modules, then $\text{Hom}_K(\text{Mod}(\Lambda))(X^*, Y^*)$ may be calculated as the degree zero homology of the ‘completed’ total complex of the double complex $\text{Hom}_\Lambda(X^*, Y^*)$ (i.e., the variation of the total complex where the terms are formed by taking direct *products* rather than the direct *sums* of diagonals in the double complex). In fact, the fact that P^* is bounded ensures that for $X^* = P^*$ and $Y^* = M^*$ or vice versa, there is only a finite number of non-zero terms on each diagonal, and so the completed total complex is the same as the usual total complex.

Condition (d) of Theorem 3.1 implies that the double complexes $\text{Hom}_\Lambda(P^*, M^*)$ and $\text{Hom}_\Lambda(M^*, P^*)$ are naturally dual. By the remark at the end of the last paragraph, their total complexes are complexes of finite dimensional vector spaces and are also naturally dual. Taking degree zero homology, the corollary follows. \square

4. Preliminaries on homotopy colimits

In this section, Λ will be an arbitrary ring. Let

$$X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots$$

be a sequence of maps in a triangulated category \mathcal{T} with countable coproducts. Recall [1] that the *homotopy colimit* $\text{hocolim}(X_i)$ of this sequence is defined by forming the distinguished triangle

$$\bigoplus_{i=0}^{\infty} X_i \longrightarrow \bigoplus_{i=0}^{\infty} X_i \longrightarrow \text{hocolim}(X_i) \longrightarrow \bigoplus_{i=0}^{\infty} X_i[1], \tag{4.1}$$

where the restriction of the first map to X_i is $\text{id}_{X_i} - \alpha_i$. This defines $\text{hocolim}(X_i)$ up to isomorphism, but not usually up to *unique* isomorphism.

If \mathcal{T} is the derived category $D(\text{Mod}(\Lambda))$, and if we can choose chain maps β_i representing the maps α_i of the derived category, then $\text{hocolim}(X_i)$ is isomorphic to the usual colimit in the category of chain complexes of the sequence

$$X_0 \xrightarrow{\beta_0} X_1 \xrightarrow{\beta_1} X_2 \xrightarrow{\beta_2} \dots$$

This is an easy consequence of the fact that the coproduct, in the derived category, of a family of complexes is the same as the coproduct in the category of chain complexes.

An object C of the triangulated category \mathcal{T} is called *compact* if the functor $\text{Hom}(C, ?)$ commutes with arbitrary coproducts: more precisely, if the natural map

$$\bigoplus_{i \in I} \text{Hom}(C, X_i) \longrightarrow \text{Hom}\left(C, \bigoplus_{i \in I} X_i\right)$$

is an isomorphism whenever $\{X_i: i \in I\}$ is a set of objects of \mathcal{T} whose coproduct exists in \mathcal{T} .

In the derived category $D(\text{Mod}(\Lambda))$ of a module category, the compact objects are precisely those that are isomorphic to bounded complexes of finitely generated projective modules.

An easy consequence of the definition is that if C is a compact object, then there is a natural isomorphism

$$\text{Hom}(C, \text{hocolim}(X_i)) \longrightarrow \text{colim}(\text{Hom}(C, X_i)).$$

We shall need a generalization of this. Let us say that a family \mathcal{X} of objects of the derived category $D(\text{Mod}(\Lambda))$ is *uniformly bounded below* if there is some $n \in \mathbb{Z}$ such that the degree j cohomology $H^j(X)$ of X is zero for all $X \in \mathcal{X}$ and all $j < n$.

Proposition 4.1. *Let C be an object of $D(\text{Mod}(\Lambda))$ isomorphic to a complex of finitely generated projective Λ -modules that is bounded above. For example, if Λ is a finite-dimensional k -algebra, let C be any object of $D^-(\text{mod}(\Lambda))$.*

(a) *Let \mathcal{X} be a family of objects of $D(\text{Mod}(\Lambda))$ that is uniformly bounded below. Then the natural map*

$$\bigoplus_{X \in \mathcal{X}} \text{Hom}(C, X) \longrightarrow \text{Hom}\left(C, \bigoplus_{X \in \mathcal{X}} X\right)$$

is an isomorphism.

(b) *Let $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots$ be a sequence of maps in the derived category $D(\text{Mod}(\Lambda))$, and suppose that $\{X_i: i \in \mathbb{N}\}$ is uniformly bounded below. Then*

$$\text{Hom}(C, \text{hocolim}(X_i)) \cong \text{colim}(\text{Hom}(C, X_i)).$$

Proof. If Λ is a finite-dimensional k -algebra, then any object of $D^-(\text{mod}(\Lambda))$ is isomorphic to its minimal projective resolution, which is a complex of finitely generated projective Λ -modules that is bounded above.

(a) Without loss of generality, we shall assume that every $X \in \mathcal{X}$ is a complex with no non-zero terms in negative degrees, and that C is the bounded above complex

$$\dots \rightarrow C^{-2} \rightarrow C^{-1} \rightarrow C^0 \rightarrow C^1 \rightarrow \dots,$$

of finitely generated projectives. Let \tilde{C} be the truncated complex

$$\dots \rightarrow 0 \rightarrow C^{-1} \rightarrow C^0 \rightarrow C^1 \rightarrow \dots,$$

which is a bounded complex of finitely generated projectives. The inclusion $\tilde{C} \rightarrow C$ of complexes induces an isomorphism

$$\text{Hom}_{D(\text{Mod}(\Lambda))}(C, Y) \rightarrow \text{Hom}_{D(\text{Mod}(\Lambda))}(\tilde{C}, Y)$$

for $Y \in \mathcal{X}$ and for $Y = \bigoplus_{X \in \mathcal{X}} X$. The proposition follows because \tilde{C} is compact.

(b) This follows because, by (a), the natural map

$$\bigoplus_{i \in \mathbb{N}} \text{Hom}(C, X_i[m]) \rightarrow \text{Hom}\left(C, \bigoplus_{i \in \mathbb{N}} X_i[m]\right)$$

is an isomorphism for every m , and so the long exact sequence obtained by applying the functor $\text{Hom}(C, ?)$ to the triangle (4.1) breaks up into a sequence of short exact sequences, including one isomorphic to

$$\begin{aligned} 0 \rightarrow \bigoplus_{i \in \mathbb{N}} \text{Hom}(C, X_i) &\rightarrow \bigoplus_{i \in \mathbb{N}} \text{Hom}(C, X_i) \rightarrow \text{Hom}(C, \text{hocolim}(X_i)) \\ &\rightarrow 0, \end{aligned}$$

which expresses $\text{Hom}(C, \text{hocolim}(X_i))$ as the colimit of $\text{Hom}(C, X_i)$. \square

5. The main theorem

Let Λ and Γ be two rings. A necessary and sufficient condition [8] for the derived categories $D(\text{Mod}(\Lambda))$ and $D(\text{Mod}(\Gamma))$ to be equivalent as triangulated categories is that there should be a *tilting complex* in $D(\text{Mod}(\Lambda))$ whose endomorphism ring is isomorphic to Γ : i.e., a bounded complex T of finitely generated projective Λ -modules such that

- (i) $\text{Hom}(T, T[m]) = 0$ for $m \neq 0$, and
- (ii) the direct summands of T generate $K^b(\text{proj}(\Lambda))$ as a triangulated category,

with $\text{End}(T) \cong \Gamma$.

If T is such a tilting complex, then there is an equivalence $D(\text{Mod}(\Lambda)) \approx D(\text{Mod}(\Gamma))$ that sends T to the free Γ -module of rank one, so the indecomposable summands of T correspond to the indecomposable projective Γ -modules.

We shall use later the fact [8] that, for a finite-dimensional algebra Λ , condition (ii) can be replaced by the condition that, for any non-zero object C of $D^-(\text{mod}(\Lambda))$, $\text{Hom}(T, C[m]) \neq 0$ for some $m \in \mathbb{Z}$.

In this section we shall consider instead the objects X_0, \dots, X_r of $D(\text{Mod}(\Lambda))$ that correspond to the simple Γ -modules in the case that Λ and Γ are finite-dimensional symmetric k -algebras. Since an equivalence $D(\text{Mod}(\Lambda)) \approx D(\text{Mod}(\Gamma))$ restricts to an equivalence between the full subcategories of objects isomorphic to bounded complexes of finitely generated modules, X_0, \dots, X_r must (up to isomorphism) be objects of $D^b(\text{mod}(\Lambda))$. Also, for $0 \leq i, j \leq r$, they must satisfy

- (a) $\text{Hom}(X_i, X_j[m]) = 0$ for $m < 0$,
- (b) $\text{Hom}(X_i, X_j) = \begin{cases} 0 & \text{if } i \neq j, \\ k & \text{if } i = j, \end{cases}$ and
- (c) X_0, \dots, X_r generate $D^b(\text{mod}(\Lambda))$ as a triangulated category,

since the simple Γ -modules satisfy the corresponding properties.

In this section we shall prove a partial converse to this.

Theorem 5.1. *Let Λ be a finite-dimensional symmetric k -algebra. Let X_0, \dots, X_r be objects of $D^b(\text{mod}(\Lambda))$ satisfying conditions (a)–(c) above. Then there is another finite-dimensional symmetric k -algebra Γ and an equivalence of triangulated categories*

$$D(\text{Mod}(\Lambda)) \approx D(\text{Mod}(\Gamma))$$

sending X_0, \dots, X_r to the simple Γ -modules.

We shall give the proof as a sequence of lemmas.

What we shall do is construct a tilting complex $T = T_0 \oplus \dots \oplus T_r$ for Λ such that, for $0 \leq i, j \leq r$ and $m \in \mathbb{Z}$,

$$\text{Hom}(T_i, X_j[m]) = \begin{cases} k & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Lemma 5.2. *Suppose there is a tilting complex $T = T_0 \oplus \dots \oplus T_r$ for Λ satisfying the property (5.1). Then Theorem 5.1 is true.*

Proof. For $\Gamma = \text{End}(T)$, there is an equivalence of triangulated categories

$$F : D(\text{Mod}(\Lambda)) \rightarrow D(\text{Mod}(\Gamma))$$

sending T_0, \dots, T_r to the indecomposable projective Γ -modules P_0, \dots, P_r . Since the simple Γ -modules S_0, \dots, S_r , numbered so that P_i is the projective

cover of S_i , are characterized up to isomorphism in $D(\text{Mod}(\Gamma))$ by the fact that, for $0 \leq i, j \leq r$ and $m \in \mathbb{Z}$,

$$\text{Hom}(P_i, S_j[m]) = \begin{cases} k & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise,} \end{cases}$$

the equivalence F sends X_0, \dots, X_r to the simple modules S_0, \dots, S_r .

Since a ring whose derived category is equivalent to a finite-dimensional symmetric k -algebra is itself a finite-dimensional symmetric k -algebra [9], Theorem 5.1 follows. \square

Now let us construct the summands T_i of the tilting complex T . Set $X_i^{(0)} := X_i$. By induction on n , we shall construct a sequence

$$X_i^{(0)} \rightarrow X_i^{(1)} \rightarrow \dots \rightarrow X_i^{(n-1)} \rightarrow X_i^{(n)} \rightarrow \dots$$

of objects and maps in $D(\text{Mod}(\Lambda))$.

Suppose we have constructed $X_i^{(n-1)}$. For each $0 \leq j \leq r$ and $t < 0$, choose a basis $B_i^{(n-1)}(j, t)$ of $\text{Hom}(X_j[t], X_i^{(n-1)})$, let $Z_i^{(n-1)}(j, t)$ be a direct sum of copies of $X_j[t]$ indexed by $B_i^{(n-1)}(j, t)$, and let

$$\alpha_i^{(n-1)}(j, t) : Z_i^{(n-1)}(j, t) \rightarrow X_i^{(n-1)}$$

be the map whose restriction to the summand of $Z_i^{(n-1)}(j, t)$ corresponding to an element $\beta \in B_i^{(n-1)}(j, t)$ is just β . Now let

$$Z_i^{(n-1)} = \bigoplus_{\substack{0 \leq j \leq r \\ t < 0}} Z_i^{(n-1)}(j, t),$$

and let

$$\alpha_i^{(n-1)} : Z_i^{(n-1)} \rightarrow X_i^{(n-1)}$$

be the map whose restriction to the summand $Z_i^{(n-1)}(j, t)$ is $\alpha_i^{(n-1)}(j, t)$.

All that will matter about this map is that

- (a) $Z_i^{(n-1)}$ is a (possibly infinite) direct sum of objects, each of which is isomorphic to $X_j[t]$ for some $0 \leq j \leq r$ and some $t < 0$;
- (b) the map $\text{Hom}(X_j[t], Z_i^{(n-1)}) \rightarrow \text{Hom}(X_j[t], X_i^{(n-1)})$ induced by $\alpha_i^{(n-1)}$ is surjective for every $0 \leq j \leq r$ and $t < 0$; and
- (c) the map $\text{Hom}(X_j[-1], Z_i^{(n-1)}) \rightarrow \text{Hom}(X_j[-1], X_i^{(n-1)})$ induced by $\alpha_i^{(n-1)}$ is an isomorphism for every $0 \leq j \leq r$.

The map we have constructed clearly satisfies properties (a) and (b). Since

$$\text{Hom}(X_j[-1], X_{j'}[t]) = \text{Hom}(X_j, X_{j'}[t + 1]) = 0$$

for all $0 \leq j, j' \leq r$ and $t < 0$ except for $j = j'$ and $t = -1$, property (c) is also satisfied.

Now define $X_i^{(n)}$, together with a map from $X_i^{(n-1)}$, by forming the distinguished triangle

$$Z_i^{(n-1)} \xrightarrow{\alpha_i^{(n-1)}} X_i^{(n-1)} \longrightarrow X_i^{(n)} \longrightarrow Z_i^{(n-1)}[1]. \tag{5.2}$$

Finally, define $T_i := \text{hocolim}(X_i^{(n)})$.

Remark. In general it will not be possible to choose $Z_i^{(n-1)}$ in the bounded derived category $D^b(\text{mod}(\Lambda))$ that satisfy properties (a)–(c), even if $X_i^{(n-1)}$ is in $D^b(\text{mod}(\Lambda))$. However, if Λ is the group algebra of a finite group, then one can prove, using the fact that the cohomology algebra is finitely generated, that we can do so, and so, by induction on n , we may force every $X_i^{(n)}$ to be in $D^b(\text{mod}(\Lambda))$. However, it is still not obvious a priori that the homotopy colimit T_i will be in $D^b(\text{mod}(\Lambda))$.

Lemma 5.3. *If Y is an object of $D^-(\text{mod}(\Lambda))$, then*

$$\text{Hom}(Y, T_i) \cong \text{colim}(\text{Hom}(Y, X_i^{(n)})).$$

Proof. A projective resolution of Y is a complex of finitely generated projective Λ -modules that is bounded above, so the lemma will follow from Proposition 4.1(b) if we can show that $\{X_i^{(n)} : n \geq 0\}$ is uniformly bounded below.

Since $\{X_i : 0 \leq i \leq r\}$ is a finite set of objects of $D^b(\text{mod}(\Lambda))$, we shall assume, without loss of generality, that the cohomology $H^m(X_i)$ vanishes for all $m < 0$. But then $H^m(Z_i^{(n)}) = 0$ for all i, n , and all $m < 1$, and so, by induction on n and the long exact sequence of homology for the distinguished triangle (5.2), we have $H^m(X_i^{(n)}) = 0$ for all $m < 0$ and all n . In other words, $\{X_i^{(n)} : n \geq 0\}$ is uniformly bounded below, as required. \square

Lemma 5.4. *For $0 \leq i, j \leq r$, and $m \in \mathbb{Z}$,*

$$\text{Hom}(X_j, T_i[m]) = \begin{cases} k & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We shall prove this lemma by considering the natural map

$$\text{Hom}(X_j, X_i^{(n-1)}[m]) \longrightarrow \text{Hom}(X_j, X_i^{(n)}[m]) \tag{5.3}$$

by using the long exact sequence obtained by applying the functor $\text{Hom}(X_j, ?)$ to the distinguished triangle (5.2).

Since, by Proposition 4.1(a), $\text{Hom}(X_j, Z_i^{(n-1)}[m]) = 0$ for $m \leq 0$, the map (5.3) is an isomorphism for $m < 0$ and is injective for $m = 0$.

By property (c) of $\alpha_i^{(n-1)} : Z_i^{(n-1)} \rightarrow X_i^{(n-1)}$, the map (5.3) is also surjective for $m = 0$.

Since, for $m \leq 0$,

$$\text{Hom}(X_j, X_i^{(0)}[m]) = \begin{cases} k & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise,} \end{cases}$$

Lemma 5.3 implies the cases of the lemma involving $m \leq 0$.

By property (b) of $\alpha_i^{(n-1)}$, the map (5.3) is zero for $m > 0$. So Lemma 5.3 implies that $\text{Hom}(X_j, T_i[m]) = 0$ for $m > 0$. \square

Lemma 5.5. *For each $0 \leq i \leq r$, T_i is a compact object of $D(\text{Mod}(\Lambda))$: i.e., it is isomorphic to a bounded complex of finitely generated projectives.*

Proof. For an object Y of $D(\text{Mod}(\Lambda))$, consider the vector space

$$\bigoplus_{m \in \mathbb{Z}} \text{Hom}(Y, T_i[m]).$$

By Lemma 5.4 this is finite-dimensional for $Y = X_j$, for any $0 \leq j \leq r$. The class of objects Y for which it is finite-dimensional form a full triangulated subcategory of $D(\text{Mod}(\Lambda))$, and so, since $\{X_j : 0 \leq j \leq r\}$ generates $D^b(\text{mod}(\Lambda))$ as a triangulated category, it is finite-dimensional for any Y in $D^b(\text{mod}(\Lambda))$.

In particular,

$$\bigoplus_{m \in \mathbb{Z}} H^m(T_i) \cong \bigoplus_{m \in \mathbb{Z}} \text{Hom}(\Lambda, T_i[m])$$

is finite-dimensional, so T_i is isomorphic to an object of $D^b(\text{mod}(\Lambda))$. Then, for any simple module S , $\bigoplus_{m \in \mathbb{Z}} \text{Hom}(S, T_i[m])$ is finite-dimensional, so a minimal injective resolution of T_i contains only a finite number of copies of the injective hull of S . Thus this injective resolution is a bounded complex of finitely generated injectives, which are also projective, since Λ is symmetric. \square

Lemma 5.6. *For $0 \leq i, j \leq r$ and $m \in \mathbb{Z}$,*

$$\text{Hom}(T_i, X_j[m]) = \begin{cases} k & \text{if } i = j \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since, by Lemma 5.5, we now know that T_i is isomorphic to a bounded complex of projectives, this follows by combining Lemma 5.4 and Corollary 3.2. \square

Lemma 5.7. *For $0 \leq i, j \leq r$ and $m \neq 0$,*

$$\text{Hom}(T_i, T_j[m]) = 0.$$

Proof. By Lemma 5.6, $\text{Hom}(T_i, X_l[m]) = 0$ for all $m < 0$ and $0 \leq l \leq r$. Hence, for any n , $\text{Hom}(T_i, Z_j^{(n)}[m]) = 0$ for $m \leq 0$, since T_i is compact (by Lemma 5.5) and $Z_j^{(n)}$ is a direct sum of copies of *negative* shifts of copies of various X_l s. Applying the functor $\text{Hom}(T_i, ?)$ to the triangle (5.2), we get an exact sequence

$$\begin{aligned} \text{Hom}(T_i, X_j^{(n-1)}[m]) &\rightarrow \text{Hom}(T_i, X_j^{(n)}[m]) \\ &\rightarrow \text{Hom}(T_i, Z_j^{(n-1)}[m+1]) = 0 \end{aligned}$$

for every $m < 0$. So by induction on n , $\text{Hom}(T_i, X_j^{(n)}[m]) = 0$ if $m < 0$.

By Proposition 4.1(b), $\text{Hom}(T_i, T_j[m]) = 0$ if $m < 0$. For $m > 0$ it follows that $\text{Hom}(T_j, T_i[m]) = 0$ by Corollary 3.2. \square

Lemma 5.8. *Let C be an object of $D^-(\text{mod}(\Lambda))$. If $C \not\cong 0$, then for some $0 \leq i \leq r$ and some $m \in \mathbb{Z}$, $\text{Hom}(C, T_i[m]) \neq 0$.*

Proof. Since C is bounded above and X_i is bounded, $\text{Hom}(C, X_i[m]) = 0$ for $m \ll 0$.

However, if $\text{Hom}(C, X_i[m]) = 0$ for all i and m , then $\text{Hom}(C, X) = 0$ for every object X of $D^b(\text{mod}(\Lambda))$, since $\{X_i : 0 \leq i \leq r\}$ generates $D^b(\text{mod}(\Lambda))$ as a triangulated category. In particular, $\text{Hom}(C, \Lambda[m]) = 0$ for all m , and so $C \cong 0$.

Thus, if $C \not\cong 0$, we can choose m and i so that $\text{Hom}(C, X_i[m]) \neq 0$ and m is minimal: i.e., $\text{Hom}(C, X_j[m']) = 0$ for all $0 \leq j \leq r$ and all $m' < m$.

Let us apply the functor $\text{Hom}(C, ?)$ to the triangle (5.2). Since, by Proposition 4.1(a), $\text{Hom}(C, Z_i^{(n-1)}[m]) = 0$, we get an exact sequence

$$0 = \text{Hom}(C, Z_i^{(n-1)}[m]) \rightarrow \text{Hom}(C, X_i^{(n-1)}[m]) \rightarrow \text{Hom}(C, X_i^{(n)}[m]).$$

In other words,

$$\text{Hom}(C, X_i^{(n-1)}[m]) \rightarrow \text{Hom}(C, X_i^{(n)}[m])$$

is injective for every $n \geq 1$. Hence $\text{colim}(\text{Hom}(C, X_i^{(n)}[m])) \neq 0$, since $\text{Hom}(C, X_i^{(0)}[m]) \neq 0$. So by Lemma 5.3, $\text{Hom}(C, T_i[m]) \neq 0$. \square

We now have all the ingredients to complete the proof of our main theorem.

Proof of Theorem 5.1. By Lemmas 5.2 and 5.6, it is sufficient to show that $T = \bigoplus_{0 \leq i \leq r} T_i$ is a tilting complex for Λ .

Lemma 5.5 shows that T is isomorphic in $D(\text{Mod}(\Lambda))$ to a bounded complex of finitely generated projective modules.

Lemma 5.7 shows that $\text{Hom}(T, T[m]) = 0$ for $m \neq 0$.

We just need to show that if C is an object of $D^-(\text{mod}(\Lambda))$ such that $\text{Hom}(T, C[m]) = 0$ for all $m \in \mathbb{Z}$, then $C \cong 0$. Let C be such an object. By Corollary 3.2,

$$\text{Hom}(C, T[-m]) \cong \text{Hom}(T, C[m])^\vee \cong 0$$

for all $m \in \mathbb{Z}$, and so $C \cong 0$ by Lemma 5.8. \square

6. Lifting stable equivalences

Suppose Γ' is another algebra satisfying the conclusion of Theorem 5.1. Then we have an equivalence of derived categories

$$D^b(\text{mod}(\Gamma)) \approx D^b(\text{mod}(\Gamma'))$$

that takes the simple Γ -modules to the simple Γ' -modules. Since the finitely generated projective modules P are characterized up to isomorphism in $D^b(\text{mod}(\Gamma))$ by the fact that $\text{Hom}(P, S[m]) = 0$ for every simple module S and every integer $m \neq 0$, this equivalence of derived categories restricts to give an equivalence

$$\text{proj}(\Gamma) \approx \text{proj}(\Gamma').$$

Hence the algebra Γ is determined up to Morita equivalence by the objects X_0, \dots, X_r .

In general, however, the algebra Γ might be hard to identify. In this section we shall build on an idea of Okuyama to show how this algebra can be identified in certain cases.

First we shall briefly describe Okuyama’s method [6] of lifting stable equivalences to equivalences of derived categories.

6.1. Okuyama’s method

Let A and B be finite-dimensional self-injective k -algebras.

Recall that the stable module category $\text{stmod}(A)$ of A is equivalent to the quotient of triangulated categories

$$D^b(\text{mod}(A))/K^b(\text{proj}(A)).$$

Thus any object X of $D^b(\text{mod}(A))$ determines an object of the stable module category $\text{stmod}(A)$. For example, if X is the bounded complex

$$\dots \rightarrow X^{i-1} \rightarrow X^i \rightarrow X^{i+1} \rightarrow \dots,$$

and if X^i is projective for $i \neq 0$, then this object of $\text{stmod}(A)$ is just X^0 . A little more generally, if all the terms of X except for X^n are projective, then the corresponding object of $\text{stmod}(A)$ is $\Omega^n(X^n)$. If two objects of $D^b(\text{mod}(A))$ become isomorphic in $\text{stmod}(A)$, we shall say that they are *stably isomorphic*.

Since any equivalence of derived categories

$$D^b(\text{mod}(A)) \approx D^b(\text{mod}(B))$$

restricts to an equivalence

$$K^b(\text{proj}(A)) \approx K^b(\text{proj}(B)),$$

an equivalence of derived categories induces an equivalence of stable module categories. This stable equivalence is ‘of Morita type’; i.e., it is induced (up to isomorphism) by an exact functor between the module categories $\text{mod}(A)$ and $\text{mod}(B)$.

Suppose we know that there is a stable equivalence of Morita type

$$F: \text{stmod}(A) \approx \text{stmod}(B),$$

and suppose we can produce an equivalence of derived categories

$$G: D^b(\text{mod}(B)) \approx D^b(\text{mod}(C)),$$

where C is a third self-injective k -algebra: for example, we might construct a tilting complex for B and take C to be its endomorphism algebra. Okuyama’s idea was to use a theorem [5, Theorem 2.1] of Linckelmann, which states that if there is a stable equivalence of Morita type

$$\text{stmod}(A) \approx \text{stmod}(C),$$

for self-injective algebras A and C , that takes the simple A -modules to the simple C -modules, then A and C are Morita equivalent, and so their derived categories are certainly equivalent. Thus if we can choose G above so that the induced equivalence between $\text{stmod}(A)$ and $\text{stmod}(C)$ has this property, then

$$D^b(\text{mod}(A)) \approx D^b(\text{mod}(C)) \approx D^b(\text{mod}(B)).$$

In a typical case of Broué’s conjecture, where A is a block algebra of a group G with abelian defect group D , and B is the Brauer correspondent block algebra of $N_G(D)$, the structure of B is much easier to determine and understand than that of A . This is the reason Okuyama’s method is so useful for proving special cases of Broué’s conjecture: so long as we already know that there is a stable equivalence of Morita type between A and B , and so long as we can determine the images of the simple modules under this equivalence, the method requires no more information about A .

Okuyama has successfully used his method to verify several cases of Broué’s conjecture: for example, many cases involving defect group $C_3 \times C_3$.

6.2. Combining Okuyama’s method with Theorem 5.1

Since Theorem 5.1 not only provides an algebra Γ whose derived category is equivalent to that of A , but also identifies the objects that are sent to the

simple Γ -modules, it immediately combines with Okuyama’s method to give the following theorem.

Theorem 6.1. *Let Λ and Γ be finite-dimensional symmetric k -algebras, let*

$$F : \text{mod}(\Gamma) \rightarrow \text{mod}(\Lambda)$$

be an exact functor inducing a stable equivalence of Morita type, and let $\{S_0, \dots, S_r\}$ be a set of representatives for the isomorphism classes of simple Γ -modules.

If there are objects X_0, \dots, X_r of $D^b(\text{mod}(\Lambda))$ such that, for each $0 \leq i \leq r$, X_i is stably isomorphic to $F(S_i)$, and such that

(a) $\text{Hom}(X_i, X_j[m]) = 0$ for $m < 0$,

(b) $\text{Hom}(X_i, X_j) = \begin{cases} 0 & \text{if } i \neq j \\ k & \text{if } i = j \end{cases}$, and

(c) X_0, \dots, X_r generate $D^b(\text{mod}(\Lambda))$ as a triangulated category,

then $D(\text{Mod}(\Lambda))$ and $D(\text{Mod}(\Gamma))$ are equivalent as triangulated categories.

6.3. Objects with homology concentrated in one degree

As in Theorem 6.1, let Λ and Γ be finite-dimensional symmetric k -algebras, let

$$F : \text{mod}(\Gamma) \rightarrow \text{mod}(\Lambda)$$

be an exact functor inducing a stable equivalence of Morita type, and let $\{S_0, \dots, S_r\}$ be a set of representatives for the isomorphism classes of simple Γ -modules. We shall assume that $F(S_i)$ is indecomposable for every i : if this is not the case, then F has a summand that induces an isomorphic stable equivalence and which does send simple modules to indecomposable modules, so this assumption does not involve any real loss of generality.

In order to use Theorem 6.1 to prove that Λ and Γ have equivalent derived categories, we need to find suitable objects X_0, \dots, X_r of $D^b(\text{mod}(\Lambda))$ that are stably isomorphic to $F(S_0), \dots, F(S_r)$. We shall consider the case where each of these objects has non-zero homology in only one degree. Since X_0, \dots, X_r must certainly be indecomposable if condition (b) of the theorem is to be satisfied, we must then have

$$X_i \cong \Omega^{n_i} F(S_i)[n_i] \tag{6.1}$$

for some integers n_0, \dots, n_r .

Let us consider what is required in order for these objects to satisfy conditions (a) and (b) of Theorem 6.1. It turns out that many of the conditions are satisfied automatically, especially if n_0, \dots, n_r are small.

Recall that if M and N are modules for a finite-dimensional self-injective k -algebra A , then for $m > 0$ there are natural isomorphisms

$$\text{Ext}_A^m(M, N) \cong \text{Hom}_{D(\text{Mod}(A))}(M, N[m]) \cong \underline{\text{Hom}}_A(\Omega^m M, N).$$

Proposition 6.2. *Conditions (a) and (b) of Theorem 6.1 are satisfied by the objects X_i of (6.1) if*

- (i) $\text{End}_A(\Omega^{n_i} F(S_i)) \cong k$ for $0 \leq i \leq r$;
- (ii) $\text{Hom}_A(\Omega^{n_i} F(S_i), \Omega^{n_j} F(S_j)) = 0$ whenever $i \neq j$ and $n_i \leq n_j$;

and if the four equivalent conditions

- (iii) $\underline{\text{Hom}}_A(\Omega^m F(S_i), F(S_j)) = 0$ whenever $-1 > m > n_i - n_j$;
- (iv) $\underline{\text{Hom}}_A(\Omega^m S_i, S_j) = 0$ whenever $-1 > m > n_i - n_j$;
- (v) $\text{Ext}_A^p(F(S_j), F(S_i)) = 0$ whenever $0 < p < n_j - n_i - 1$;
- (vi) $\text{Ext}_A^p(S_j, S_i) = 0$ whenever $0 < p < n_j - n_i - 1$;

are satisfied.

Before we prove this, let us point out how few conditions this leaves us to check when n_0, \dots, n_r do not vary very much.

Corollary 6.3. *Suppose*

$$\max_{0 \leq i, j \leq r} |n_i - n_j| \leq 2.$$

Then conditions (a) and (b) of Theorem 6.1 are satisfied by the objects X_i of (6.1) if

- (i) $\text{End}_A(\Omega^{n_i} F(S_i)) \cong k$ for $0 \leq i \leq r$, and
- (ii) $\text{Hom}_A(\Omega^{n_i} F(S_i), \Omega^{n_j} F(S_j)) = 0$ whenever $i \neq j$ and $n_i \leq n_j$.

Proof of Proposition 6.2. The case $i = j$ of condition (b) of Theorem 6.1 is just condition (i) of the proposition.

Since

$$\text{Hom}(X_i, X_j[m]) = \text{Hom}(\Omega^{n_i} F(S_i), \Omega^{n_j} F(S_j)[m + n_j - n_i]),$$

we have $\text{Hom}(X_i, X_j[m]) = 0$ when $m + n_j - n_i < 0$, so we need only consider values of m with $n_i - n_j \leq m \leq 0$.

The case $0 \geq m = n_i - n_j$ requires

$$0 = \text{Hom}_\Lambda(\Omega^{n_i} F(S_i), \Omega^{n_j} F(S_j))$$

when $i \neq j$, which is condition (ii) of the proposition.

Since F induces a stable equivalence,

$$\underline{\text{Hom}}_\Lambda(\Omega^m F(S_i), F(S_j)) \cong \underline{\text{Hom}}_\Gamma(\Omega^m S_i, S_j)$$

for any m , and by Tate duality these spaces are dual to

$$\underline{\text{Hom}}_\Lambda(\Omega^p F(S_j), F(S_i)) \cong \underline{\text{Hom}}_\Gamma(\Omega^p S_j, S_i)$$

for $p = -1 - m$.

What remains to be checked is that these spaces are zero for $0 \geq m > n_i - n_j$, or $-1 + n_j - n_i < p \leq -1$ (in which case $n_i \neq n_j$, so $i \neq j$). Since S_i and S_j are non-isomorphic simple modules,

$$\underline{\text{Hom}}_\Gamma(S_i, S_j) = 0 = \underline{\text{Hom}}_\Gamma(S_j, S_i),$$

so the cases $m = 0$, $p = -1$, and $m = -1$, $p = 0$ are automatically satisfied. The remaining cases are just conditions (iii) to (vi) of the proposition. \square

In fact, projective maps are the only obstruction to the automatic satisfaction of condition (i) and, if $n_j - n_i < 2$, of condition (ii). For Λ -modules M and N , write $\text{PHom}_\Lambda(M, N)$ for the space of projective maps from M to N , so that

$$\underline{\text{Hom}}_\Lambda(M, N) = \text{Hom}_\Lambda(M, N) / \text{PHom}_\Lambda(M, N).$$

Proposition 6.4. (a) Condition (i) of Proposition 6.2 (or Corollary 6.3) is satisfied for a given i if and only if

$$\text{PHom}_\Lambda(\Omega^{n_i} F(S_i), \Omega^{n_i} F(S_i)) = 0.$$

(b) Condition (ii) is satisfied for a given i and j if and only if

$$\text{PHom}_\Lambda(\Omega^{n_i} F(S_i), \Omega^{n_j} F(S_j)) = 0$$

and either

- (I) $n_j - n_i < 2$, or
- (II) $n_j - n_i \geq 2$ and $\text{Ext}_\Gamma^{n_j - n_i - 1}(S_j, S_i) = 0$.

Proof. Since F induces a stable equivalence,

$$\underline{\text{Hom}}_\Lambda(\Omega^{n_i} F(S_i), \Omega^{n_j} F(S_j)) \cong \underline{\text{Hom}}_\Gamma(S_i, \Omega^{n_j - n_i} S_j),$$

which is isomorphic to k if $i = j$, is zero if $i \neq j$ and $n_i = n_j$, is dual to $\underline{\text{Hom}}_\Gamma(S_j, S_i) = 0$ if $n_j - n_i = 1$, and is dual to $\text{Ext}_\Gamma^{n_j - n_i - 1}(S_j, S_i)$ if $n_j - n_i > 1$. \square

Remark. Since there are no non-zero projective maps from a simple module to a module with no projective summands, the vanishing of $\text{PHom}_\Lambda(\Omega^{n_i} F(S_i), \Omega^{n_j} F(S_j)) = 0$ is automatic if $\Omega^{n_i} F(S_i)$ is a simple Γ -module.

7. Some examples

We shall give some examples to show how Theorem 6.1 can be used to verify specific cases of Broué's conjecture on equivalences of derived categories for blocks with abelian defect group.

Conjecture 7.1 (Broué). *Let A be a block algebra of a finite group algebra kG , where k is an algebraically closed field of characteristic $p > 0$. Suppose that a defect group D of A is abelian. Let $H = N_G(D)$, and let B be the Brauer correspondent of A : a block algebra of kH . Then there is an equivalence $D(\text{Mod}(A)) \approx D(\text{Mod}(B))$ of triangulated categories.*

We shall use the notation (G, H, A, B, D, k, p) of Conjecture 7.1 in all the examples. In each case we shall describe a set $\mathcal{X} = \{X_i: 0 \leq i \leq r\}$ of objects of $D^b(\text{mod}(B))$ that are stably isomorphic to the images of the simple A -modules under a stable equivalence. We shall then need to check the conditions of Theorem 6.1. The last of these conditions is that $D^b(\text{mod}(B))$ is generated as a triangulated category by the elements of \mathcal{X} . We shall use the notation $\langle \mathcal{X} \rangle$ to refer to the triangulated category generated by \mathcal{X} , and to prove that this is the whole of $D^b(\text{mod}(B))$ we shall show that each of the simple B -modules is in $\langle \mathcal{X} \rangle$.

All of these examples are already known by other methods, but Theorem 6.1 provides a simpler proof. In these cases we shall give references to previous proofs.

In all the examples we give, it is well known that there is a stable equivalence which coincides with Green correspondence on objects.

We start with a very simple example.

7.1. Principal block of $G = A_5$, $p = 2$

The alternating group $G = A_5 \cong SL(2, 4) \cong PSL(2, 5)$ has Sylow 2-subgroup $P \cong C_2 \times C_2$, with normalizer $H = N_G(P) \cong A_4 \cong P \rtimes C_3$.

The principal block A of kG has three simple modules: the trivial module and two 2-dimensional modules.

$B = kH$ has three 1-dimensional simple modules, which we will denote by k , 1, and 2.

The restrictions of the simple A -modules have the following structures.

$$Y_0 := k, \quad Y_1 := \begin{matrix} 1 \\ 2 \end{matrix}, \quad Y_2 := \begin{matrix} 2 \\ 1 \end{matrix}.$$

To check condition (c) of Theorem 6.1, note that the simples k , 1, and 3 are certainly in $\langle \mathcal{X} \rangle$, and hence $M := \Omega Y_2 / \text{soc}(\Omega Y_2)$ is in $\langle \mathcal{X} \rangle$, since each composition factor is isomorphic to one of k , 1, 3. Therefore the simple 2 is in $\langle \mathcal{X} \rangle$, since it is the kernel of the natural surjection $\Omega Y_2 \rightarrow M$.

Okuyama [6] gave the first proof of this example.

7.3. Principal block of $G = A_8$, $p = 3$

The alternating group $G = A_8$ has Sylow 3-subgroup $P \cong C_3 \times C_3$, with normalizer $H = N_G(P) \cong P \rtimes D_8$.

The principal block A of kG has five simple modules: the trivial module and modules with dimensions 7, 13, 28, and 35.

$B = kH$ has four 1-dimensional modules, which we shall denote by k , 1, 2, and 3. We shall choose the names so that the kernel of the action of D_8 on the simple 2 is cyclic, whereas the kernels of the actions on 1 and 3 are elementary abelian of rank two. There is also a 2-dimensional simple module S .

The Green correspondents of the simple A -modules have the following Loewy structures:

$$Y_0 := k, \quad Y_1 := 2, \quad Y_2 := \begin{matrix} 1 \\ S \\ 1 \end{matrix}, \quad Y_3 := 3, \quad Y_4 := S.$$

We shall take $X_i := Y_i$ for $i \in \{0, 1, 3, 4\}$ and $X_2 := \Omega Y_2[1]$.

The Loewy structure of the projective cover of the simple module 1 is

$$\begin{matrix} & & 1 \\ & & S \\ k & 1 & 2, \\ & & S \\ & & 1 \end{matrix}$$

and so ΩY_2 has the structure

$$\begin{matrix} k & & 2 \\ & & S \\ & & 1 \end{matrix}.$$

Since ΩY_2 has simple socle 1, and since 1 only occurs with multiplicity one as a composition factor, $\text{End}_B(\Omega Y_2) \cong k$ and $\text{Hom}_B(X_i, \Omega Y_2) = 0$ for $i \in \{0, 1, 3, 4\}$. Hence the conditions of Corollary 6.3 are satisfied.

Condition (c) of Theorem 6.1 also holds, because clearly all the simples other than 1 are in $\langle \mathcal{X} \rangle$, and since ΩY_2 , which contains 1 as a composition factor with multiplicity one, is also in $\langle \mathcal{X} \rangle$, it follows that 1 is in $\langle \mathcal{X} \rangle$.

Okuyama [6] gave the first proof of this example.

8. General coefficient fields

The only use we made of the condition that k is algebraically closed was to assume that the endomorphism ring of a simple module for a finite-dimensional k -algebra was just k . Our main theorem, Theorem 5.1, is true a little more generally for a field k that is not algebraically closed, as we can allow $\text{End}(X_i)$ to be any finite-dimensional division algebra over k . The only difference in the construction of the objects T_i is that rather than using a k -basis $B_i^{(n-1)}(j, t)$ of $\text{Hom}(X_j[t], X_i^{(n-1)})$ in order to form the object $Z_i^{(n-1)}(j, t)$, we should use a basis as a left $\text{End}(X_j)$ -module.

Theorem 6.1 generalizes similarly.

References

- [1] M. Bökstedt, A. Neeman, Homotopy limits in triangulated categories, *Compositio Math.* 86 (2) (1993) 209–234.
- [2] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, *Astérisque* 181–182 (1990) 61–92.
- [3] J. Chuang, Derived equivalence in $SL_2(p^2)$, *Trans. Amer. Math. Soc.* 353 (7) (2001) 2897–2913, electronic.
- [4] M. Holloway, Derived Equivalences for Group Algebras, PhD thesis, Bristol, 2001.
- [5] M. Linckelmann, Stable equivalences of Morita type for self-injective algebras and p -groups, *Math. Z.* 223 (1) (1996) 87–100.
- [6] T. Okuyama, Some examples of derived equivalent blocks of finite groups, preprint, Hokkaido, 1998.
- [7] T. Okuyama, Derived equivalence in $SL(2, q)$, preprint, Hokkaido, 2000.
- [8] J. Rickard, Morita theory for derived categories, *J. London Math. Soc.* (2) 39 (3) (1989) 436–456.
- [9] J. Rickard, Derived equivalences as derived functors, *J. London Math. Soc.* (2) 43 (1) (1991) 37–48.
- [10] J. Rickard, Splendid equivalences: derived categories and permutation modules, *Proc. London Math. Soc.* (3) 72 (2) (1996) 331–358.
- [11] R. Rouquier, Gluing p -permutation modules, preprint, Paris, 1988.
- [12] R. Rouquier, Block Theory Via Stable and Rickard Equivalences, in: *Modular Representation Theory of Finite Groups*, de Gruyter, Berlin, 2001, pp. 101–146.