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# Wreath operations in the group of automorphisms of the binary tree

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**Abstract**

A new operation called *tree-wreathing* is defined on groups of automorphisms of the binary tree. Given a countable residually finite 2-group $H$ and a free abelian group $K$ of finite rank $r$ this operation produces uniformly copies of these as automorphism groups of the binary tree such that the group generated by them is an over-group of the restricted wreath product $H \wr K$. Indeed, $G$ contains a normal subgroup $N$ which is an infinite direct sum of copies of the derived group $H'$ and the quotient group $G/N$ is isomorphic to $H \wr K$. The tree-wreathing construction preserves the properties of solvability, torsion-freeness and of having finite state (i.e., generated by finite automata). A faithful representation of any free metabelian group of finite rank is obtained as a finite-state group of automorphisms of the binary tree.


*Keywords:* Wreath product; Binary tree; Automata; Pro-$p$ group

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## 1. Introduction

Automorphisms of the one-rooted regular $n$-ary tree have a natural interpretation as input–output automata on the alphabet $\{0, 1, \ldots, n-1\}$ and the automorphisms which correspond to finite-state automata form an enumerable group $\mathcal{F}_n$ called the group of finite-state automorphisms [2]. The present paper continues the study of the group of automorphisms $\mathcal{A}$ of the binary tree and especially of its subgroup of finite-state automorphisms $\mathcal{F}$ [12]. For an overview of the present situation for groups generated by input–output automata, see [8].

The identification of the elements of $\mathcal{F}$ with finite automata provides a proof that the word problem in $\mathcal{F}$ is solvable. Given a countable set of subgroups of $\mathcal{F}$ it is possible using the tree structure to embed their direct sum into $\mathcal{F}$. Likewise, given a countable subgroup $H$ and a finite 2-subgroup $K$ of $\mathcal{F}$, it may be shown that the restricted wreath product $H \wr K$ embeds into $\mathcal{F}$. In this paper, $H \wr K$ indicates the restricted wreath product obtained from the regular representation of $K$. Residual finiteness of wreath products of groups is governed by Gruenberg's theorem: the group $G = H \wr K$ is residually finite if and only if $H$, $K$ are residually finite and $K$ finite or $H$ abelian [6]. Thus, in considering the question of which pairs $(H, K)$ of subgroups of $\mathcal{F}$ the restricted wreath product $H \wr K$ is embeddable in $\mathcal{F}$, we have to consider only the second alternative in Gruenberg's theorem and this has been the motivation behind our work.

We define a new operation on subgroups of $\mathcal{A}$, which we call *tree-wreathing*. Given a subgroup $H$ of $\mathcal{A}$ and a free abelian group $K$ of finite rank $r$ this operation produces uniformly copies of these in $\mathcal{A}$ such that the group $G$ generated by them, indicated by $H \,\overline{\wr}\, K$, is an over-group of the restricted wreath product $H \wr K$. Indeed, $G$ contains a normal subgroup $N$ which is an infinite direct sum of copies of the derived group $H'$ and the quotient group $G/N$ is isomorphic to $H \wr K$. The tree-wreathing construction preserves the properties of solvability, torsion-freeness and of having finite state. When $H$ is an abelian subgroup of $\mathcal{F}$, we obtain an embedding of $H \wr K$ in $\mathcal{F}$. An application of this result is an embedding of the free metabelian group of finite rank into $\mathcal{F}$.

The group $K$ is shown to be generated by a certain set $\{\alpha(i) \mid 0 \leqslant i \leqslant r-1\}$ such that the closure $\widehat{K}$ of $K$ in $\mathcal{A}$ with respect to its pro-2 topology as $\widehat{K} = \sum\{\widehat{\langle\alpha(i)\rangle} \mid 0 \leqslant i \leqslant r-1\}$ where each $\widehat{\langle\alpha(i)\rangle}$ is isomorphic to the dyadic integers $\mathbb{Z}_2$. Moreover, the tree-wreath construction is extended to $H \,\overline{\wr}\, \widehat{K}$. We find that the normalizer in $\mathcal{A}$ of each $\widehat{\langle\alpha(i)\rangle}$ contains a subgroup $\Lambda_i$ isomorphic to the group of units of $\mathbb{Z}_2$. Furthermore, the group $\Lambda$ generated by the $\Lambda_i$'s is their direct sum and $\Lambda$ normalizes $H \,\overline{\wr}\, \widehat{K}$.

Our construction considerably enlarges the class of known residually finite 2-groups and in particular of those which afford finite-state representations. However, the concrete realization as finite-state automorphism groups of most of the groups in the second alternative of Gruenberg's theorem remains open. For example, does $C \wr (C \wr C)$ have a finite-state representation, where $C$ is an infinite cyclic group?

A faithful representation of the affine group $\mathbb{Z}_2^m GL(m, \mathbb{Z}_2)$ for $m \geqslant 1$ was obtained in [3] as a group acting on the $2^m$-ary regular tree. Let $\mathbb{Z}_{(2)}$ be the localization of the rational numbers at the prime 2. It was also shown in the same paper that the restriction of the representation to the subgroup $\mathbb{Z}_{(2)}^m GL(m, \mathbb{Z}_{(2)})$ produced a faithful finite-state representation of this subgroup. In a later work [5] it was proven that for $m \geqslant 1$, the group of finite-state automorphisms of the binary tree $\mathcal{F}$ embeds the affine group $\mathbb{Z}^m B(m, \mathbb{Z})$ where $B(m, \mathbb{Z})$ consists of those invertible matrices $(a_{ij})$ with $a_{ij}$ even for all $j > i$, which is therefore of finite index in $GL(m, \mathbb{Z})$. In particular then, $\mathcal{F}$ embeds the free group of rank 2 and by a result of Malcev [10, Section 17.2] it also embeds any finitely generated torsion-free nilpotent group. We note that the 2-generator free metabelian group is not linear over $\mathbb{Z}$ (since it contains free abelian subgroups of infinite rank).

It is by now a well-known fact that $\mathcal{F}$ embeds Burnside 2-groups with branching subgroup structure and, as was shown more recently, it also embeds torsion-free groups sharing such a property [4]. These groups cannot admit faithful finite-dimensional linear representations. A comprehensive exposition on the topic of Branch Groups is forthcoming [1].

As large as the class of finitely generated subgroups of $\mathcal{F}$ may be, this group is not as universal as $\mathcal{A}$. The known argument in support of the assertion is based on cardinality considerations. For, by a variation on a construction of Hall [9], there exist $2^{\omega_0}$ isomorphism classes of 2-generated center-by-metabelian residually finite 2-groups; yet clearly, there are only a countable number of 2-generated subgroups in $\mathcal{F}$. The class of Grigorchuk groups provides another proof; see [7].

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## 2. Preliminaries

The binary tree $\mathcal{T}$ can be identified with the free monoid $\mathcal{M} = \langle 0, 1\rangle^*$ of finite sequences on $0, 1$, ordered by $v \leqslant u$ provided $u$ is an initial subword of $v$. Let $\sigma$ be the transposition $(0, 1)$ whose action is extended to the tree by $\sigma : 0u \leftrightarrow 1u$. Then a general automorphism $\alpha \in \mathcal{A}$ can be represented as $\alpha = (\alpha_0, \alpha_1)\sigma_\phi$ where $\sigma_\phi = \sigma^i$, $i = 0, 1$ with $\phi$ the empty word, and where $\alpha_0, \alpha_1$ are automorphisms of the subtrees headed by $0, 1$, respectively. As these subtrees are isomorphic to $\mathcal{T}$ by simply deleting the first letter from the labels of vertices in the subtrees, we may by using this isomorphism consider $\alpha_0, \alpha_1 \in \mathcal{A}$. The automorphism $\alpha$ is said to be active provided $\sigma_\phi = \sigma$, otherwise it is inactive. Proceeding with the development of $\alpha$, we produce the set of states $Q = \{\alpha_u \mid u \in \mathcal{M}\}$ where $\alpha_\phi = \alpha$ and the set of activities $\{\sigma_u \mid u \in \mathcal{M}\}$.

A subgroup of $\mathcal{A}$ which fixes all the vertices outside the subtree headed by the index $u$ and projects onto a group $H$ at the vertex $u$ will be indicated by $u * H$ and the elements of the latter will be denoted by $u * h$. The set $\{1^i * \sigma \mid 0 \leqslant i \leqslant k\}$ generates a group $P_k$ which is the 2-Sylow subgroup of the symmetric group on the set $\{u \in \mathcal{M} \mid |u| = k+1\}$ and is isomorphic to $(((C_2 \wr \cdots) \wr C_2) \wr C_2)$, the $k$-fold wreath product of cyclic groups of order 2.

The following constructions preserve subgroups of both $\mathcal{A}$ and $\mathcal{F}$:

(i) Let $\{H_i \mid i \geqslant 0\}$ be subgroups of $\mathcal{A}$ , then the group generated by $\{1^i 0 * H_i \mid i \geqslant 0\}$ is a direct sum of copies of $H_i$;
(ii) Given $H$ a subgroup of $\mathcal{A}$ and an integer $k \geqslant 0$, the group generated by $1^k * H$ and $P_k$ is a subgroup of $\mathcal{A}$ isomorphic to $H \wr P_k$;
(iii) Given a group $R \leqslant \mathcal{A}$, we define inductively the following subgroup of $\mathcal{A}$, whose elements have finite support:

$$\nu(R) = \bigl(\nu(R) \times R\bigr) \times \bigl(\nu(R) \times R\bigr).$$

The group $\nu(R)$ is generated by $u * R$ for all $u \in U = (\{0^2, 10\}^*)\{01, 11\}$.

The automorphism $\alpha$ has activity growth $\theta(n) = \#\{u \mid |u| = n,\ \sigma_u \neq e\}$. Also, $\alpha$ has $m$-circuit type provided the length of the longest circuit in the graph of the automata corresponding to $\alpha$ is $m$; the length of a circuit is measured by the number of distinct vertices lying on it. If the only circuit in the graph corresponding to $\alpha$ occurs at the identity element $e$ then $\alpha$ is of 0-circuit type. Given a sufficiently natural measure of activity growth, the set of automorphisms with growth limited above by this measure forms a subgroup. The finite-state automorphisms of bounded growth form the subgroup $\mathcal{F}_0$. Those elements of 0-circuit type form the subgroup of finitary automorphisms $\mathcal{F}_{0,0} = \bigcup\{P_k \mid k \geqslant 1\}$. Furthermore, if $m \geqslant 1$, then those which have 0-circuit type or $k$-circuit type where $k$ divides $m$ form the group $\mathcal{F}_{0,m}$; see [12].

The tree $\mathcal{T}$ is the inverse limit of its truncations at the $n$th levels. Thus the group $\mathcal{A}$ is the inverse limit of the permutation groups it induces on the $n$th level vertices. This endows $\mathcal{A}$ with a pro-2 topological group structure. An infinite product of elements $\mathcal{A}$ is a well-defined element of $\mathcal{A}$ provided for any given level $n$, only finitely many of the elements in the product have non-trivial action on vertices at level $n$. Let $H$ be a subgroup of $\mathcal{A}$. The closure of a subgroup $H$ in the topological group $\mathcal{A}$ will be indicated by $\widehat{H}$. We note that if $H$ is abelian then

$$\widehat{H} = \bigl\{h^{\xi} \bigm| h \in H,\ \xi \text{ a dyadic integer}\bigr\}$$

which is also an abelian group.

## 3. The tree-wreath product $H \wr C$

### 3.1. The translation operator

Let $\alpha$ be the automorphism of the binary tree defined specifically and recursively by $\alpha = (e, (\alpha, e))\sigma$. Then $\alpha$ will serve from now on as the translation operator in the construction of a tree-wreath product. We note that $\alpha$ has 3 states, has bounded growth and is of 2-circuit type.

### 3.2. Copying subgroups

Let $H$ be some group acting on the tree. We seek to construct a copy $\widetilde{H}$ of $H$ which will be compatible with the translation $\alpha$, in the sense that $\widetilde{H}$ should commute with all (or as many as possible) of its conjugates by elements from $\langle\alpha\rangle$.

For every $h \in H$ define the automorphism $\tilde{h}$ of the tree recursively as $\tilde{h} = ((\tilde{h}, h), (e, e))$, or simply as $\tilde{h} = ((\tilde{h}, h), e)$. It is clear that the set $\widetilde{H} = \{\tilde{h} \mid h \in H\}$ is a group of automorphisms of the tree and that $\widetilde{H}$ is isomorphic to $H$. Also, if $H$ is a finite-state group then so is $\widetilde{H}$. We note that for all $h \in H$, the element $\tilde{h}$ provides a partial mapping $\tilde{h} : \mathcal{M} \to H$ from the labels of the tree into $H$, defined by $(u)\tilde{h} = e$ if $u = 0^i 1$, $i$ even, and by $(u)\tilde{h} = h$ if $u = 0^i 1$, $i$ odd. In this sense, $\tilde{h}$ has infinite support for all $h \neq e$.

### 3.3. Tree-wreathing

Let $G$ be the group generated by $\widetilde{H}$ and $\alpha$. We say that $G$ is $H$ *tree-wreathed* by the infinite cyclic group $C$ generated by $\alpha$ and use the notation $G = H \wr C$.

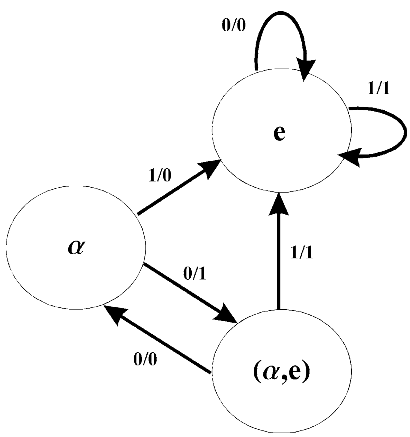


Fig. 1. The automaton $\alpha$.

**Theorem 1.** *Let $G$ be the tree-wreath product $H \,\bar{\wr}\, C$ defined above. Then $G$ satisfies the following properties*:

(i) *the subgroup*

$$N = \langle [\widetilde{H}^{\alpha^i}, \widetilde{H}^{\alpha^j}] \mid 0 \leqslant i < j \rangle$$

*which is normal in $G$ can be expressed in its action on the tree as $N = \nu(H') = (N \times H') \times (N \times H')$*;

(ii) *the quotient group $G/N$ is isomorphic to the restricted wreath product $H \wr C$*;

(iii) *the subgroup of $G$ generated by $(0^{2^m-1}1) * H'$ and $\alpha$ is isomorphic to a central extension of $H' \wr C_{2^m}$*;

(iv) *if $J$ is a subgroup of $H$ then $\langle \widetilde{J}, \alpha \rangle$, considered as a subgroup of $G = H \,\bar{\wr}\, C$, is isomorphic to $J \,\bar{\wr}\, C$; in particular, if $J$ is abelian then $\langle \widetilde{J}, \alpha \rangle$ is isomorphic to $J \wr C$*;

(v) *the group $G$ is finite-state* (*solvable, torsion-free*) *if and only if $H$ is finite-state* (*solvable, torsion-free*).

**Proof.** I. Define $H_i = \widetilde{H}^{\alpha^i}$ and $H_{i,j} = [\widetilde{H}^{\alpha^i}, \widetilde{H}^{\alpha^j}]$ for all $0 \leqslant i < j$. Furthermore, define $N = \langle H_{i,j} \mid 0 \leqslant i < j \rangle$ and its stratification by $N_0 = \{e\}$, $N_s = \langle H_{i,j} \mid 0 \leqslant i < j < 2^{s+1},\ j - i \leqslant 2^s \rangle$ for all $s > 1$. We will show that

$$N_s \text{ is a normal subgroup of } G,$$

$$N_s = P_s \times P_s \quad \text{where } P_s = N_{s-1} \times H',$$

and

$$[N_s, \alpha^{2^s}] = e$$

for all $s \geqslant 1$. Also, $\langle N_s, \alpha \rangle$ is a central extension of $H' \wr C_{2^s}$.

(i) Let $h, k \in H$ and let $i$ be any integer then

$$\alpha^{2i} = ((\alpha^i, e), (\alpha^i, e)), \qquad \alpha^{2i+1} = ((\alpha^i, e), (\alpha^{i+1}, e))\sigma,$$

$$\tilde{h}^{\alpha^{2i}} = ((\tilde{h}^{\alpha^i}, h), e), \qquad \tilde{h}^{\alpha^{2i+1}} = (e, (\tilde{h}^{\alpha^i}, h)),$$

$$[\tilde{h}, \tilde{k}^{4i+2}] = [((\tilde{h}, h), e), ((\tilde{k}^{\alpha^{2i+1}}, k), e)] = (([\tilde{h}, \tilde{k}^{\alpha^{2i+1}}], [h,k]), e)$$
$$= ((e, [h,k]), e).$$

It follows that

$$H_{0,2i+1} = \{e\},$$

$$H_{0,4i+2} = (\{e\} \times H') \times \{e\} = (01) * H',$$

$$H_{1,4i+3} = \{e\} \times (\{e\} \times H') = (11) * H'.$$

Thus we verify in the first stage of generation of $N$,

$$N_1 = \langle H_{0,2}, H_{1,3}\rangle = P_1 \times P_1,$$
$$P_1 = \{e\} \times H',$$
$$\left[N_1, \alpha^2\right] = \{e\},\ N_1 \lhd G$$

and $\langle (01) * H', \alpha\rangle$ is a central extension of $H' \wr C_2$.

(ii) Let $n = 2^s m$, $s \geqslant 1$, $m$ odd. Then for all $h, k \in H$,

$$c = \left[\tilde{h}, \tilde{k}^{\alpha^n}\right] = \left[\left(\left(\tilde{h}, h\right), e\right), \left(\left(\tilde{k}^{\alpha^{n/2}}, k\right), e\right)\right] = \left(\left(\left[\tilde{h}, \tilde{k}^{\alpha^{n/2}}\right], [h,k]\right), e\right) = \cdots$$
$$= \left(\left(\left(\left((e, \ldots), [h,k]\right), e\right), [h,k]\right), e\right).$$

Therefore, $c$ is defined as a partial mapping on $\mathcal{M}$ by

$$\left(0^i 1\right)c = \begin{cases} [h,k] & \text{for all } i \text{ odd and } 1 \leqslant i \leqslant 2^s - 1, \\ e & \text{for all } i \text{ even and } 0 \leqslant i \leqslant 2^s - 2, \end{cases}$$
$$\left(0^{2^s}\right)c = e;$$

also,

$$c = \left[\tilde{h}, \tilde{k}^{\alpha^n}\right] = \left[\tilde{h}, \tilde{k}^{\alpha^{2^s}}\right], \qquad \left[c, \alpha^{2^s}\right] = e.$$

Inductively, we produce $(0^i 1) * H'$ for all $i$ odd and $1 \leqslant i \leqslant 2^s - 1$.

(iii) Let $P = \bigcup P_s$. Then, $N = P \times P$, $P = N \times H'$. The elements $c$ of $N_s$ are partial mappings from the monoid $\mathcal{M}$ into $H'$, defined on the sequences of length $2^s$ in

$$\left(\left\{0^2, 10\right\}^*\right)\{01, 11\}$$

and as such have finite support. Therefore, all elements of $N$ have finite support.

II. We will now show that $G/N$ is isomorphic to $H \wr C$. It is clear that the coset representatives of $N$ in $G$ can be chosen as expressions having the form

$$w = \left(\tilde{h}_1^{\alpha^{2i_1}} \tilde{h}_2^{\alpha^{2i_2}} \cdots \tilde{h}_s^{\alpha^{2i_s}}\right)\left(\tilde{k}_1^{\alpha^{2j_1+1}} \tilde{k}_2^{\alpha^{2j_2+1}} \cdots \tilde{k}_t^{\alpha^{2j_t+1}}\right)\alpha^m$$

with distinct integers $i_1, i_2, \ldots, i_s$, distinct integers $j_1, j_2, \ldots, j_s$ and $m = 2m' + \varepsilon$ with $\varepsilon \in \{0, 1\}$. Let $l(m)$ be the 2-valuation of $m$. We call semi-normal a form for $w$ having $(l(m), s+t)$ minimal under lexicographical ordering. The element $w$ can be developed in its action on the tree as $w = (u, v)\sigma^{\varepsilon}$ where

$$u = \left(\tilde{h}_1^{\alpha^{i_1}} \tilde{h}_2^{\alpha^{i_2}} \cdots \tilde{h}_s^{\alpha^{i_s}} \alpha^{m'}, h_1 h_2 \cdots h_s\right),$$
$$v = \left(\tilde{k}_1^{\alpha^{j_1}} \tilde{k}_2^{\alpha^{j_2}} \cdots \tilde{k}_s^{\alpha^{j_t}} \alpha^{m'}, k_1 k_2 \cdots k_t\right).$$

Suppose $w \in N$ is as above, in semi-normal form, and $w \neq e$. Choose $w$ having these properties and being minimal with respect to the ordered pair of integers $(l(m), s+t)$. Since $N$ stabilizes the first level of the tree, we have $\varepsilon = 0$. Thus, either $s + t$ or $m$ is different from 0. Therefore, as $N = P \times P$, it follows that $u, v \in P$. Likewise, as $P = N \times H'$, it follows that

$$\tilde{h}_1^{\alpha^{i_1}}\tilde{h}_2^{\alpha^{i_2}}\cdots\tilde{h}_s^{\alpha^{i_s}}\alpha^{m'},\ \tilde{k}_1^{\alpha^{j_1}}\tilde{k}_2^{\alpha^{j_2}}\cdots\tilde{k}_s^{\alpha^{j_t}}\alpha^{m'}\in N,$$
$$h_1h_2\cdots h_s,\ k_1k_2\cdots k_t\in H'.$$

By the minimality condition, $l(m')=0$ and as $\alpha^{m'}$ is inactive, $m'$ is even; thus

$$m'=0.$$

Therefore,

$$\tilde{h}_1^{\alpha^{i_1}}\tilde{h}_2^{\alpha^{i_2}}\cdots\tilde{h}_s^{\alpha^{i_s}},\ \tilde{k}_1^{\alpha^{j_1}}\tilde{k}_2^{\alpha^{j_2}}\cdots\tilde{k}_s^{\alpha^{j_t}}\in N.$$

Again, by the minimality condition, we have $s=0$, or $t=0$. Thus, by a repetition of this argument we reach $s=0$, $t=1$, or $s=1$, $t=0$. On conjugating $w$ by an adequate power of $\alpha$ we have $w=\tilde{h}=((\tilde{h},h),e)$ and $h\in H'$. Since $\tilde{h}$ has infinite support whereas elements of $N$ have finite support, a contradiction is reached. □

**Remark 1.** (i) It is clear from the last part of the proof that if $h\in H'$ then $\tilde{h}$ is an element of the topological closure $\widehat{N}$ of $N$ in $\mathcal{A}$. Therefore, the quotient group $\widehat{G}/\widehat{N}$ is metabelian.

(ii) Since $C\wr(C\wr C)$ is residually a finite 2-group, we ask whether there could exist a copy of such a group within $H\,\overline{\wr}\,C$ for some group $H$. Let $h_1,h_2\in H$. Then,

$$\tilde{h}_1^{\alpha}=\bigl(e,\bigl(\tilde{h}_1^{\alpha},h_1\bigr)\bigr),\qquad \bigl(\tilde{h}_2\bigr)^{\tilde{h}_1^{\alpha}}=\tilde{h}_2.$$

Thus, there do not exist non-trivial $h_1,h_2\in H$ such that

$$\langle\tilde{h}_1,\tilde{h}_2,\alpha\rangle=\langle\tilde{h}_1\rangle\wr\langle\tilde{h}_2,\alpha\rangle.$$

(iii) The translation $\alpha=(e,(\alpha,e))\sigma$ has bounded growth and has 2-circuit type. If $H$ is also of bounded growth then the tree-wreath product $H\,\overline{\wr}\,\langle\alpha\rangle$ also has bounded growth. Furthermore, if $H$ is generated by automorphisms with a bounded circuit structure then $H\,\overline{\wr}\,\langle\alpha\rangle$ also has bounded circuit structure. So, the tree-wreath construction preserves subgroups of $\mathcal{F}_{0,2}$, i.e., those formed by automorphisms having bounded growth and 2-circuit type.

**Proposition 1.** *Let $L$ be the normal closure of $\widetilde{H}$ in $G$. If $H$ is solvable* (*nilpotent*), *then $L$ and $H$ have equal solvability degree* (*nilpotency class*).

**Proof.** Let $R$ be a subgroup of $\mathcal{A}$ and $\nu(R)=(\nu(R)\times R)\times(\nu(R)\times R)$ as defined previously, and $\tilde{h}\in\widetilde{H}$. Then we have

$$\bigl[\nu(R),\tilde{h}\bigr]=\bigl(\bigl[\nu(R)\times R,\bigl(\tilde{h},h\bigr)\bigr],e\bigr)=\bigl(\bigl[\nu(R),\tilde{h}\bigr]\times[R,h],e\bigr).$$

In particular, for $R=H'$,

$$\bigl[\nu(H'),L\bigr]=\nu\bigl(\gamma_3(H)\bigr),\qquad \bigl[\nu(H'),L'\bigr]=\nu(H'').$$

Since $L$ is the normal closure of $\widetilde{H}$ in $G$, clearly, $L = \nu(H')\langle H_i \mid i = 0, 1, \ldots\rangle$, where $H_i = \widetilde{H}^{\alpha^i}$ as before, and we have the following formulas for the derived series of $L$,

$$L' = \nu(H'')\big\langle\big[\nu(H'), H_i\big], H_i' \mid i \geqslant 0\big\rangle = \nu\big(\gamma_3(H)\big)\big\langle H_i' \mid i \geqslant 0\big\rangle$$

and for $j \geqslant 1$,

$$L^{(j)} = \nu\big(\big[\gamma_3(H), H', \ldots, H^{(j-1)}\big]\big)\big\langle H_i^{(j)} \mid i \geqslant 0\big\rangle.$$

The formulas for the lower central series of $L$ for $j \geqslant 2$ are

$$\gamma_2(L) = \nu\big(\gamma_3(H)\big)\big\langle H_i' \mid i \geqslant 0\big\rangle,$$
$$\gamma_j(L) = \nu\big(\gamma_j(H)\big)\big\langle \gamma_j(H_i) \mid i \geqslant 0\big\rangle. \qquad \square$$

## 4. The tree-wreath product $H \wr K$

We determine the structure of the centralizer of $\alpha$ and then choose within it a convenient abelian free subgroup $K(= K(r))$ of rank $r$; we then consider the topological closure $\widehat{K}$ and also the normalizer of $\widehat{K}$.

### *4.1. The centralizer of* $\alpha$

Let $\gamma$ commute with $\alpha$. Since $\gamma$ also commutes with $\gamma\alpha$, we may assume $\gamma$ to be inactive; that is, $\gamma = (\gamma_0, \gamma_1)$. Then it is direct to see that $\gamma_0 = \gamma_1$, that $\gamma_0$ commutes with $(\alpha, e)$ and that $\gamma_0 = (\gamma_{00}, \gamma_{01})$ where $\gamma_{00}$ commutes with $\alpha$ and $\gamma_{01}$ is an arbitrary element of $\mathcal{A}$. We use the notation that for $\beta \in \mathcal{A}$ we define $\beta^{(1)} = (\beta, \beta)$, an element also in $\mathcal{A}$. We conclude that $C_{\mathcal{A}}(\alpha)$ has the following decomposition in its action on the tree

$$C_{\mathcal{A}}(\alpha) = C_{\mathcal{A}}\big((\alpha, e)\big)^{(1)}\langle\alpha\rangle, \qquad C_{\mathcal{A}}\big((\alpha, e)\big) = C_{\mathcal{A}}(\alpha) \times \mathcal{A}.$$

Also, given $\delta \in \mathcal{A}$, we define $\delta' = (\delta', \delta)^{(1)}$ in $\mathcal{A}$. The following calculation proves $\delta' \in C_{\mathcal{A}}(\alpha)$:

$$\alpha^{\delta'} = \big(e, (\alpha, e)^{(\delta', \delta)}\big)\sigma = \big(e, \big(\alpha^{\delta'}, e\big)\big)\sigma = \alpha.$$

Using this last definition, we produce inductively the following sequence of elements in $C_{\mathcal{A}}(\alpha)$:

$$\alpha(0) = \alpha, \qquad \alpha(i) = \big(\alpha(i), \alpha(i-1)\big)^{(1)}$$

for all $i \geqslant 1$. Define $K(r)$ to be the group generated by $\{\alpha(i) \mid 0 \leqslant i \leqslant r\}$.

**Lemma 1.** *The group $K(r)$ defined above is free abelian of rank $r$.*

**Proof.** Straightforward. $\square$

*4.2. The topological closure of $K(r)$ and its normalizer*

**Lemma 2.** *The closure of $K(r)$ is the direct sum*

$$\widehat{K(r)} = \sum\left\{\left\langle\widehat{\alpha(i)}\right\rangle \mid 0 \leqslant i \leqslant r\right\}.$$

**Proof.** Straightforward. □

Now we produce some elements of $\mathcal{A}$ which normalize $\widehat{K(r)}$. For every dyadic unit $\xi = 1 + \sum\{a_i 2^i \mid i \geqslant 1\}$ with $a_i = 0, 1$, we define in $\mathcal{A}$ the element

$$\lambda_\xi = \left((\lambda_\xi, e), \left(\lambda_\xi \alpha^{(\xi-1)/2}, e\right)\right),$$

and define the set $\Lambda = \{\lambda_\xi \mid \xi \text{ a dyadic unit}\}$. Then it is direct to verify that $\lambda_\xi$ conjugates $\alpha$ to $\alpha^\xi$; i.e., $\alpha^{\lambda_\xi} = \alpha^\xi$. Since $\lambda_\xi \lambda_\mu = \lambda_{\xi+\mu}$ for all dyadics $\xi, \mu$, it follows that $\mathrm{Aut}(\langle \widehat{\alpha} \rangle) \cong \Lambda$. Now define

$$\lambda_\xi(0) = \lambda_\xi = \left((\lambda_\xi, e), \left(\lambda_\xi \alpha^{(\xi-1)/2}, e\right)\right),$$

and define inductively

$$\lambda_\xi(i) = \left(\lambda_\xi(i), \lambda_\xi(i-1)\right)^{(1)}$$

for all $i \geqslant 1$.

Let $\Lambda_i = \{\lambda_\xi(i) \mid \xi \text{ a dyadic integer}\}$ and $\Lambda(r) = \langle \Lambda_i \mid 0 \leqslant i \leqslant r\rangle$. Then, $\mathrm{Aut}(\langle\widehat{\alpha(i)}\rangle) \cong \Lambda_i$:

$$\begin{aligned} \alpha(i)^{\lambda_\xi(i)} &= \left(\alpha(i)^{\lambda_\xi(i)}, \alpha(i-1)^{\lambda_\xi(i-1)}\right)^{(1)} = \left(\alpha(i)^{\lambda_\xi(i)}, \alpha(i-1)^\xi\right)^{(1)} \\ &= \alpha(i)^\xi. \end{aligned}$$

**Proposition 2.** *The group $\Lambda(r) = \langle \Lambda_i \mid 0 \leqslant i \leqslant r\rangle$ normalizes $\widehat{K(r)}$ and the group $\langle \widehat{K(r)}, \Lambda(r)\rangle$ is the direct sum*

$$\sum\left\{\left\langle\widehat{\alpha(i)}\right\rangle \Lambda_i \mid 0 \leqslant i \leqslant r\right\}.$$

**Proof.** (i) As $\lambda_\xi(i)$ is of type $\delta'$ seen above, it commutes with $\alpha$ and thus

$$\lambda_\xi(i) \in C_{\mathcal{A}}(\alpha) \quad \text{for all } i \geqslant 1.$$

(ii) We compute the commutator

$$\begin{aligned} \left[\lambda_\xi(0), \lambda_\xi(i)\right] &= \left(\left(\left[\lambda_\xi, \lambda_\xi(i)\right], e\right), \left(\left[\lambda_\xi \alpha^{(\xi-1)/2}, \lambda_\xi(i)\right], e\right)\right) \\ &= \left(\left(\left[\lambda_\xi, \lambda_\xi(i)\right], e\right), \left(\left[\lambda_\xi, \lambda_\xi(i)\right]^{\alpha^{(\xi-1)/2}}, e\right)\right) = e, \end{aligned}$$

for all $i \geqslant 1$, and we prove inductively that $[\lambda_\xi(j), \lambda_\xi(i)] = e$ for all $j, i$. It follows directly that $\Lambda(r) = \sum\{\Lambda_i \mid 0 \leqslant i \leqslant r\}$.

(iii) We compute the conjugates of $\alpha(i)$ for $i \geqslant 1$

$$\begin{aligned}\alpha(i)^{\lambda_\xi(0)} &= \big(\big(\alpha(i)^{\lambda_\xi}, \alpha(i-1)\big), \big(\alpha(i)^{\lambda_\xi \alpha^{(\xi-1)/2}}, \alpha(i-1)\big)\big) \\ &= \big(\big(\alpha(i)^{\lambda_\xi}, \alpha(i-1)\big), \big(\alpha(i)^{\lambda_\xi}, \alpha(i-1)\big)\big) = \alpha(i),\end{aligned}$$

and by induction on $j$, for $j \neq i$, compute

$$\begin{aligned}\alpha(i)^{\lambda_\xi(j)} &= \big(\alpha(i)^{\lambda_\xi(j)}, \alpha(i-1)^{\lambda_\xi(j-1)}\big)^{(1)} \\ &= \big(\alpha(i)^{\lambda_\xi(j)}, \alpha(i-1)\big)^{(1)} = \alpha(i). \qquad \square\end{aligned}$$

*4.3. The group $H \,\bar{\wr}\, \widehat{K(r)}$*

The copying process of $H$ used in the previous section can be iterated to remain compatible with wreathing by $K(r)$ as follows: define for every $h \in H$,

$$\tilde{h}(0) = h, \qquad \tilde{h}(1) = \tilde{h} = \big(\big(\tilde{h}(1), h\big), e\big),$$

and inductively for all $i \geqslant 1$

$$\tilde{h}(i) = \big(\big(\tilde{h}(i), \tilde{h}(i-1)\big), e\big).$$

Then clearly, $\widetilde{H}(i) = \{\tilde{h}(i) \mid h \in H\}$ is a group isomorphic to $H$. Now, we prove

**Theorem 2.** *The group $G = \langle \widetilde{H}(r), \widehat{K(r)} \rangle$ is a tree-wreath product $H \,\bar{\wr}\, \widehat{K(r)}$.*

**Proof.** Let $N(r)$ be the subgroup of $G$ generated by the commutator groups $[\widetilde{H}(r), \widetilde{H}(r)^\gamma]$ for all $\gamma \in \widehat{K(r)}$. We need to show that the commutator quotient $G/N(r)$ is isomorphic to $\widetilde{H}(r) \wr \widehat{K(r)}$ and that $N(r)$ is a direct sum of an infinite number of copies of the derived group $H'$.

Let $\xi$ be a dyadic integer and denote by $l(\xi)$ is its 2-valuation. We also recall the generators of $K(r)$,

$$\alpha(0) = \alpha, \qquad \alpha(i) = \big(\alpha(i), \alpha(i-1)\big)^{(1)},$$

$0 \leqslant i \leqslant r-1$. The following is a sketch of the proof.

(i) Let $\xi = \varepsilon + 2\xi'$, $\varepsilon = 0, 1$. Then,

$$\alpha^\xi = \begin{cases} \big(\alpha^{\xi'}, e\big)^{(1)}, & \text{if } \varepsilon = 0, \\ \big(\big(\alpha^{\xi'}, e\big), \big(\alpha^{1+\xi'}, e\big)\big)\sigma, & \text{if } \varepsilon = 1. \end{cases}$$

(ii) Let

$$\begin{aligned}&\xi_i = \varepsilon_i + 2\xi_i', \qquad \varepsilon_i = 0, 1, \\ &\gamma = \alpha(0)^{\xi_0}\alpha(1)^{\xi_1}\cdots\alpha(r)^{\xi_r}, \qquad \beta = \alpha(1)^{\xi_1}\cdots\alpha(r)^{\xi_r}.\end{aligned}$$

Then,

$$\beta = (\beta, \beta')^{(1)}, \qquad \beta' = \alpha(0)^{\xi_1}\alpha(1)^{\xi_2}\cdots\alpha(r-1)^{\xi_r},$$

and $\gamma$ is active if and only if $\varepsilon_0 = 1$.

Let

$$\tilde{h}(r) = \big(\big(\tilde{h}(r), \tilde{h}(r-1)\big), e\big), \qquad \tilde{b}(r) = \big(\big(\tilde{b}(r), \tilde{b}(r-1)\big), e\big) \in \widetilde{H}(r).$$

Then,

$$\tilde{h}(r)^{\gamma} = \begin{cases} \tilde{h}(r)^{\alpha(0)^{\xi_0}\beta} = \big(\big(\tilde{h}(r)^{\alpha(0)^{\xi_0'}\beta}, \tilde{h}(r-1)^{\beta'}\big), e\big) & \text{if } \varepsilon_0 = 0, \\ \big(e, \big(\tilde{h}(r)^{\alpha(0)^{\xi_0'}\beta}, \tilde{h}(r-1)^{\beta'}\big)\big) & \text{if } \varepsilon_0 = 1. \end{cases}$$

Note that

$$\big[\tilde{h}(r), \tilde{b}(r)^{\gamma}\big] = \begin{cases} e, \quad \text{if } \varepsilon_0 = 1, \\ \big(\big(\big[\tilde{h}(r), \tilde{b}(r)^{\alpha(0)^{\xi_0'}\beta}\big], \big[\tilde{h}(r-1), \tilde{b}(r-1)^{\beta'}\big]\big), e\big), \\ \quad \text{if } \varepsilon_0 = 0. \end{cases}$$

(iii) We argue by induction on $r$ that $[\tilde{h}(r), \tilde{b}(r)^{\gamma}]$, seen as a partial map from the monoid $\mathcal{M}$ into $H'$, has a finite number of non-trivial entries in places with indices from $U^r$ where $U = (\{0^2, 10\}^*)\{01, 11\}$ and that all of these entries are equal to $[h, b]$. The case $r = 1$ is argued as in part (ii) of the theorem. By the inductive hypothesis, the second term $[\tilde{h}(r-1), \tilde{b}(r-1)^{\beta'}]$ in $[\tilde{h}(r), \tilde{b}(r)^{\gamma}]$ conforms to the assertion. Now, if the conjugator $\alpha(0)^{\xi_0'\beta}$ is active (that is, $\xi_0'$ is a dyadic unit) then the first term $[\tilde{h}(r), \tilde{b}(r)^{\alpha(0)^{\xi_0'}\beta}] = e$ and

$$\big[\tilde{h}(r), \tilde{b}(r)^{\gamma}\big] = \big(\big(e, \big[\tilde{h}(r-1), \tilde{b}(r-1)^{\beta'}\big]\big), e\big).$$

On the other hand, $[\tilde{h}(r), \tilde{b}(r)^{\alpha(0)^{\xi_0'}\beta}]$ is non-trivial if and only if $l(\xi_0') \geqslant 1$ and then in this case we repeat the argument as in the above paragraph. If $\xi_0 \neq 0$ then $l(\xi_0) = s$ for some finite $s$ and so this development stops after at most $s$ steps.

(iv) Let $N(r)$ be subgroup generated by the commutators between the $\widehat{K(r)}$ conjugates of $\widetilde{H}(r)$. Then, $N(r) = P(r) \times P(r)$ and $P(r) = N(r) \times N(r-1)$, where $N(0) = H'$.

(v) The argument for showing that $G/N(r)$ is isomorphic to $H \wr \widehat{K(r)}$ follows closely that of the theorem.

The coset representatives of $N(r)$ in $G$ can be chosen as expressions of form

$$w = \big(\widetilde{h_1(r)}^{\gamma_1} \widetilde{h_2(r)}^{\gamma_2} \cdots \widetilde{h_s(r)}^{\gamma_s}\big)\big(\widetilde{b_1(r)}^{\mu_1} \widetilde{b_2(r)}^{\mu_2} \cdots \widetilde{b_t(r)}^{\mu_t}\big) \\ \times \big(\alpha(0)^{\xi_0}\alpha(1)^{\xi_1} \cdots \alpha(r)^{\xi_r}\big),$$

where $\gamma_1, \gamma_2, \ldots, \gamma_s$ are inactive elements of $\widehat{K(r)}$ and $\mu_1, \mu_2, \ldots, \mu_t$ are active elements of $\widehat{K(r)}$ .

Let us call semi-normal a form for $w$ having minimal $(\sum\{l(\xi_i) \mid i \geqslant 0\}, s + t)$. Suppose $w \in N(r)$ is as above, in semi-normal form and that $w \neq e$. Since $N(r)$ stabilizes the first level of the tree, we have $l(\xi) > 0$. Therefore, as

$N(r) = P(r) \times P(r)$, it follows that $w = (u, v)$ and $u, v \in P(r)$. Likewise, as $P(r) = N(r) \times N(r-1)$, by considering the respective coordinates of $u, v$ and the minimality of $w$, we reach that $w = h(r) = ((h(r), h(r-1), e)$ and $h(r-1) \in H(r-1)'$. Again, we appeal to the fact that elements of $N(r)$ have finite support whereas nontrivial elements of $H(r)$ have infinite support to reach a contradiction. □

**Corollary 1.** *The group $\Lambda(r)$ normalizes the tree-wreath product $H \wr \widehat{K(r)}$.*

**Proof.** First we note that for

$$\tilde{h} = \big((\tilde{h}, h), e\big) \quad \text{and} \quad \lambda_\xi = \big((\lambda_\xi, e), \big(\lambda_\xi \alpha^{(\xi-1)/2}, e\big)\big),$$

we have $\tilde{h}^{\lambda_\xi} = ((\tilde{h}^{\lambda_\xi}, h), e) = \tilde{h}$ and thus $\lambda_\xi$ centralizes $\widetilde{H}$. Inductively, it is easy to see that $\Lambda(r)$ centralizes $\widetilde{H}(r)$. We conclude that $\Lambda(r)$ normalizes the group $\langle \widetilde{H}(r), \widehat{K(r)} \rangle$. □

**Corollary 2.** *The free metabelian group $\mathbb{M}$ of rank $r$ has a faithful representation as a group of finite-state automorphisms.*

**Proof.** Let $H = \langle x_1, x_2, \ldots, x_r \rangle$ and $K = \langle y_1, y_2, \ldots, y_r \rangle$ be two free abelian groups each of rank $r$ and let $G = H \wr K$. Then, using the Magnus embedding of wreath products into $2 \times 2$ matrices, it can be seen that the subgroup $\langle x_1 y_1, x_2 y_2, \ldots, x_r y_r \rangle$ is isomorphic to the free metabelian group $\mathbb{M}$ of rank $r$ [11]. Since we have produced in the previous theorem a faithful representation of $G$ into $\mathcal{F}$, the proof follows. □

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