Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Algebra 273 (2004) 780–805
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Character formula for infinite-dimensional unitarizable modules of the general linear superalgebra

Shun-Jen Cheng,[a,1] Ngau Lam,[b,*,2] and R.B. Zhang[c,3]

[a] *Department of Mathematics, National Taiwan University, Taipei, Taiwan 106*
[b] *Department of Mathematics, National Cheng Kung University, Tainan, Taiwan 701*
[c] *School of Mathematics and Statistics, University of Sydney, New South Wales 2006, Sydney, Australia*

Received 22 January 2003

Communicated by Masaki Kashiwara

Abstract

The Fock space of $m + p$ bosonic and $n + q$ fermionic quantum oscillators forms a unitarizable module of the general linear superalgebra $gl_{m+p|n+q}$. Its tensor powers decompose into direct sums of infinite-dimensional irreducible highest-weight $gl_{m+p|n+q}$-modules. We obtain an explicit decomposition of any tensor power of this Fock space into irreducibles, and develop a character formula for the irreducible $gl_{m+p|n+q}$-modules arising in this way.

Keywords: Lie superalgebra; Unitarizable representations; Howe duality; Character formula

1. Introduction

The Fock space of $m + p$ bosonic and $n + q$ fermionic quantum oscillators (see Section 3.3 for definition) with the standard inner product furnishes a unitarizable complex representation of the real form $u(m, p|n, q)$ of the general linear superalgebra $gl_{m+p|n+q}$.

* Corresponding author.
E-mail addresses: chengsj@math.ntu.edu.tw (S.-J. Cheng), nlam@mail.ncku.edu.tw (N. Lam), rzhang@maths.usyd.edu.au (R.B. Zhang).
[1] Partially supported by NSC-grant 91-2115-M-002-007 of the ROC.
[2] Partially supported by NSC-grant 91-2115-M-006-013 of the ROC.
[3] Supported by the Australian Research Council.

0021-8693/$ – see front matter
doi:10.1016/S0021-8693(03)00538-6

This representation decomposes into a direct sum of infinite-dimensional irreducible representations which are of highest-weight type with respect to an appropriate choice of a Borel subalgebra. Because of the unitarity, any tensor power of the representation is also semi-simple with all irreducible sub-representations being unitarizable highest-weight representations. We shall characterize the irreducible sub-representations and determine their structure.

In recent years there have been considerable activities (see, e.g., [7] for references) in the physics community to study unitarizable highest-weight representations of Lie superalgebras. This is motivated by applications of such representations in quantum field theory. For example, the symmetry algebra of the yet largely conjectural M-theory is closely related to $osp_{1|32}(\mathbb{R})$ [23]. An understanding of the unitarizable highest-weight representations of this Lie superalgebra will help to solve mysteries of M-theory. It has also been recognized [9] that some real forms of simple basic classical Lie superalgebras provide the conformal superalgebras of higher-dimensional space–time manifolds with extended supersymmetries. The unitarizable highest-weight representations of these Lie superalgebras thus describe the spectra of possible elementary particles existing in such space–times.

The problem of determining the possible unitarizable irreducible highest-weight representations of real forms of simple Lie superalgebras was investigated by a number of people (see [7] and references therein), with the most systematical study given in [16]. However, a classification analogous to the Enright–Howe–Wallach [11] classification of unitarizable positive energy irreducible representations for ordinary real simple Lie algebras has yet to be achieved (see Section 3.3).

A demanding but physically more important problem is to understand the structure of the unitarizable irreducible representations. Recall that a character formula for the unitarizable irreducible highest-weight representations of real forms of simple Lie algebras [11] was given in [10] some fifteen years ago. In a recent publication [6], two of the authors studied the irreducible representations arising from the decomposition of the tensor powers of the oscillator representations of the orthosymplectic superalgebras. By using results of [8,10], a character formula for these irreducible representations was derived. In this paper we investigate the case of the general linear superalgebra.

It is known from [14] that $u(d)$ and $u(m, p|n, q)$ form a dual reductive pair on the dth tensor power of the Fock space of $m+p$ bosonic and $n+q$ fermionic quantum oscillators. We explore the duality between the complexifications of these Lie (super)algebras to obtain in Theorem 3.3 an explicit decomposition of the tensor power of the Fock space into irreducible $gl_d \times gl_{m+p|n+q}$-modules. The Howe duality again as in [2,6] forms the key ingredient and further enables us to compute the characters for the irreducible $gl_{m+p|n+q}$-representations of Theorem 3.3. This result is presented in Theorem 5.3. Another application of the Howe duality is the computation of the tensor product decomposition of any two such unitarizable modules, which is the contents of Theorem 6.1.

Here is an outline of the paper. In Section 2 we discuss the $(gl_d, gl_{m|n})$-dualities on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ and its graded dual space. The material is largely known, but the highest-weight vectors in the graded dual of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ given in Lemma 2.2 have not been computed previously as far as we are aware of. In Section 3 we study the $gl_{m+p|n+q}$-representations furnished by tensor powers of the Fock space of $m + p$ bosonic and

$n + q$ fermionic quantum oscillators. In Section 3.2 we show that such representations are unitarizable and their irreducible sub-representations are infinite-dimensional highest-weight representations, and in Section 3.4 we obtain the explicit decomposition of the dth tensor power of the Fock space with respect to the semi-simple multiplicity free action of $gl_d \times gl_{m+p|n+q}$. Section 4 gives the $gl_{m+p|n+q} \to gl_{p|q} \times gl_{m|n}$ branching rule for the infinite-dimensional unitarizable irreducible $gl_{m+p|n+q}$-representations arising from the decomposition of tensor powers of the Fock space. In Section 5 we develop a character formula for these infinite-dimensional irreducible $gl_{m+p|n+q}$-representations in terms of hook Schur functions. Finally, in Section 6 we calculate the tensor product decomposition of two such irreducible $gl_{m+p|n+q}$-modules that appear in our decompositions of tensor powers.

2. Tensorial representations of general linear superalgebra

This section presents some results on the $(gl_d, gl_{m|n})$-dualities on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ and its graded dual vector space. The material contained here will be important for the remainder of the paper.

2.1. Preliminaries

We work over the field $\mathbb{C}$ of complex numbers throughout the paper. Let gl_d denote the Lie algebra of all complex $d \times d$ matrices. Let $\{e^1, \dots, e^d\}$ be the standard basis for $\mathbb{C}^d$. Denote by e_{ij} the elementary matrix with 1 in the ith row and jth column and 0 elsewhere. Then $\mathfrak{h}_d = \sum_{i=1}^{d} \mathbb{C}e_{ii}$ is a Cartan subalgebra, while $\mathfrak{b}_d = \sum_{1 \leqslant i \leqslant j \leqslant d} \mathbb{C}e_{ij}$ is the standard Borel subalgebra containing $\mathfrak{h}_d$. The weight of e^i is denoted by $\tilde{\varepsilon}_i$ for $1 \leqslant i \leqslant d$.

Let $\mathbb{C}^{m|n} = \mathbb{C}^{m|0} \oplus \mathbb{C}^{0|n}$ denote the $m|n$-dimensional superspace. The superspace of complex linear transformations on $\mathbb{C}^{m|n}$ has a natural structure of a Lie superalgebra [17], which we will denote by $gl_{m|n}$. Choose a basis $\{e_1, \dots, e_m\}$ for the even subspace $\mathbb{C}^{m|0}$ and a basis $\{f_1, \dots, f_n\}$ for the odd subspace $\mathbb{C}^{0|n}$, then $\{e_1, \dots, e_m, f_1, \dots, f_n\}$ is a homogeneous basis for $\mathbb{C}^{m|n}$. We may regard $gl_{m|n}$ as consisting of $(m + n) \times (m + n)$ matrices relative to this basis. Denote by E_{ij} the elementary matrix with 1 in the ith row and jth column and 0 elsewhere. Then $\mathfrak{h}_{m|n} = \sum_{i=1}^{m+n} \mathbb{C}E_{ii}$ is a Cartan subalgebra, while $\mathfrak{b}_{m|n} = \sum_{1 \leqslant i \leqslant j \leqslant m+n} \mathbb{C}E_{ij}$ is the standard Borel subalgebra containing $\mathfrak{h}_{m|n}$. We shall denote the weights of e_i and f_j by ε_i and δ_j, respectively, for $i = 1, \dots, m$, and $j = 1, \dots, n$.

By a partition λ of length k we mean a non-increasing finite sequence of non-negative integers $(\lambda_1, \dots, \lambda_k)$. We will let λ' denote the transpose of the partition λ. For example, if $\lambda = (4, 3, 1, 0, 0)$, then the length of λ is 5 and $\lambda' = (3, 2, 2, 1)$. By a generalized partition of length k, we shall mean a non-increasing finite sequence of integers $(\lambda_1, \dots, \lambda_k)$. In particular, every partition is a generalized partition of non-negative integers. Corresponding to each generalized partition $\lambda = (\lambda_1, \dots, \lambda_d)$, we will define $\lambda^* := (-\lambda_d, \dots, -\lambda_1)$. Then λ^* is also a generalized partition.

We regard a finite sequence $\lambda = (\lambda_1, \dots, \lambda_d)$ of complex numbers as an element of the dual vector space $\mathfrak{h}_d^*$ of $\mathfrak{h}_d$ defined by $\lambda(e_{ii}) = \lambda_i$, for $i = 1, \dots, d$. Denote by V_d^λ the irreducible gl_d-module with highest weight λ relative to the standard Borel subalgebra $\mathfrak{b}_d$. Similarly, we shall also regard a finite sequence of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_{m+n})$ as an element of the dual vector space $\mathfrak{h}^*_{m|n}$ of $\mathfrak{h}_{m|n}$ such that $\lambda(E_{jj}) = \lambda_j$, $1 \leqslant j \leqslant m+n$. We denote by $V^\lambda_{m|n}$ the irreducible $gl_{m|n}$-module with highest weight λ relative to the standard Borel subalgebra $\mathfrak{b}_{m+n}$.

2.2. *The $(gl_d, gl_{m|n})$-duality on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$*

Recall that the natural action of the Lie superalgebra $gl_d \times gl_{m|n}$ on $\mathbb{C}^d \otimes \mathbb{C}^{m|n}$ induces an action on the supersymmetric tensor algebra $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. This action is completely reducible and multiplicity free [3,4,14,22]. Indeed the pair $(gl_d, gl_{m|n})$ forms a dual reductive pair on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ in the sense of Howe [14,15].

Theorem 2.1 [3]. *Under the $gl_d \times gl_{m|n}$-action, $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ decomposes into*

$$S\big(\mathbb{C}^d \otimes \mathbb{C}^{m|n}\big) \cong \sum_\lambda V^\lambda_d \otimes V^{\tilde{\lambda}}_{m|n}, \tag{2.1}$$

where the sum in (2.1) *is over all partitions λ of length d subject to the condition $\lambda_{m+1} \leqslant n$, and*

$$\tilde{\lambda} = \big(\lambda_1, \ldots, \lambda_m; \langle \lambda'_1 - m\rangle, \ldots, \langle \lambda'_n - m\rangle\big). \tag{2.2}$$

Here λ' is the transpose partition of λ, and $\langle r\rangle$ stands for r, if $r \in \mathbb{N}$, and 0 *otherwise.*

Remark 2.1. The condition $\lambda_{m+1} \leqslant n$ is considered to be automatically satisfied by every generalized partition λ of length d if $m \geqslant d$.

We shall need an explicit formula for the joint highest-weight vectors of the irreducible $gl_d \times gl_{m|n}$-module $V^\lambda_d \otimes V^{\tilde{\lambda}}_{m|n}$ inside $S(\mathbb{C} \otimes \mathbb{C}^{m|n})$. (See also [20,21] for different descriptions of these vectors.) We set

$$x^i_l := e^i \otimes e_l, \qquad \eta^i_k := e^i \otimes f_k, \tag{2.3}$$

for $i = 1, \ldots, d$, $l = 1, \ldots, m$, and $k = 1, \ldots, n$. We will denote by $\mathbb{C}[\mathbf{x}, \eta]$ the polynomial superalgebra generated by (2.3). By identifying $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ with $\mathbb{C}[\mathbf{x}, \eta]$ the commuting pair $(gl_d, gl_{m|n})$ can be realized in terms of the following first-order differential operators acting on the left ($1 \leqslant i, i' \leqslant d$, $1 \leqslant s, s' \leqslant m$ and $1 \leqslant k, k' \leqslant n$):

$$\phi(e_{ii'}) := \sum_{j=1}^m x^i_j \frac{\partial}{\partial x^{i'}_j} + \sum_{j=1}^n \eta^i_j \frac{\partial}{\partial \eta^{i'}_j}, \tag{2.4}$$

$$\phi(E_{ss'}) := \sum_{j=1}^d x^j_s \frac{\partial}{\partial x^j_{s'}}, \qquad \phi(E_{m+k,m+k'}) := \sum_{j=1}^d \eta^j_k \frac{\partial}{\partial \eta^j_{k'}}, \tag{2.5}$$

$$\phi(E_{s,m+k}) := \sum_{j=1}^{d} x_s^j \frac{\partial}{\partial \eta_k^j}, \qquad \phi(E_{m+k,s}) := \sum_{j=1}^{d} \eta_k^j \frac{\partial}{\partial x_s^j}.$$

Straightforward calculations show that $\phi(e_{ij})$, $1 \leqslant i, j \leqslant d$, and $\phi(E_{ab})$, $1 \leqslant a, b \leqslant m+n$, satisfy the same commutation relations as the elementary matrices e_{ij} and E_{ab}, respectively. Thus (2.4) spans a copy of gl_d, and (2.5) a copy of $gl_{m|n}$.

For $1 \leqslant r \leqslant \min(d, m)$, we define

$$\Delta_r := \det \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_r^1 \\ x_1^2 & x_2^2 & \cdots & x_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^r & x_2^r & \cdots & x_r^r \end{pmatrix}. \tag{2.6}$$

If $d > m$, we consider the following determinant of an $r \times r$ matrix for every $m < r \leqslant d$:

$$\Delta_{k,r} := \det \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^r \\ x_2^1 & x_2^2 & \cdots & x_2^r \\ \vdots & \vdots & \cdots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^r \\ \eta_k^1 & \eta_k^2 & \cdots & \eta_k^r \\ \eta_k^1 & \eta_k^2 & \cdots & \eta_k^r \\ \vdots & \vdots & \cdots & \vdots \\ \eta_k^1 & \eta_k^2 & \cdots & \eta_k^r \end{pmatrix}, \quad k = 1, \ldots, n. \tag{2.7}$$

That is, the first m rows are filled by the vectors $(x_j^1, \ldots, x_j^r)$, for $j = 1, \ldots, m$, in increasing order and the last $r - m$ rows are filled with the same vector $(\eta_k^1, \ldots, \eta_k^r)$. Here the determinant of an $r \times r$ matrix

$$A := \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^r \\ a_2^1 & a_2^2 & \cdots & a_2^r \\ \vdots & \vdots & \cdots & \vdots \\ a_r^1 & a_r^2 & \cdots & a_r^r \end{pmatrix},$$

with matrix entries possibly involving Grassmann variables, is by definition the expression $\sum_{\sigma \in S_r} (-1)^{l(\sigma)} a_1^{\sigma(1)} a_2^{\sigma(2)} \cdots a_r^{\sigma(r)}$, where $l(\sigma)$ is the length of σ in the symmetric group S_r.

Observe that both Δ_r and $\Delta_{k,r}$ (if non-zero) are weight vectors of $gl_d \times gl_{m|n}$. Their gl_d-weights are respectively

$$\begin{aligned} wt_d(\Delta_r) &= (1, \ldots, \underbrace{1}_{r}, 0, \ldots, 0), \\ wt_d(\Delta_{k,r}) &= (1, \ldots, \underbrace{1}_{r}, 0, \ldots, 0), \end{aligned} \tag{2.8}$$

while the $gl_{m|n}$-weights are respectively

$$\begin{aligned} wt_{m|n}(\Delta_r) &= (1,\ldots,\underbrace{1}_{r},0,\ldots,0),\\ wt_{m|n}(\Delta_{k,r}) &= (1,\ldots,\underbrace{1}_{m},0,\ldots,0,\underbrace{r-m}_{m+k},0,\ldots,0), \end{aligned} \tag{2.9}$$

In correspondence to each partition λ of length d satisfying the condition $\lambda_{m+1} \leqslant n$, we define

$$\Delta_\lambda := \begin{cases} \Delta_{\lambda'_1}\Delta_{\lambda'_2}\cdots\Delta_{\lambda'_{\lambda_1}}, & \text{if } \lambda'_1 \leqslant m,\\ \prod_{k=1}^{\lambda_{m+1}}\Delta_{k,\lambda'_k}\prod_{j=1+\lambda_{m+1}}^{\lambda_1}\Delta_{\lambda'_j}, & \text{if } \lambda'_1 > m. \end{cases} \tag{2.10}$$

Lemma 2.1 [3]. *The space of $gl_d \times gl_{m|n}$ highest-weight vectors in the submodule $V_d^\lambda \otimes V_{m|n}^{\tilde\lambda}$ of $\mathbb{C}[\mathbf{x},\eta]$ is $\mathbb{C}\Delta_\lambda$.*

2.3. *The $(gl_d, gl_{p|q})$-duality on $S(\mathbb{C}^{d^*}\otimes\mathbb{C}^{p|q^*})$*

Let us denote by $\mathbb{C}^{p|q^*}$ the dual of the natural $gl_{p|q}$-module $\mathbb{C}^{p|q}$, and by $\mathbb{C}^{d^*}$ the dual of the natural gl_d-module $\mathbb{C}^d$. Then the $gl_d \times gl_{p|q}$-action on $\mathbb{C}^{d^*}\otimes\mathbb{C}^{p|q^*}$ induces a $gl_d \times gl_{p|q}$-action on $S(\mathbb{C}^{d^*}\otimes\mathbb{C}^{p|q^*})$

If $S^k(W)$ denotes the set of all homogeneous elements of degree k in the supersymmetric tensor algebra of the superspace W, then $S^k(\mathbb{C}^{d^*}\otimes\mathbb{C}^{p|q^*}) \cong S^k(\mathbb{C}^d\otimes\mathbb{C}^{p|q})^*$, and thus $S(\mathbb{C}^{d^*}\otimes\mathbb{C}^{p|q^*}) \cong \sum_k S^k(\mathbb{C}^d\otimes\mathbb{C}^{p|q})^*$. Therefore it follows from the decomposition (2.1) that the $gl_d \times gl_{p|q}$-action on $S(\mathbb{C}^{d^*}\otimes\mathbb{C}^{p|q^*})$ is also semi-simple and multiplicity free. Furthermore, we have the following decomposition $S(\mathbb{C}^{d^*}\otimes\mathbb{C}^{p|q^*}) \cong \sum_\lambda (V_d^\lambda)^* \otimes (V_{p|q}^{\tilde\lambda})^*$, where λ is summed over all partitions of length d subject to the condition $\lambda_{p+1} \leqslant q$. Clearly, $(V_d^\lambda)^* \cong V_d^{\lambda^*}$. Also, $(V_{p|q}^{\tilde\lambda})^* \cong V_{p|q}^{\widehat{\lambda^*}}$, where $\widehat{\lambda^*}$ is the negative of the lowest weight of $V_{p|q}^{\tilde\lambda}$. We shall give an explicit formula for $\widehat{\lambda^*}$ in Eq. (2.20).

Since the supertrace Str is trivial on the derived algebra of $gl_{p|q}$, one may twist any action of $gl_{p|q}$ by any scalar multiple of the supertrace. This is to say that, if $X \in gl_{p|q}$ acts on a space, then we may define a new action of X on this space by $X + \gamma\,\mathrm{Str}(X)$ instead, where $\gamma \in \mathbb{C}$. This in particular allows us to twist the standard action of $gl_{p|q}$ on $S(\mathbb{C}^{d*}\otimes\mathbb{C}^{p|q*})$ by $-d\mathrm{Str}$. Under this twisted action of $gl_{p|q}$, the space $S(\mathbb{C}^{d^*}\otimes\mathbb{C}^{p|q^*})$ decomposes into

$$S\big(\mathbb{C}^{d^*}\otimes\mathbb{C}^{p|q^*}\big) \cong \sum_\lambda V_d^\lambda \otimes V_{p|q}^{-d\mathbf{1}+\hat\lambda}, \tag{2.11}$$

where

$$\mathbf{1} := (1,\ldots,\underbrace{1}_{p},-1,\ldots,-1).$$

Here the summation in λ is over all generalized partitions of non-positive integers with length d subject to $\lambda_{d-p} \geqslant -q$. Observe that λ^* is a partition of length d satisfying $(\lambda^*)_{p+1} \leqslant q$.

Remark 2.2. For any generalized partition λ of length d, the condition $\lambda_{d-p} \geqslant -q$ is considered to be automatically satisfied if $p \geqslant d$.

Remark 2.3. Hereafter we shall always mean this twisted action of $gl_{p|q}$ when considering $S(\mathbb{C}^{d*} \otimes \mathbb{C}^{p|q*})$ as a $gl_d \times gl_{p|q}$-module.

Let $e^1, \dots, e^d$ be the standard basis of $\mathbb{C}^d$. Let $e^{1*}, \dots, e^{d*}$ be a basis for the contragredient gl_d-module $\mathbb{C}^{d*}$. We require the two bases to be dual in the sense that $e^{i*}(e^j) = \delta_{ij}$ for all $i, j \in \{1, \dots, d\}$. Similarly, we let $e_1, \dots, e_p$, $f_1, \dots, f_q$ denote the standard homogeneous basis for the natural $gl_{p|q}$-module $\mathbb{C}^{p|q}$ and $e_1^*, \dots, e_p^*$, $f_1^*, \dots, f_q^*$ denote the dual basis for the contragredient $gl_{p|q}$-module $\mathbb{C}^{p|q*}$. For $1 \leqslant l \leqslant d$, $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$, we set

$$y_i^l := e^{l*} \otimes e_i^*, \qquad \zeta_j^l := e^{l*} \otimes f_j^*, \tag{2.12}$$

which form a basis for $\mathbb{C}^{d*} \otimes \mathbb{C}^{p|q*}$.

We will denote by $\mathbb{C}[\mathbf{y}, \zeta]$ the polynomial superalgebra generated by (2.12). Let e_{ij}, $1 \leqslant i, j \leqslant d$ and E_{ab}, $1 \leqslant a, b \leqslant p+q$ be the bases respectively for gl_d and $gl_{p|q}$ consisting of elementary matrices. Then the action of the commuting pair $(gl_d, gl_{p|q})$ on $\mathbb{C}[\mathbf{y}, \zeta]$ can be realized in terms of first-order differential operators as follows ($1 \leqslant i, j \leqslant d$, $1 \leqslant r, r' \leqslant p$ and $1 \leqslant s, s' \leqslant q$):

$$\bar{\phi}(e_{ij}) := -\sum_{k=1}^{p} y_k^j \frac{\partial}{\partial y_k^i} - \sum_{k=1}^{q} \zeta_k^j \frac{\partial}{\partial \zeta_k^i}, \tag{2.13}$$

$$\bar{\phi}(E_{rr'}) := -\sum_{l=1}^{d} \frac{\partial}{\partial y_r^l} y_{r'}^l, \qquad \bar{\phi}(E_{s+p,s'+p}) := \sum_{l=1}^{d} \frac{\partial}{\partial \zeta_s^l} \zeta_{s'}^l, \tag{2.14}$$

$$\bar{\phi}(E_{s+p,r}) := -\sum_{l=1}^{d} \frac{\partial}{\partial \zeta_s^l} y_r^l, \qquad \bar{\phi}(E_{r,s+p}) := \sum_{l=1}^{d} \frac{\partial}{\partial y_r^l} \zeta_s^l.$$

It is straightforward to show that the $\bar{\phi}(e_{ij})$ and $\bar{\phi}(E_{ab})$ satisfy the same commutation relations as the e_{ij} and E_{ab}, respectively. Furthermore, the elements of (2.13) commute with those of (2.14).

For $1 \leqslant r \leqslant \min(d, p)$, we define the following determinant of an $r \times r$ matrix:

$$\Delta_r^* := \det \begin{pmatrix} y_p^d & y_{p-1}^d & \cdots & y_{p-r+1}^d \\ y_p^{d-1} & y_{p-1}^{d-1} & \cdots & y_{p-r+1}^{d-1} \\ \vdots & \vdots & \vdots & \vdots \\ y_p^{d-r+1} & y_{p-1}^{d-r+1} & \cdots & y_{p-r+1}^{d-r+1} \end{pmatrix}. \tag{2.15}$$

For $1 \leqslant r \leqslant d$, we define

$$\Delta_{k,r}^* := \zeta_k^d \zeta_k^{d-1} \cdots \zeta_k^{d-r+1}, \quad k = 1, \ldots, q. \tag{2.16}$$

It is clear that the Δ_r^* and $\Delta_{k,r}^*$ are all gl_d highest-weight vectors with respect to the standard Borel subalgebra $\mathfrak{b}_d$. They are also weight vectors under the action of $gl_d \times gl_{p|q}$ with the gl_d-weights respectively given by

$$\begin{aligned} wt_d\big(\Delta_r^*\big) &= (0, \ldots, 0, \underbrace{-1}_{d+1-r}, \ldots, -1), \\ wt_d\big(\Delta_{k,r}^*\big) &= (0, \ldots, 0, \underbrace{-1}_{d+1-r}, \ldots, -1), \end{aligned} \tag{2.17}$$

and the $gl_{p|q}$-weights respectively given by

$$\begin{aligned} wt_{p|q}\big(\Delta_r^*\big) &= -d\mathbf{1} + (0, \ldots, 0, \underbrace{-1}_{p+1-r}, \ldots, \underbrace{-1}_{p}, 0, \ldots, 0), \\ wt_{p|q}\big(\Delta_{k,r}^*\big) &= -d\mathbf{1} + (0, \ldots, 0, \underbrace{-r}_{p+k}, 0, \ldots, 0). \end{aligned} \tag{2.18}$$

Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a generalized partition of non-positive integers subject to the condition $\lambda_{d-p} \geqslant -q$. Then $\mu := \lambda^*$ is a partition satisfying the condition $\mu_{p+1} \leqslant q$. We let μ' denote the transpose partition of μ. Define

$$\Delta_\lambda^* := \begin{cases} \prod_{k=1}^{\mu_1} \Delta_{q+1-k,\mu_k'}^*, & \text{if } \mu_1 \leqslant q, \\ \prod_{k=1}^{q} \Delta_{q+1-k,\mu_k'}^* \prod_{l=q+1}^{\mu_1} \Delta_{\mu_l'}^*, & \text{if } \mu_1 > q. \end{cases} \tag{2.19}$$

Lemma 2.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a generalized partition of non-positive integers subject to the condition $\lambda_{d-p} \geqslant -q$. Then Δ_λ^* is a non-zero highest-weight vector in $\mathbb{C}[\mathbf{y}, \zeta]$ with respect to the joint action of gl_d and $gl_{p|q}$. The gl_d-weight of Δ_λ^* is λ, while the $gl_{p|q}$-weight of Δ_λ^* is $-d\mathbf{1} + \hat{\lambda}$ with*

$$\hat{\lambda} := -\big(\langle \mu_p - q\rangle, \langle \mu_{p-1} - q\rangle, \ldots, \langle \mu_1 - q\rangle, \mu_q', \mu_{q-1}', \ldots, \mu_1'\big), \tag{2.20}$$

where $\mu = \lambda^*$.

Proof. Note that $\bar{\phi}(e_{ij})$, for all $1 \leqslant i, j \leqslant d$, act on $\mathbb{C}[\mathbf{y}, \zeta]$ by derivations. Thus the product of any subset of the gl_d highest-weight vectors Δ_r^*, $\Delta_{k,r}^*$, $r = 1, 2, \ldots, \min(d, p)$, $k = 1, 2, \cdots, q$, is also a gl_d highest-weight vector. Hence so is Δ_λ^*.

Obviously Δ_λ^* is a highest-weight vector with respect to the action of the subalgebra $gl_p \times gl_q$ of $gl_{p|q}$. Consider the action of $\bar{\phi}(E_{p,p+1}) = \sum_{l=1}^d \partial\zeta_1^l/\partial y_p^l$ on Δ_λ^*. When $-\lambda_d = \mu_1 \leqslant q$, Δ_λ^* does not involve any of the variables y_p^l, thus $\bar{\phi}(E_{p,p+1})\Delta_\lambda^* = 0$. By using the equation

$$\Delta_{1,r}^* \sum_{l=1}^d \zeta_1^l \frac{\partial}{\partial y_p^l} \Delta_s = 0, \quad r \geqslant s,$$

we also easily show that Δ_λ^* is annihilated by $\bar{\phi}(E_{p,p+1})$ when $\mu_1 > q$. This proves that Δ_λ^* is indeed a $gl_d \times gl_{p|q}$ highest-weight vector. The rest of the lemma easily follows from Eqs. (2.17) and (2.18). $\square$

To summarize this subsection, we combine Lemma 2.2 with Eq. (2.1) into the following theorem.

Theorem 2.2. *Under the $gl_d \times gl_{p|q}$-action, $\mathbb{C}[\mathbf{y}, \zeta]$ decomposes into*

$$\mathbb{C}[\mathbf{y}, \zeta] \cong \sum_\lambda V_d^\lambda \otimes V_{p|q}^{-d\mathbf{1}+\hat{\lambda}},$$

where λ is summed over all generalized partitions of non-positive integers with length d subject to $\lambda_{d-p} \geqslant -q$. The space of highest-weight vectors in $V_d^\lambda \otimes V_{p|q}^{-d\mathbf{1}+\hat{\lambda}}$ is given by $\mathbb{C}\Delta_\lambda^*$.

3. The $(gl_d, gl_{m+p|n+q})$-duality on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{d^*} \otimes \mathbb{C}^{p|q^*})$

3.1. The $gl_d \times gl_{m+p|n+q}$-action on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{d^*} \otimes \mathbb{C}^{p|q^*})$

We described the semi-simple multiplicity free actions of $gl_d \times gl_{m|n}$ on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ and $gl_d \times gl_{p|q}$ on $S(\mathbb{C}^{d^*} \otimes \mathbb{C}^{p|q^*})$ in the last section. Through the obvious isomorphism between $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \otimes_{\mathbb{C}} S(\mathbb{C}^{d^*} \otimes \mathbb{C}^{p|q^*})$ and $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{d^*} \otimes \mathbb{C}^{p|q^*})$, these actions lead to a $gl_d \times gl_{m|n} \times gl_{p|q}$-action on the latter, where gl_d acts diagonally. It is not immediately obvious, but nevertheless true [14], that $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{d^*} \otimes \mathbb{C}^{p|q^*})$ also admits an action of the larger algebra $gl_d \times gl_{m+p|n+q}$.

For the purpose of describing this action, it is convenient to introduce a basis for $gl_{m+p|n+q}$ different from that given in Section 2.1. Set $\mathbf{I} = \{1, 2, \ldots, m+p+n+q\}$. Let $\{v_A \mid A \in \mathbf{I}\}$ be a basis of $\mathbb{C}^{p|q} \oplus \mathbb{C}^{m|n}$ such that $\{v_a \mid 1 \leqslant a \leqslant p+q\}$ and $\{v_{p+q+c} \mid 1 \leqslant c \leqslant m+n\}$ are respectively the standard bases for $\mathbb{C}^{p|q}$ and $\mathbb{C}^{m|n}$ described in Section 2.1. Let E_B^A, $A, B \in \mathbf{I}$, be the set of the elementary matrices satisfying $E_B^A v_C = \delta_{BC} v_A$.

These matrices form a homogeneous basis of $gl_{m+p|n+q}$ with the following commutation relations:

$$\left[E_B^A, E_D^C\right] = \delta_{BC} E_D^A - (-1)^{\deg E_B^A \deg E_D^C} \delta_{AD} E_B^C, \tag{3.1}$$

where $\deg E_B^A$ is the $\mathbb{Z}_2$-degree of E_B^A.

Let $\mathcal{B} := \sum_{A \leqslant B;\ A,B \in \mathbf{I}} \mathbb{C} E_B^A$, and $\mathfrak{h}_{m+p|n+q} := \sum_{A \in \mathbf{I}} \mathbb{C} E_A^A$. Then $\mathcal{B}$ forms a (non-standard) Borel subalgebra of $gl_{m+p|n+q}$ with the Cartan subalgebra $\mathfrak{h}_{m+p|n+q}$. Note that $\sum_{a,b=1}^{p+q} \mathbb{C} E_b^a$ forms a subalgebra isomorphic to $gl_{p|q}$, and $\sum_{u,v=1}^{m+n} \mathbb{C} E_{p+q+v}^{p+q+u}$ forms a subalgebra isomorphic to $gl_{m|n}$, and these two subalgebras mutually commute. Together they form $gl_{p|q} \times gl_{m|n}$, which is a regular subalgebra of $gl_{m+p|n+q}$ in the sense that its standard Borel subalgebra $\mathfrak{b}_{p|q} \times \mathfrak{b}_{m|n}$ is contained in $\mathcal{B}$, and the corresponding Cartan subalgebra $\mathfrak{h}_{p|q} \times \mathfrak{h}_{m|n}$ is identified with $\mathfrak{h}_{m+p|n+q}$. This identification leads to canonical embeddings of $\mathfrak{h}^*_{p|q}$ and $\mathfrak{h}^*_{m|n}$ in $\mathfrak{h}^*_{m+p|n+q}$. Choose a basis $\{\hat{\varepsilon}_A \mid A \in \mathbf{I}\}$ for $\mathfrak{h}^*_{m+p|n+q}$ such that $\hat{\varepsilon}_A(E_{BB}) = \delta_{AB}$, for all $A, B \in \mathbf{I}$. An element $\Lambda = \sum_{A \in \mathbf{I}} \Lambda_A \hat{\varepsilon}_A$ of $\mathfrak{h}^*_{m+p|n+q}$ will also be written as $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_{p+q+m+n})$. Now any pair of elements $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{p+q}) \in \mathfrak{h}^*_{p|q}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{m+n}) \in \mathfrak{h}^*_{m|n}$ gives rise to an element $(\lambda; \mu) \in \mathfrak{h}^*_{m+p|n+q}$ defined by

$$(\lambda; \mu) := (\lambda_1, \lambda_2, \ldots, \lambda_{p+q}, \mu_1, \mu_2, \ldots, \mu_{m+n}). \tag{3.2}$$

We retain the notations $\mathbb{C}[\mathbf{x}, \eta]$ and $\mathbb{C}[\mathbf{y}, \zeta]$ from Section 2, and denote by $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ the polynomial superalgebra $\mathbb{C}[\mathbf{x}, \eta] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{y}, \zeta]$. Let $\mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ denote the oscillator superalgebra generated by the variables $x_i^l, \eta_j^l, y_r^l, \zeta_s^l$, and their derivatives $\partial/\partial x_i^l$, $\partial/\partial \eta_j^l$, $\partial/\partial y_r^l$, $\partial/\partial \zeta_s^l$, where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $1 \leqslant r \leqslant p$, $1 \leqslant s \leqslant q$, and $1 \leqslant l \leqslant d$. Then $\mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ naturally acts on $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$, thus forms a subalgebra of $\mathrm{End}(\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta])$.

The general linear group $GL(d)$ acts on $\mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ by conjugation. The corresponding action of the Lie algebra gl_d is realized in terms of the following first-order differential operators ($1 \leqslant i, j \leqslant d$):

$$\Phi(e_{ij}) = \sum_{k=1}^m x_k^i \frac{\partial}{\partial x_k^j} + \sum_{k=1}^n \eta_k^i \frac{\partial}{\partial \eta_k^j} - \sum_{k=1}^p y_k^j \frac{\partial}{\partial y_k^i} - \sum_{k=1}^q \zeta_k^j \frac{\partial}{\partial \zeta_k^i}. \tag{3.3}$$

Let $\mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta]^{GL(d)}$ denote the $GL(d)$-invariant subalgebra of $\mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$. The $GL(d)$-action is semi-simple. Thus from the first fundamental theorem of the invariant theory of the general linear group (see, e.g., Chapter 4 of [12]) we deduce that $\mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta]^{GL(d)}$ is generated by the following operators:

$$\Phi\left(E_b^a\right) := \bar{\phi}(E_{ab}), \quad 1 \leqslant a, b \leqslant p+q, \tag{3.4}$$

$$\Phi\left(E_{p+q+v}^{p+q+u}\right) := \phi(E_{uv}), \quad 1 \leqslant u, v \leqslant m+n, \tag{3.5}$$

$$\Phi\big(E^r_{p+q+i}\big) := \sum_{l=1}^d \frac{\partial}{\partial y^l_r}\frac{\partial}{\partial x^l_i}, \qquad \Phi\big(E^r_{m+p+q+j}\big) := \sum_{l=1}^d \frac{\partial}{\partial y^l_r}\frac{\partial}{\partial \eta^l_j},$$

$$\Phi\big(E^{p+s}_{p+q+i}\big) := \sum_{l=1}^d \frac{\partial}{\partial \zeta^l_s}\frac{\partial}{\partial x^l_i}, \qquad \Phi\big(E^{p+s}_{m+p+q+j}\big) := \sum_{l=1}^d \frac{\partial}{\partial \zeta^l_s}\frac{\partial}{\partial \eta^l_j}, \tag{3.6}$$

$$\Phi\big(E^{p+q+i}_r\big) := -\sum_{l=1}^d x^l_i y^l_r, \qquad \Phi\big(E^{p+q+i}_{p+s}\big) := \sum_{l=1}^d x^l_i \zeta^l_s,$$

$$\Phi\big(E^{m+p+q+j}_r\big) := -\sum_{l=1}^d \eta^l_j y^l_r, \qquad \Phi\big(E^{m+p+q+j}_{p+s}\big) := \sum_{l=1}^d \eta^l_j \zeta^l_s, \tag{3.7}$$

where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $1 \leqslant r \leqslant p$, and $1 \leqslant s \leqslant q$. It is an easy exercise to show that the space spanned by $\Phi(e_{ij})$, $1 \leqslant i, j \leqslant d$, and $\Phi(E^A_B)$, $A, B \in \mathbf{I}$ is a homomorphic image of $gl_d \times gl_{m+p|n+q}$ in $\mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$. As every Lie superalgebra map uniquely extends to a homomorphism of its universal enveloping algebra, we have an associative superalgebra homomorphism

$$\Phi : \mathrm{U}(gl_d \times gl_{m+p|n+q}) \to \mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta].$$

Now by identifying $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{d^*} \otimes \mathbb{C}^{p|q^*})$ with the polynomial superalgebra $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$, we obtain an action of $gl_d \times gl_{m+p|n+q}$ on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n} + \mathbb{C}^{d^*} \otimes \mathbb{C}^{p|q^*})$. It can be extracted from [14] that this action is semi-simple and multiplicity free. We state this as a theorem for convenience of reference.

Theorem 3.1 [14]. *The pair* $(gl_d, gl_{m+p|n+q})$ *of Lie* (*super*)*algebras forms a dual reductive pair on* $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$.

3.2. Unitarity

We first recall some basic facts about $*$-superalgebras and their unitarizable representations. A $*$-superalgebra is an associative superalgebra A together with an anti-linear anti-involution $\omega : A \to A$. Here we should emphasize that for any $a, b \in A$, we have $\omega(ab) = \omega(b)\omega(a)$, where no sign factors are involved. A $*$-superalgebra homomorphism $f : (A, \omega) \to (A', \omega')$ is a superalgebra homomorphism obeying $f \circ \omega = \omega' \circ f$. Let (A, ω) be a $*$-superalgebra, and let V be a $\mathbb{Z}_2$-graded A-module. A Hermitian form $\langle \cdot | \cdot \rangle$ on V is said to be contravariant if $\langle av | v' \rangle = \langle v | \omega(a)v' \rangle$, for all $a \in A$, $v, v' \in V$. An A-module equipped with a positive definite contravariant Hermitian form is called a unitarizable A-module. It is clear that any unitarizable A-module is completely reducible.

The oscillator superalgebra $\mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ admits the $*$-structure ω defined by

$$x^l_i \mapsto \frac{\partial}{\partial x^l_i}, \quad \frac{\partial}{\partial x^l_i} \mapsto x^l_i, \qquad \eta^l_j \mapsto \frac{\partial}{\partial \eta^l_j}, \quad \frac{\partial}{\partial \eta^l_j} \mapsto \eta^l_j,$$

$$y_r^l \mapsto \frac{\partial}{\partial y_r^l} \quad \frac{\partial}{\partial y_r^l} \mapsto y_r^l, \qquad \zeta_s^l \mapsto \frac{\partial}{\partial \zeta_s^l}, \quad \frac{\partial}{\partial \zeta_s^l} \mapsto \zeta_s^l,$$

where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $1 \leqslant r \leqslant p$, $1 \leqslant s \leqslant q$, and $1 \leqslant l \leqslant d$. There exits a unique contravariant Hermitian form $\langle \cdot | \cdot \rangle$ on $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ with $\langle 1 | 1 \rangle = 1$. By using the 'particle number' basis relative to which the operators $x_i^l \partial/\partial x_i^l$, $\eta_j^l\, \partial/\partial \eta_j^l$, $y_r^l\, \partial/\partial y_r^l$, $\zeta_s^l\, \partial/\partial \zeta_s^l$, for all i, j, r, s, l are simultaneously diagonalizable, one can easily show that the form $\langle \cdot | \cdot \rangle$ is positive definite. The polynomial superalgebra $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ with this inner product (after completion) is the Fock space of $d(m+p)$ bosonic and $d(n+q)$ fermionic quantum oscillators. When $d = 1$, we denote it by $\mathcal{F}_{m+p|n+q}$. Then it is clear that for arbitrary d we have

$$\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta] \cong (\mathcal{F}_{m+p|n+q})^{\otimes d}.$$

What presented in this paragraph is standard material on Fock spaces, which is part of the basic ingredients of second quantization.

We now consider a $*$-structure of $\mathrm{U}(gl_d \times gl_{m+p|n+q})$. We shall regard $gl_d \times gl_{m+p|n+q}$ as embedded in its universal enveloping algebra. Consider the anti-linear anti-involution σ of $\mathrm{U}(gl_d \times gl_{m+p|n+q})$ defined, for all $1 \leqslant a, b \leqslant p+q$, $p+q+1 \leqslant r, s \leqslant p+q+m+n$, and $1 \leqslant i, j \leqslant d$, by

$$\begin{aligned} &E_b^a \mapsto (-1)^{[a]+[b]} E_a^b, \qquad E_s^r \mapsto E_r^s, \\ &E_s^a \mapsto -(-1)^{[a]} E_a^s, \qquad E_b^r \mapsto -(-1)^{[b]} E_r^b, \\ &e_{ij} \mapsto e_{ji}, \end{aligned}$$

where

$$[a] = \begin{cases} 0, & 1 \leqslant a \leqslant p, \\ 1, & 1 \leqslant a - p \leqslant q. \end{cases}$$

By direct calculations we can show that this anti-linear map respects the commutation relations (3.1), thus indeed defines an anti-linear anti-involution of $\mathrm{U}(gl_d \times gl_{m+p|n+q})$. Now σ gives rise to a $*$-structure for $\mathrm{U}(gl_d \times gl_{m+p|n+q})$.

Theorem 3.2.

(1) *The map Φ is a $*$-superalgebra homomorphism from* $(\mathrm{U}(gl_d \times gl_{m+p|n+q}), \sigma)$ *to the oscillator superalgebra* $(\mathbb{D}[\mathbf{x}, \mathbf{y}, \eta, \zeta], \omega)$.
(2) $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ *is a unitarizable* $(U(gl_d \times gl_{m+p|n+q}), \sigma)$*-module with respect to the Hermitian form* $\langle \cdot , \cdot \rangle$.

Proof. Using Eqs. (3.3)–(3.7), we can show by direct calculations that for all $X \in gl_d \times gl_{m+p|n+q}$, we have $\Phi\sigma(X) = \omega\Phi(X)$. This proves part (1). Part (2) immediately follows from part (1). $\square$

We also have the following result.

Lemma 3.1. *All the irreducible $gl_d \times gl_{m+p|n+q}$-submodules of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ are of highest-weight type with respect to the Borel subalgebra $\mathfrak{b}_d \times \mathcal{B}$.*

Proof. Let H be the harmonic subspace of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$, i.e., the subspace consisting of such polynomials that are annihilated by all the elements $\Phi(E^A_B)$, $A \leqslant p+q$, $B > p+q$, of (3.6). Then H forms a module of the subalgebra $gl_d \times gl_{p|q} \times gl_{m|n}$ spanned by elements of (3.3), (3.4) and (3.5). It follows from the $(gl_d, gl_{m|n})$-duality on $\mathbb{C}[\mathbf{x}, \eta]$ described in Theorem 2.1 and the $(gl_d, gl_{p|q})$-duality on $\mathbb{C}[\mathbf{y}, \zeta]$ described in Theorem 2.2 that H decomposes into a direct sum of finite-dimensional irreducible $gl_d \times gl_{p|q} \times gl_{m|n}$-modules.

Let W be an irreducible $gl_d \times gl_{m+p|n+q}$-submodule of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$. Let $H_W = W \cap H$. Then $H_W \neq 0$, as the lowest-order polynomials of W are all contained in H_W. Now H_W forms a module of the subalgebra $gl_d \times gl_{p|q} \times gl_{m|n}$, which in fact is irreducible. To see this, we note that if H_W were reducible with respect to $gl_d \times gl_{p|q} \times gl_{m|n}$, then due to complete reducibility H_W would contain more than one linearly independent $gl_d \times gl_{p|q} \times gl_{m|n}$ highest-weight vectors. However, since they lie in H_W, they would also be $gl_d \times gl_{m+p|n+q}$ highest-weight vectors with respect to $\mathfrak{b}_d \times \mathcal{B}$, thus contradicting the irreducibility of W. Thus W is generated by a $gl_d \times gl_{m+p|n+q}$ highest-weight vector with respect to $\mathfrak{b}_d \times \mathcal{B}$, as claimed. □

Remark 3.1. From Theorem 3.1 and the proof of the lemma we can deduce that the action of $gl_d \times gl_{p|q} \times gl_{m|n}$ on the harmonic subspace H of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ is semi-simple and multiplicity free.

Let $\mathfrak{g}^{\mathbb{R}}$ be the real superspace spanned by $\{X \in (gl_{m+p|n+q})_{\bar{0}} \mid \sigma(X) = -X\} \cup \sqrt{\mathrm{i}}\{X \in (gl_{m+p|n+q})_{\bar{1}} \mid \sigma(X) = -X\}$. Then $\mathfrak{g}^{\mathbb{R}}$ is a real form of $gl_{m+p|n+q}$, that is, $\mathfrak{g}^{\mathbb{R}}$ forms a real Lie superalgebra with the complexification being $gl_{m+p|n+q}$ itself. The usual notation for this real form is $u(m, p|n, q)$. Note that the maximal even subalgebra of $\mathfrak{g}^{\mathbb{R}}$ is $u(m, p) \times u(n, q)$. Thus every non-trivial unitarizable $u(m, p|n, q)$-module must be infinite-dimensional.

3.3. Comments on unitarizable modules

At this point, we should relate to results in the literature. Note that the restrictions of σ to the subalgebras $gl_{m|n}$ and $gl_{p|q}$ act differently on the odd subspaces. They respectively give rise to two different real forms $u_+(m|n)$ and $u_-(p|q)$ of the subalgebras. Now $\mathfrak{g}^{\mathbb{R}}$ contains the subalgebra $u_-(p|q) \times u_+(m|n)$, which one would like to regard as the 'maximal compact subalgebra'. Unfortunately finite-dimensional representations of $u_+(m|n)$ and $u_-(p|q)$ are not necessarily unitarizable. In fact it has long been known [13] that the only finite-dimensional unitarizable irreducible representations of $u_+(m|n)$ (respectively $u_-(p|q)$) are the tensor products of the irreducible representations appearing in Theorem 2.1 (respectively Theorem 2.2) with some 1-dimensional representations (upon restricting modules of the general linear superalgebra to modules of its real form), which constitute a small class of the finite-dimensional irreducible representations. Thus the situation is very different from the case of the compact real Lie algebra $u(k)$.

The intersection of $u(m,p|n,q)$ with $sl_{m+p|n+p}$ gives rise to the real Lie superalgebra $su(m,p|n,q)$. It was shown in [16] that $su(m,p|n,q)$ admits no unitarizable highest- or lowest-weight representations with respect to the standard Borel subalgebra if all the integers m, n, p and q are non-zero. Since [16] was only concerned with irreducible unitarizable highest-weight modules of simple basic classical Lie superalgebras with respect to their standard Borel subalgebras [17], the irreducible $gl_{m+p|n+q}$-modules appearing in the decomposition of $\mathbb{C}[\mathbf{x},\mathbf{y},\eta,\zeta]$ were ignored. In fact, with respect to the standard Borel subalgebra of $gl_{m+p|n+q}$, the unitarizable irreducible representations studied in this paper are neither highest-weight nor lowest-weight type unless some of the integers m, n, p and q are zero.

A final comment is that when both of the integers n and q are zero, the general linear superalgebra $gl_{m+p|n+q}$ reduces to the ordinary Lie algebra gl_{m+p}, and $\mathbb{C}[\mathbf{x},\mathbf{y},\eta,\zeta]$ to the ordinary polynomial algebra $\mathbb{C}[\mathbf{x},\mathbf{y}]$ in the two sets of variables $\mathbf{x}$ and $\mathbf{y}$. Then the irreducible gl_{m+p}-modules appearing in $\mathbb{C}[\mathbf{x},\mathbf{y}]$ are the unitarizable irreducible $u(m,p)$-modules studied by Kashiwara and Vergne in [18]. It is known [8,18] that the unitarizable irreducible $u(m,p)$-module at every reduction point [11] is a submodule in $\mathbb{C}[\mathbf{x},\mathbf{y}]$ for some d. However, it is not known whether this is still true in the super case.

3.4. *The $(gl_d, gl_{m+p|n+q})$-duality on $\mathbb{C}[\mathbf{x},\mathbf{y},\eta,\zeta]$*

Each generalized partition $\lambda=(\lambda_1,\cdots,\lambda_d)$ of length d can be uniquely expressed as $\lambda=\lambda^+ + \lambda^-$, with

$$\lambda^+ := \big(\max\{\lambda_1,0\},\ldots,\max\{\lambda_d,0\}\big), \qquad \lambda^- := \big(\min\{\lambda_1,0\},\ldots,\min\{\lambda_d,0\}\big).$$

Note that λ^+ is a partition of length d , while λ^- is a generalized partition of non-positive integers with length d, such that $(\lambda^-)^*$ is a partition. Furthermore,

$$\text{depth of } \lambda^+ + \text{depth of } (\lambda^-)^* \leqslant d, \tag{3.8}$$

where the depth of a partition is the number of positive integers in it.

The generalized partition $\lambda=(\lambda_1,\ldots,\lambda_d)$ satisfies the conditions $\lambda_{m+1}\leqslant n$ and $\lambda_{d-p}\geqslant -q$ if and only if $\lambda^+_{m+1}\leqslant n$ and $(\lambda^-)^*_{p+1}\leqslant q$. Corresponding to each such generalized partition λ, we define

$$\square_\lambda := \Delta^*_{\lambda^-}\Delta_{\lambda^+}. \tag{3.9}$$

Lemma 3.2. *If the generalized partition λ satisfies the conditions $\lambda_{m+1}\leqslant n$ and $\lambda_{d-p}\geqslant -q$, then $\square_\lambda$ is a non-zero $gl_d\times gl_{m+p|n+q}$ highest-weight vector with respect to the Borel subalgebra $\mathfrak{b}_d\times\mathcal{B}$. The gl_d-weight of $\square_\lambda$ is λ, and the $gl_{m+p|n+q}$-weight is given by*

$$\Lambda(\lambda) := \big(-d\mathbf{1}+\widehat{\lambda^-};\ \widetilde{\lambda^+}\big), \tag{3.10}$$

where the expression on the right-hand side is as explained by (3.2).

Proof. By construction, $\Box_\lambda$ is a highest-weight vector of the subalgebra $gl_d \times gl_{p|q} \times gl_{m|n}$ with respect to the standard Borel subalgebra $\mathfrak{b}_d \times \mathfrak{b}_{p|q} \times \mathfrak{b}_{m|n}$. Therefore, we only need to show that

$$\Phi(E_{p+q,p+q+1}) = \sum_{k=1}^{d} \frac{\partial}{\partial \zeta_q^k} \frac{\partial}{\partial x_1^k}$$

annihilates $\Box_\lambda$ in order to prove that $\Box_\lambda$ is a $gl_d \times gl_{m+p|n+q}$ highest-weight vector with respect to the Borel subalgebra $\mathfrak{b}_d \times \mathcal{B}$. If $\Phi(E_{p+q,p+q+1})\Box_\lambda \neq 0$, then there must exist at least one integer $i \in \{1, 2, \ldots, d\}$ such that $x_1^i \zeta_q^i$ appears in $\Box_\lambda$. Let $ht(\lambda^+)$ denote the depth of λ^+, and $ht((\lambda^-)^*)$ denote the depth of $(\lambda^-)^*$. By examining its explicit form, we can see that $\Box_\lambda$ does not involve any of the variables x_1^k, $d \geqslant k > ht(\lambda^+)$, and ζ_q^l, $1 \leqslant l < d + 1 - ht((\lambda^-)^*)$. Therefore in order for $x_1^i \zeta_q^i$ to appear in $\Box_\lambda$, the integer i must satisfy $d + 1 - ht((\lambda^-)^*) \leqslant i \leqslant ht(\lambda^+)$. But this is impossible since (3.8) requires $ht(\lambda^+) + ht((\lambda^-)^*) \leqslant d$. Therefore, $\Phi(E_{p+q,p+q+1})\Box_\lambda = 0$, and thus $\Box_\lambda$ is a $gl_d \times gl_{m+p|n+q}$ highest-weight vector with respect to the Borel subalgebra $\mathfrak{b}_d \times \mathcal{B}$.

The gl_d-weight of $\Box_\lambda$ is obviously λ. From Lemmas 2.1 and 2.2, we easily see that the $gl_{m+p|n+q}$-weight of $\Box_\lambda$ is indeed $(-d\mathbf{1} + \widehat{\lambda^-}; \widetilde{\lambda^+})$. $\Box$

We shall denote by $W^{\Lambda}_{m+p|n+q}$ the irreducible highest-weight $gl_{m+p|n+q}$-module with highest weight Λ relative to the non-standard Borel subalgebra $\mathcal{B}$.

Theorem 3.3. *Under the $gl_d \times gl_{m+p|n+q}$-action $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ decomposes into*

$$\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta] \cong \sum_{\lambda} V_d^{\lambda} \otimes W^{\Lambda(\lambda)}_{m+p|n+q}, \tag{3.11}$$

where λ is summed over all generalized partitions of length d subject to $\lambda_{m+1} \leqslant n$ and $\lambda_{d-p} \geqslant -q$. Furthermore, $\mathbb{C}\Box_\lambda$ is the space of highest-weight vectors of the irreducible module $V_d^{\lambda} \otimes W^{\Lambda(\lambda)}_{m+p|n+q}$.

Proof. Note that every irreducible gl_d-submodule of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ is finite-dimensional. Thus it follows from Theorem 3.1 and Lemma 3.1 that the decomposition of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ under $gl_d \times gl_{m+p|n+q}$ has to be of the form (3.11), with the sum in λ ranging over some subset of generalized partitions of length d. (Here if λ happens to be a generalized partition not satisfying $\lambda_{m+1} \leqslant n$ and $\lambda_{d-p} \geqslant -q$, the expression $\Lambda(\lambda)$ stands for the highest weight for $gl_{m+p|n+q}$ corresponding to λ under this Howe duality.) In view of Lemma 3.2, we only need to show that every generalized partition λ belonging to this subset must satisfy the conditions $\lambda_{m+1} \leqslant n$ and $\lambda_{d-p} \geqslant -q$, in order to prove the theorem.

Let μ be a generalized partition of length d. Assume that either one or both of the conditions $\mu_{m+1} \leqslant n$ and $\mu_{d-p} \geqslant -q$ are violated. We choose a pair of positive integers p' and n' with $p' \geqslant p$ and $n' \geqslant n$ such that $\mu_{m+1} \leqslant n'$ and $\mu_{d-p'} \geqslant -q$. Such an n' is trivial to come by, and such a p' also exists since by Remark 2.2 the condition $\mu_{d-p'} \geqslant -q$ is always satisfied if $p' \geqslant d$. We let $\mathbb{C}[\mathbf{x}, \bar{\mathbf{y}}, \bar{\eta}, \zeta]$ denote the polynomial superalgebra generated by

$$x_i^l := e^l \otimes e_i, \qquad \bar{y}_r^l := e^{l*} \otimes e_r^*, \qquad \bar{\eta}_j^l := e^l \otimes f_j, \qquad \zeta_s^l := e^{l*} \otimes f_s^*,$$

for $1 \leqslant l \leqslant d$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n'$, $1 \leqslant r \leqslant p'$ and $1 \leqslant s \leqslant q$. Then $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ becomes a subspace of $\mathbb{C}[\mathbf{x}, \bar{\mathbf{y}}, \bar{\eta}, \zeta]$ upon identifying

$$y_r^l := \bar{y}_{p'-p+r}^l, \qquad \zeta_s^l := \zeta_s^l, \qquad x_i^l := x_i^l, \qquad \eta_j^l := \bar{\eta}_j^l, \tag{3.12}$$

for $1 \leqslant l \leqslant d$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $1 \leqslant r \leqslant p$, and $1 \leqslant s \leqslant q$. We denote this inclusion by $\iota : \mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta] \to \mathbb{C}[\mathbf{x}, \bar{\mathbf{y}}, \bar{\eta}, \zeta]$. There exist also the surjection $\pi : \mathbb{C}[\mathbf{x}, \bar{\mathbf{y}}, \bar{\eta}, \zeta] \to \mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ defined by setting $\bar{y}_r^l = 0$, $1 \leqslant r \leqslant p' - p$, and $\bar{\eta}_s^l = 0$, $n < s \leqslant n'$, for all l, then making the identification (3.12). Obviously, $\pi\iota$ is the identity map on $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$.

Now we turn to the $gl_d \times gl_{m+p'|n'+q}$-action on $\mathbb{C}[\mathbf{x}, \bar{\mathbf{y}}, \bar{\eta}, \zeta]$. Upon choosing the basis for $\mathbb{C}^{p'|q} \oplus \mathbb{C}^{m|n'}$ that is the union of the standard bases of $\mathbb{C}^{p'|q}$ and $\mathbb{C}^{m|n'}$, the general linear superalgebra $gl_{m+p'|n'+q}$ becomes the Lie superalgebra of $(m + p' + n' + q) \times (m + p' + n' + q)$-matrices. Consider the subalgebra $\mathfrak{l}$ of $gl_{m+p'|n'+q}$ consisting of matrices of the form

$$\begin{pmatrix} D & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & D' \end{pmatrix},$$

where $X \in gl_{m+p|n+q}$, and D and D' are diagonal matrices of sizes $(p' - p) \times (p' - p)$ and $(n' - n) \times (n' - n)$, respectively. Obviously, $\mathfrak{l}$ contains the $gl_{m+p|n+q}$ subalgebra

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, X \in gl_{m+p|n+q} \right\}.$$

Let $\mathfrak{p} = \mathfrak{n} + \mathfrak{l}$ be a parabolic subalgebra of $gl_{m+p'|n'+q}$ with the Levi factor $\mathfrak{l}$ and nilpotent radical $\mathfrak{n}$. We assume that $\mathfrak{p}$ contains all the upper triangular matrices. Then there exists a nilpotent subalgebra $\bar{\mathfrak{n}}$ consisting of strictly lower triangular matrices such that $gl_{m+p'|n'+q} = \mathfrak{p} + \bar{\mathfrak{n}}$. By examining Eqs. (3.3)–(3.7), we can see that ι is a $gl_d \times gl_{m+p|n+q}$-module map. Let V be any $gl_d \times gl_{m+p|n+q}$-submodule of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$. Then $\iota(V)$ is in fact a $gl_d \times \mathfrak{p}$-module with $\mathfrak{n}$ acting by zero. Thus $W = \Phi(\mathrm{U}(\bar{\mathfrak{n}}))\iota(V)$ forms a $gl_d \times gl_{m+p'|n'+q}$-submodule of $\mathbb{C}[\mathbf{x}, \bar{\mathbf{y}}, \bar{\eta}, \zeta]$. Note that W is irreducible if V is irreducible with respect to $gl_d \times gl_{m+p|n+q}$. Again by examining Eqs. (3.3)–(3.7), we can see that π is a $gl_d \times gl_{m+p|n+q}$-module map from the restriction of $\mathbb{C}[\mathbf{x}, \bar{\mathbf{y}}, \bar{\eta}, \zeta]$ to $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$, and satisfies $\pi(W) = V$. This in particular implies that $\pi\iota$ is the identity $gl_d \times gl_{m+p|n+q}$-module map on $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$.

Let $v_\mu \in \mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ be any $gl_d \times gl_{m+p|n+q}$ highest-weight vector with the gl_d-weight μ (that violates one or both of the conditions $\mu_{m+1} \leqslant n$ and $\mu_{d-p} \geqslant -q$). Then by the above discussion, $\iota(v_\mu)$ is a $gl_d \times \mathfrak{l}$ highest-weight vector in $\mathbb{C}[\mathbf{x}, \bar{y}, \bar{\eta}, \zeta]$, which has the same gl_d weight, and is also annihilated by $\mathfrak{n}$. Therefore, $\iota(v_\mu)$ is a $gl_d \times gl_{m+p'|n'+q}$

highest-weight vector in $\mathbb{C}[\mathbf{x}, \bar{y}, \bar{\eta}, \zeta]$. By Theorem 3.1 and Lemma 3.2 (with p replaced by p' and n by n'), there exists a unique non-zero $\Box_\mu \in \mathbb{C}[\mathbf{x}, \bar{\mathbf{y}}, \bar{\eta}, \zeta]$ such that $\iota(v_\mu) = c\Box_\mu$ for some complex number c.

We claim that every monomial in the polynomial $\Box_\mu$ contains at least one of the variables $\bar{y}_r^l$, $\bar{\eta}_s^l$, where $r = 1, \ldots, p' - p$, $s = n + 1, \ldots, n'$ and $l = 1, \ldots, d$. This can be seen from the explicit form (3.9) of $\Box_\lambda$. Consider first the case with $t := \mu_{m+1} > n$. Then $(\mu^+)'_1 > m$, and Δ_{μ^+} has the factor $\Delta_{t,(\mu^+)'_t}$. From (2.7) we see that $\Delta_{t,(\mu^+)'_t}$ is the determinant of a matrix with rows $(\bar{\eta}_t^1, \bar{\eta}_t^2, \ldots, \bar{\eta}_t^{(\mu^+)'_t})$ with Grassmann number entries. Now consider the case with $\mu_{d-p} < -q$. Let $\gamma = (\mu^-)^*$, then $\gamma_{p+1} > q$. Thus we must also have $\gamma_1 > q$, and $u := \gamma'_{q+1} \geqslant p + 1$. Now $\Delta^*_{\mu^-}$ has the factor Δ^*_u. From (2.16) we see that Δ^*_u is the determinant of a matrix with a column $(\bar{y}^d_{p'+1-u}, \bar{y}^{d-1}_{p'+1-u}, \ldots, \bar{y}^{d+1-u}_{p'+1-u})$. Note that $p' + 1 - u \leqslant p' - p$.

Now it is obvious that $\pi(\Box_\mu) = 0$, which in turn implies $v_\mu = 0$. Therefore, the decomposition of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ can not contain $V_d^\mu \otimes W^{\Lambda(\mu)}_{m+p|n+q}$ as an irreducible submodule if μ violates any of the conditions $\mu_{m+1} \leqslant n$ and $\mu_{d-p} \geqslant -q$. $\Box$

By using Theorems 2.1, 2.2 and the decomposition $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta] = \mathbb{C}[\mathbf{x}, \eta] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{y}, \zeta]$, we have

$$\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta] \cong \sum_{\lambda,\mu} V_d^\lambda \otimes V_d^\mu \otimes V_{m|n}^{\tilde{\lambda}} \otimes V_{p|q}^{-d\mathbf{1}+\hat{\mu}},$$

where the summation in λ is over all the partitions of length d satisfying $\lambda_{m+1} \leqslant n$, and the summation in μ is over all the generalized partitions of non-positive integers of length d satisfying $\mu_{d-p} \geqslant -q$.

The decomposition of the tensor product of any two finite-dimensional irreducible gl_d-modules is described by the Littlewood–Richardson theory. For any generalized partitions λ and μ of length d,

$$V_d^\lambda \otimes V_d^\mu \cong \sum_\nu C^\nu_{\lambda\mu} V_d^\nu, \tag{3.13}$$

where the non-negative integers $C^\nu_{\lambda\mu}$ are the so-called Littlewood–Richardson coefficients, which give the respective multiplicities of the irreducible gl_d-modules V_d^ν appearing in the tensor product.

Therefore, $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ decomposes into

$$\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta] \cong \sum_\nu V_d^\nu \otimes \sum_{\lambda,\mu} C^\nu_{\lambda\mu} V_{m|n}^{\tilde{\lambda}} \otimes V_{p|q}^{-d\mathbf{1}+\hat{\mu}}, \tag{3.14}$$

where the summation in ν is over all the generalized partitions of length d. Combining Theorem 3.3 and (3.14), the Littlewood–Richardson coefficients $C^\nu_{\lambda\mu}$ appearing in (3.14) may be non-zero only when ν satisfies the conditions $\nu_{m+1} \leqslant n$ and $\nu_{d-p} \geqslant -q$. Now, if we put $p = d - m - 1$, $n = \lambda_{m+1}$ and $q = -\mu_{d-p}$, then the conditions become $\nu_{m+1} \leqslant \lambda_{m+1}$, and $\nu_{m+1} \geqslant \mu_{m+1}$. Letting m run from 0 to $d - 1$, we have the following corollary.

Corollary 3.1. *Assume that λ is a partition of length d and μ is a generalized partition of non-positive integers of length d. If ν is a generalized partition of length d satisfying $\nu_m > \lambda_m$ or $\nu_m < \mu_m$ for some $m \in \{1, 2, \ldots, d\}$, then $C^{\nu}_{\lambda\mu} = 0$.*

Corollary 3.1 translated to ordinary partitions implies the following.

Corollary 3.2. *Let λ and μ be two partitions of length d. If ν is a partition of length d satisfying $\nu_m > \min\{\mu_m + \lambda_1, \lambda_m + \mu_1\}$ for some $m \in \{1, 2, \ldots, d\}$, then the Littlewood–Richardson coefficient $C^{\nu}_{\lambda\mu} = 0$.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ and $\mu = (\mu_1, \mu_2, \ldots \mu_d)$ be two partitions. Then $\mu - \mu_1\mathbf{1} := (\mu_1 - \mu_1, \mu_2 - \mu_1, \ldots \mu_d - \mu_1)$ is a generalized partition. By Corollary 3.1, we have $C^{\nu}_{\lambda\mu} = C^{\nu-\mu_1\mathbf{1}}_{\lambda,\mu-\mu_1\mathbf{1}} = 0$ if $\nu_m - \mu_1 > \lambda_m$ for some $m \in \{1, 2, \ldots, d\}$. Therefore, the corollary follows from the symmetry property of the Littlewood–Richardson coefficients. $\square$

Remark 3.2. An alternative method to prove Theorem 3.3 is the following. One can first prove Corollary 3.2, using, for example, the celebrated combinatorial algorithm known as the Littlewood–Richardson rule (see, e.g., [19]). Using Corollary 3.2 it can then be derived that in the tensor product decomposition of $V^{\lambda}_d \otimes V^{\mu}_d$, with λ a partition of length d satisfying $\lambda_{m+1} \leqslant n$, and μ a generalized partition of non-positive integers of length d satisfying $\mu_{d-p} \geqslant -q$, only gl_d-modules associated to generalized partitions ν with $\nu_{m+1} \leqslant n$ and $\nu_{d-p} \geqslant -q$ can occur. Using this fact together with Lemma 3.2 it is then not difficult to prove Theorem 3.3.

4. Branching rules of unitarizable irreducible $gl_{m+p|n+q}$-modules

As an easy application of Theorem 3.3, we derive the $gl_{m+p|n+q} \to gl_{p|q} \times gl_{m|n}$ branching rule for the infinite-dimensional unitarizable irreducible $gl_{m+p|n+q}$ representations arising from the decomposition of tensor powers of the Fock space of $m+p$ bosonic and $n+q$ fermionic quantum oscillators. Results of this section will be important for developing a character formula for these unitarizable irreducible $gl_{m+p|n+q}$-modules.

Let us denote by $\mathfrak{k}^{\mathbb{C}}$ the subalgebra $gl_{p|q} \times gl_{m|n}$ of $gl_{m+p|n+q}$. Recall the decomposition of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$ as a $gl_d \times \mathfrak{k}^{\mathbb{C}}$-module (3.14). Denote by $W^{\Lambda(\nu)}_{m+p|n+q}|_{\mathfrak{k}^C}$ the restriction of $W^{\Lambda(\nu)}_{m+p|n+q}$ to a $\mathfrak{k}^{\mathbb{C}}$-module. Let us now consider Theorem 3.3 by restricting both sides of Eq. (3.11) to $gl_d \times \mathfrak{k}^{\mathbb{C}}$-modules. Using (3.14) we obtain

$$\sum_{\nu} V^{\nu}_d \otimes W^{\Lambda(\nu)}_{m+p|n+q}\big|_{\mathfrak{k}^C} \cong \sum_{\nu} V^{\nu}_d \otimes \sum_{\lambda,\mu} C^{\nu}_{\lambda\mu} V^{\tilde{\lambda}}_{m|n} \otimes V^{-d\mathbf{1}+\hat{\mu}}_{p|q},$$

where the summation in ν on the left-hand side is over the generalized partitions of length d satisfying the conditions $\nu_{m+1} \leqslant n$ and $\nu_{d-p} \geqslant -q$. The above equation immediately leads to the following $gl_{m+p|n+q} \to gl_{p|q} \times gl_{m|n}$ branching rule:

Theorem 4.1. *Let ν be a generalized partition of length d subject to the conditions $\nu_{m+1} \leqslant n$ and $\nu_{d-p} \geqslant -q$. We have*

$$W^{\Lambda(\nu)}_{m+p|n+q}\big|_{gl_{p|q} \times gl_{m|n}} \cong \sum_{\lambda,\mu} C^{\nu}_{\lambda\mu} V^{\tilde{\lambda}}_{m|n} \otimes V^{-d\mathbf{1}+\hat{\mu}}_{p|q}, \tag{4.1}$$

where the summation in λ is over all the partitions of length d satisfying $\lambda_{m+1} \leqslant n$, and the summation in μ is over all the generalized partitions of non-positive integers with length d satisfying $\mu_{d-p} \geqslant -q$.

Recall that $\Lambda(\nu)$ is defined by (3.10).

5. Character formula for unitarizable irreducible $gl_{m+p|n+q}$-modules

In this section, we shall develop a character formula for the infinite-dimensional unitarizable irreducible $gl_{m+p|n+q}$-modules appearing in the decomposition of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$. Let us first present some background material on Schur functions and the so-called hook Schur functions of Berele–Regev [1]. A comprehensive reference on Schur functions is [19].

5.1. Hook Schur function

Let $\mathbf{x} = \{x_1, x_2, \ldots, x_m\}$ be a set of m variables. To a partition λ of non-negative integers we may associate the Schur function $s_\lambda(x_1, x_2, \ldots, x_m)$. We will write $s_\lambda(\mathbf{x})$ for $s_\lambda(x_1, x_2, \ldots, x_m)$. For a partition $\mu \subseteq \lambda$, we let $s_{\lambda/\mu}(\mathbf{x})$ denote the corresponding skew Schur function. Denote by μ' the conjugate partition of a partition μ. The *hook Schur function* [1] corresponding to a partition λ is defined by

$$HS_\lambda(\mathbf{x}; \mathbf{y}) := \sum_{\mu \subset \lambda} s_\mu(\mathbf{x}) s_{\lambda'/\mu'}(\mathbf{y}), \tag{5.1}$$

where as usual $\mathbf{y} = \{y_1, y_2, \ldots, y_n\}$.

Let λ be a partition and $\mu \subseteq \lambda$. We fill the boxes in μ with entries from the linearly ordered set $\{x_1 < x_2 < \cdots < x_m\}$ so that the resulting tableau is semi-standard. Recall that this means that the rows are non-decreasing, while the columns are strictly increasing. Next we fill the skew partition λ/μ with entries from the linearly ordered set $\{y_1 < y_2 < \cdots < y_n\}$ so that it is conjugate semi-standard, which means that the rows are strictly increasing, while its columns are non-decreasing. We will refer to such a tableau as an $(m|n)$-*semi-standard tableau* (cf. [1]). To each such tableau T we may associate a polynomial $(\mathbf{xy})^T$, which is obtained by taking the products of all the entries in T. Then we have [1]

$$HS_\lambda(\mathbf{x}; \mathbf{y}) = \sum_T (\mathbf{xy})^T, \tag{5.2}$$

where the summation is over all $(m|n)$-semi-standard tableaux of shape λ.

We recall the following combinatorial identity involving hook Schur functions, that is of crucial importance in the sequel. Note that $HS_\lambda(\mathbf{x};\mathbf{y}) \neq 0$ iff $\lambda_{m+1} \leqslant n$.

Proposition 5.1 [2]. *Let $\mathbf{x} = \{x_1, x_2, \dots, x_m\}$, $\eta = \{\eta_1, \eta_2, \dots, \eta_n\}$ be two sets of variables and $\mathbf{z} = \{z_1, z_2, \dots, z_d\}$ be d variables. Then*

$$\prod_{i,j,k}(1 - x_i z_k)^{-1}(1 + \eta_j z_k) = \sum_\lambda HS_\lambda(\mathbf{x};\eta)s_\lambda(\mathbf{z}),$$

where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant d$, and λ is summed over all partitions with length d subject to $\lambda_{m+1} \leqslant n$.

Replacing x_i, η_j, and z_k in Proposition 5.1 by y_i^{-1}, ζ_j^{-1}, and z_k^{-1}, we obtain the following result.

Proposition 5.2 [2]. *Let $\mathbf{y} = \{y_1, y_2, \dots, y_p\}$, $\zeta = \{\zeta_1, \zeta_2, \dots, \zeta_q\}$, and $\mathbf{z} = \{z_1, z_2, \dots, z_d\}$. Denote $\mathbf{y}^{-1} = \{y_1^{-1}, y_2^{-1}, \dots, y_p^{-1}\}$, $\zeta^{-1} = \{\zeta_1^{-1}, \zeta_2^{-1}, \dots, \zeta_q^{-1}\}$, and $\mathbf{z}^{-1} = \{z_1^{-1}, z_2^{-1}, \dots, z_d^{-1}\}$. Then*

$$\prod_{i,j,k}\bigl(1 - y_i^{-1}z_k^{-1}\bigr)^{-1}\bigl(1 + \zeta_j^{-1}z_k^{-1}\bigr) = \sum_\lambda HS_\lambda\bigl(\mathbf{y}^{-1};\zeta^{-1}\bigr)s_\lambda\bigl(\mathbf{z}^{-1}\bigr),$$

where $1 \leqslant k \leqslant d$, $1 \leqslant i \leqslant p$, and $1 \leqslant j \leqslant q$, and λ is summed over all partitions with length d subject to $\lambda_{p+1} \leqslant q$.

We recall the following lemma which plays a crucial role in developing a character formula for unitarizable irreducible $gl_{m+p|n+q}$-modules by using Howe duality. Denote by $\mathrm{ch}(V_d^\lambda)$ the formal character of the irreducible gl_d-module V_d^λ.

Lemma 5.1 [2]. *Let q be an indeterminate and suppose that $\sum_\lambda \phi_\lambda(q)\,\mathrm{ch}\,V_d^\lambda = 0$, where $\phi_\lambda(q)$ are power series in q and λ above is summed over all generalized partitions of length d. Then $\phi_\lambda(q) = 0$ for all λ.*

5.2. Character formula for finite-dimensional modules

Recall from Section 2.1 that $\tilde\varepsilon_1, \dots, \tilde\varepsilon_d$, are the weights of the natural gl_d-module $\mathbb{C}^d$, and $\varepsilon_1, \dots, \varepsilon_m$, $\delta_1, \dots, \delta_n$ are the weights of the natural $gl_{m|n}$-module $\mathbb{C}^{m|n}$. Let e be a formal indeterminate. For $k = 1, \dots, d$, $i = 1, \dots, m$, and $j = 1, \dots, n$, we set

$$\bar z_k = e^{\tilde\varepsilon_k}, \qquad \bar x_i = e^{\varepsilon_i}, \qquad \bar\eta_j = e^{\delta_j}, \tag{5.3}$$

and let $\bar{\mathbf{x}} = \{\bar x_1, \bar x_2, \dots, \bar x_m\}$, $\bar\eta = \{\bar\eta_1, \bar\eta_2, \dots, \bar\eta_n\}$, and $\bar{\mathbf{z}} = \{\bar z_1, \bar z_2, \dots, \bar z_d\}$.

Consider $\mathbb{C}[\mathbf{x},\eta]$ as a $gl_d \times gl_{m|n}$-module. Its formal character $\mathrm{ch}(\mathbb{C}[\mathbf{x},\eta])$ with respect to the Cartan subalgebra $\sum_{i=1}^{m+n}\mathbb{C}E_{ii} \oplus \sum_{k=1}^{d}\mathbb{C}e_{kk}$ can be easily computed by using Eqs. (2.4) and (2.5) to give

$$\mathrm{ch}\big(\mathbb{C}[\mathbf{x},\eta]\big) = \prod_{i,j,k}(1-\bar{x}_i\bar{z}_k)^{-1}(1+\bar{\eta}_j\bar{z}_k), \tag{5.4}$$

where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant d$. Thus, by Proposition 5.1,

$$\mathrm{ch}\big(\mathbb{C}[\mathbf{x},\eta]\big) = \sum_{\lambda} HS_\lambda(\bar{\mathbf{x}};\bar{\eta})s_\lambda(\bar{\mathbf{z}}), \tag{5.5}$$

where λ is summed over all partitions with length d subject to $\lambda_{m+1} \leqslant n$.

Let us denote by $\mathrm{ch}(V_{m|n}^{\tilde{\lambda}})$ the formal character of the irreducible $gl_{m|n}$-module. Theorem 2.1 leads to

$$\mathrm{ch}\big(\mathbb{C}[\mathbf{x},\eta]\big) = \sum_{\lambda}\mathrm{ch}\big(V_d^{\lambda}\big)\,\mathrm{ch}\big(V_{m|n}^{\tilde{\lambda}}\big),$$

where, we recall that, $\mathrm{ch}(V_d^{\lambda}) = s_\lambda(\bar{\mathbf{z}})$. By using Lemma 5.1, we obtain the following well-known result [1].

Theorem 5.1. *For each partition λ of length d subject to the condition $\lambda_{m+1} \leqslant n$,*

$$\mathrm{ch}\, V_{m|n}^{\tilde{\lambda}} = HS_\lambda(\bar{\mathbf{x}};\bar{\eta}),$$

where $\tilde{\lambda}$ is defined by (2.2).

Keep the notations of this subsection but replace m by p and n by q. Let $\bar{\mathbf{x}}^{-1} = \{\bar{x}_1^{-1}, \bar{x}_2^{-1}, \ldots, \bar{x}_p^{-1}\}$, $\bar{\eta}^{-1} = \{\bar{\eta}_1^{-1}, \bar{\eta}_2^{-1}, \ldots, \bar{\eta}_q^{-1}\}$, and $\bar{\mathbf{z}}^{-1} = \{\bar{z}_1^{-1}, \bar{z}_2^{-1}, \ldots, \bar{z}_d^{-1}\}$. Using (2.13) and (2.14), we can easily compute the formal character of the $gl_d \times gl_{p|q}$-module $\mathbb{C}[\mathbf{y},\zeta]$ with respect to the Cartan subalgebra $\sum_{i=1}^{p+q}\mathbb{C}E_{ii} \oplus \sum_{k=1}^{d}\mathbb{C}e_{kk}$. We have

$$\mathrm{ch}\big(\mathbb{C}[\mathbf{y},\zeta]\big) = (\bar{x}_1\cdots\bar{x}_p)^{-d}(\bar{\eta}_1\cdots\bar{\eta}_q)^{d}\prod_{i,j,k}\big(1-\bar{x}_i^{-1}\bar{z}_k^{-1}\big)^{-1}\big(1+\bar{\eta}_j^{-1}\bar{z}_k^{-1}\big), \tag{5.6}$$

where $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, and $1 \leqslant k \leqslant d$. By Proposition 5.2,

$$\mathrm{ch}\big(\mathbb{C}[\mathbf{y},\zeta]\big) = (\bar{x}_1\cdots\bar{x}_p)^{-d}(\bar{\eta}_1\cdots\bar{\eta}_q)^{d}\sum_{\lambda} HS_\lambda\big(\bar{\mathbf{x}}^{-1};\bar{\eta}^{-1}\big)s_\lambda\big(\bar{\mathbf{z}}^{-1}\big), \tag{5.7}$$

where λ is summed over all partitions with length d subject to $\lambda_{p+1} \leqslant q$.

Note that $\mathrm{ch}(V_d^{\lambda^*}) = s_\lambda(\bar{\mathbf{z}}^{-1})$. Thus the following theorem is a consequence of Theorem 2.2 obtained using Lemma 5.1 and Eq. (5.7).

Theorem 5.2. *For each partition* λ *of length* d *subject to the condition* $\lambda_{p+1} \leqslant q$,

$$\mathrm{ch}\, V_{p|q}^{-d\mathbf{1}+\widehat{\lambda^*}} = (\bar{x}_1 \cdots \bar{x}_p)^{-d} (\bar{\eta}_1 \cdots \bar{\eta}_q)^d HS_\lambda\big(\bar{\mathbf{x}}^{-1}; \bar{\eta}^{-1}\big),$$

where $\widehat{\lambda^*}$ *is as given in* (2.18).

5.3. Character formulas for unitarizable $gl_{m+p|n+q}$*-modules*

We keep the notations z_i, $1 \leqslant i \leqslant d$, and $\mathbf{z}$, $\mathbf{z}^{-1}$ from the last subsection. Let e be the formal indeterminate as before. For $1 \leqslant r \leqslant p$, $1 \leqslant s \leqslant q$, $1 \leqslant i \leqslant m$, and $1 \leqslant j \leqslant n$, we define

$$\bar{y}_r = e^{\hat{\varepsilon}_r}, \qquad \bar{\zeta}_s = e^{\hat{\varepsilon}_{s+p}}, \qquad \bar{x}_i = e^{\hat{\varepsilon}_{i+p+q}}, \qquad \bar{\eta}_j = e^{\hat{\varepsilon}_{j+m+p+q}},$$

where we recall that $\hat{\varepsilon}_A \in \mathfrak{h}^*_{m+p|n+q}$, $A \in \mathbf{I}$, are defined by $\hat{\varepsilon}_A(E^B_B) = \delta_{AB}$, $A, B \in \mathbf{I}$. Let $\bar{\mathbf{x}} = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\}$, $\bar{\eta} = \{\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_n\}$, $\bar{\mathbf{y}}^{-1} = \{\bar{y}_1^{-1}, \bar{y}_2^{-1}, \ldots, \bar{y}_p^{-1}\}$ and $\bar{\zeta}^{-1} = \{\bar{\zeta}_1^{-1}, \bar{\zeta}_2^{-1}, \ldots, \bar{\zeta}_q^{-1}\}$. We wish to compute the formal characters $\mathrm{ch}(W^{\Lambda(\lambda)}_{m+p|n+q})$ with respect to the Cartan subalgebra $\mathfrak{h}_{m+p|n+q} = \sum_{A=1}^{m+n+p+q} \mathbb{C}E^A_A$ for the unitarizable irreducible $gl_{m+p|n+q}$-modules $W^{\Lambda}_{m+p|n+q}$ appearing in the decomposition of $\mathbb{C}[\mathbf{x}, \mathbf{y}, \eta, \zeta]$.

Theorem 5.3. *For each generalized partition* λ *of length* d, *subject to the conditions* $\lambda_{m+1} \leqslant n$ *and* $\lambda_{d-p} \geqslant -q$,

$$\mathrm{ch}\big(W^{\Lambda(\lambda)}_{m+p|n+q}\big) = (\bar{y}_1 \bar{y}_2 \cdots \bar{y}_p)^{-d} (\bar{\zeta}_1 \bar{\zeta}_2 \cdots \bar{\zeta}_q)^d \sum_{\mu,\nu} C^\lambda_{\mu\nu^*} HS_\mu(\bar{\mathbf{x}}; \bar{\eta}) HS_\nu\big(\bar{\mathbf{y}}^{-1}; \bar{\zeta}^{-1}\big),$$

where μ *and* ν *are summed over all partitions of length* d *subject to the conditions* $\mu_{m+1} \leqslant n$ *and* $\nu_{p+1} \leqslant q$, *respectively. The* $C^\lambda_{\mu\nu^*}$ *are the Littlewood–Richardson coefficients.*

Proof. Consider the restriction of $W^{\Lambda(\lambda)}_{m+p|n+q}$ to a module of the subalgebra $gl_d \times gl_{m|n} \times gl_{p|q}$. Its formal character with respect to the Cartan subalgebra $\mathfrak{h}_d \times \mathfrak{h}_{p|q} \times \mathfrak{h}_{m|n}$ coincides with $\mathrm{ch}(W^{\Lambda(\lambda)}_{m+p|n+q})$. Therefore by Theorem 4.1, we have

$$\mathrm{ch}\big(W^{\Lambda(\lambda)}_{m+p|n+q}\big) = \sum_{\mu,\nu} C^\lambda_{\mu\nu^*}\, \mathrm{ch}\big(V^{\tilde{\mu}}_{m|n}\big)\, \mathrm{ch}\big(V_{p|q}^{-d\mathbf{1}+\widehat{\nu^*}}\big) \tag{5.8}$$

where μ and ν are summed over all partitions of length d subject to the conditions $\mu_{m+1} \leqslant n$ and $\nu_{p+1} \leqslant q$, respectively. Using Theorems 5.1 and 5.2 in this equation, we immediately arrive at the claimed result. □

6. Tensor product decomposition of unitarizable irreducible $gl_{m+p|n+q}$-modules

As another application of Theorem 3.3, we shall compute the tensor product decomposition

$$W^{\Lambda(\mu)}_{m+p|n+q} \otimes W^{\Lambda(\nu)}_{m+p|n+q} \cong \sum_{\lambda} a^{\lambda}_{\mu\nu} W^{\Lambda(\lambda)}_{m+p|n+q}, \tag{6.1}$$

where μ and ν are generalized partitions of length l and r, respectively, satisfying in addition the conditions $\mu_{m+1} \leqslant n$, $\nu_{m+1} \leqslant n$, $\mu_{l-p} \geqslant -q$, and $\nu_{r-p} \geqslant -q$. It will follow easily from our discussion that the summation λ in (6.1) is over all generalized partitions of length $l+r$ and satisfies $\lambda_{m+1} \leqslant n$ and $\lambda_{l+r-p} \geqslant -q$. We will compute the coefficients $a^{\lambda}_{\mu\nu}$ in terms of the usual Littlewood–Richardson coefficients (see, e.g., [19]).

We have by Theorem 3.3 for $d = l, r$, and $l+r$ respectively:

$$S\big(\mathbb{C}^l \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{l*} \otimes \mathbb{C}^{p|q*}\big) \cong \sum_{\mu} V^{\mu}_l \otimes W^{\Lambda(\mu)}_{m+p|n+q},$$

$$S\big(\mathbb{C}^r \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{r*} \otimes \mathbb{C}^{p|q*}\big) \cong \sum_{\nu} V^{\nu}_r \otimes W^{\Lambda(\nu)}_{m+p|n+q}, \quad \text{and}$$

$$S\big(\mathbb{C}^{l+r} \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{l+r*} \otimes \mathbb{C}^{p|q*}\big) \cong \sum_{\lambda} V^{\lambda}_{l+r} \otimes W^{\Lambda(\lambda)}_{m+p|n+q},$$

where μ, ν, and λ are generalized partitions satisfying the corresponding conditions described above. The isomorphism $S(\mathbb{C}^l \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{l*} \otimes \mathbb{C}^{p|q*}) \otimes S(\mathbb{C}^r \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{r*} \otimes \mathbb{C}^{p|q*}) \cong S(\mathbb{C}^{l+r} \otimes \mathbb{C}^{m|n} \oplus \mathbb{C}^{l+r*} \otimes \mathbb{C}^{p|q*})$ gives rise to

$$\sum_{\mu,\nu} V^{\mu}_l \otimes V^{\nu}_r \otimes W^{\Lambda(\mu)}_{m+p|n+q} \otimes W^{\Lambda(\nu)}_{m+p|n+q} \cong \sum_{\lambda} V^{\lambda}_{l+r} \otimes W^{\Lambda(\lambda)}_{m+p|n+q}. \tag{6.2}$$

Now suppose that V^{λ}_{l+r}, when regarded as a $gl_l \times gl_r$-module via the obvious embedding of $gl_l \times gl_r$ into gl_{l+r}, decomposes as

$$V^{\lambda}_{l+r} \cong \sum_{\mu,\nu} b^{\mu\nu}_{\lambda} V^{\mu}_l \otimes V^{\nu}_r.$$

This together with (6.1) and (6.2) give

$$a^{\lambda}_{\mu\nu} = b^{\mu\nu}_{\lambda}. \tag{6.3}$$

The duality between the branching coefficients and tensor products of a general dual pair is well known [15]. We recall that in (6.3) μ, ν and λ are generalized partitions subject to the appropriate constraints.

Now Theorem 2.1 with $n = 0$ combined with an analogous argument as the one given above implies that

$$C^{\lambda}_{\mu\nu} = b^{\mu\nu}_{\lambda}, \tag{6.4}$$

where μ, ν, λ are partitions of appropriate lengths and the $C^{\lambda}_{\mu\nu}$'s are the usual Littlewood–Richardson coefficients.

Now, for generalized partitions μ, ν, and λ, subject to appropriate constraints, the decomposition $V^{\lambda}_{l+r} \cong \sum_{\mu,\nu} b^{\mu\nu}_{\lambda} V^{\nu}_{l} \otimes V^{\mu}_{r}$ implies that $V^{\lambda+d\mathbf{1}_{l+r}}_{l+r} \cong \sum_{\mu,\nu} b^{\mu\nu}_{\lambda} V^{\mu+d\mathbf{1}_l}_{l} \otimes V^{\nu+d\mathbf{1}_r}_{r}$, where $\mathbf{1}_k$ denotes the k-tuple $(1, 1, \ldots, 1)$ regarded as a partition. Hence $b^{\mu\nu}_{\lambda} = b^{\mu+d\mathbf{1}_l,\nu+d\mathbf{1}_r}_{\lambda+d\mathbf{1}_{l+r}}$. Now, if we choose a non-negative integer d so that $\lambda + d\mathbf{1}_{l+r}$ is a partition, then $b^{\mu+d\mathbf{1}_l,\nu+d\mathbf{1}_r}_{\lambda+d\mathbf{1}_{l+r}} = C^{\lambda+d\mathbf{1}_{l+r}}_{\mu+d\mathbf{1}_l,\nu+d\mathbf{1}_r}$ and hence by (6.3) and (6.4),

$$a^{\lambda}_{\mu\nu} = C^{\lambda+d\mathbf{1}_{l+r}}_{\mu+d\mathbf{1}_l,\nu+d\mathbf{1}_r}.$$

From our discussion above we arrive at the following theorem.

Theorem 6.1. *Let μ and ν be generalized partitions of length l and r, respectively, satisfying in addition the conditions $\mu_{m+1} \leqslant n$, $\nu_{m+1} \leqslant n$, $\mu_{l-p} \geqslant -q$, and $\nu_{r-p} \geqslant -q$. Let $W^{\Lambda(\mu)}_{m+p|n+q}$ and $W^{\Lambda(\nu)}_{m+p|n+q}$ be the corresponding unitarizable $gl_{m+p|n+q}$-modules. We have the following decomposition of $W^{\Lambda(\mu)}_{m+p|n+q} \otimes W^{\Lambda(\nu)}_{m+p|n+q}$ into irreducible $gl_{m+p|n+q}$-modules*:

$$W^{\Lambda(\mu)}_{m+p|n+q} \otimes W^{\Lambda(\nu)}_{m+p|n+q} \cong \sum_{(\lambda,d)} C^{\lambda}_{\mu+d\mathbf{1}_l,\nu+d\mathbf{1}_r} W^{\Lambda(\lambda-d\mathbf{1}_{l+r})}_{m+p|n+q},$$

where the summation above is over all pairs (λ, d) subject to the following four conditions:

(i) *λ is a partition of length $l + r$ and d a non-negative integer*;
(ii) *$(\lambda - d\mathbf{1}_{l+r})_{m+1} \leqslant n$ and $(\lambda - d\mathbf{1}_{l+r})_{l+r-p} \geqslant -q$*;
(iii) *$\mu + d\mathbf{1}_l$ and $\nu + d\mathbf{1}_r$ are partitions*;
(iv) *If $d > 0$, then λ is a partition with $\lambda_{l+r} = 0$.*

Here the coefficients $C^{\lambda}_{\mu+d\mathbf{1}_l,\nu+d\mathbf{1}_r}$ are determined by the tensor product decomposition of gl_k-modules

$$V^{\mu+d\mathbf{1}_l}_{k} \otimes V^{\nu+d\mathbf{1}_r}_{k} \cong \sum_{\lambda} C^{\lambda}_{\mu+d\mathbf{1}_l,\nu+d\mathbf{1}_r} V^{\lambda}_{k}, \quad \textit{where } l + r = k.$$

Remark 6.1. In the above theorem the coefficients $C^{\lambda}_{\mu+d\mathbf{1}_l,\nu+d\mathbf{1}_r}$ are the usual Littlewood–Richardson coefficients associated to partitions, and hence can be computed via the Littlewood–Richardson rule.

Remark 6.2. We finally note the remarkable similarity between the irreducible representations of the so-called super $W_{1+\infty}$, which is the Lie superalgebra of differential operators on the circle with $N = 1$ extended symmetry, that appear in the decomposition of tensor powers of its natural representation on the infinite-dimensional Fock space [2], and the irreducible unitarizable representations of $gl_{m+p|n+q}$ of this paper. Indeed, the characters and the tensor product decomposition are virtually identical modulo some modification necessitated by the infinite-dimensional nature of the super $W_{1+\infty}$. This similarity can be explained by the existence of a Howe duality between the super $W_{1+\infty}$ and gl_d on the dth tensor power of the Fock space generated by infinitely many fermionic and bosonic quantum oscillators [5].

References

[1] A. Berele, A. Regev, Hook Young diagrams with applications to combinatorics and representations of Lie superalgebras, Adv. Math. 64 (1987) 118–175.
[2] S.-J. Cheng, N. Lam, Infinite-dimensional Lie superalgebras and Hook Schur functions, math.RT/0206034, Comm. Math. Phys. 238 (2003) 95–118.
[3] S.-J. Cheng, W. Wang, Howe duality for Lie superalgebras, Compositio Math. 128 (2001) 55–94.
[4] S.-J. Cheng, W. Wang, Remarks on the Schur–Howe–Sergeev duality, Lett. Math. Phys. 52 (2000) 143–153.
[5] S.-J. Cheng, W. Wang, Lie subalgebras of differential operators on the super circle, math.QA/0103092, Publ. Res. Inst. Math. Sci. 39 (2003) 545–600.
[6] S.-J. Cheng, R.B. Zhang, Howe duality and combinatorial character formula for orthosymplectic Lie superalgebras, math.RT/0206036; Adv. Math., in press.
[7] V.K. Dobrev, Positive energy unitary irreducible representations of $D = 6$ conformal supersymmetry, J. Phys. A 35 (2002) 7079–7100.
[8] M. Davidson, E. Enright, R. Stanke, Differential operators and highest weight representations, Mem. Amer. Math. Soc. 94 (455) (1999).
[9] R. D'Auria, S. Ferrara, M.A. Lled, V.S. Varadarajan, Spinor algebras, J. Geom. Phys. 40 (2001) 101–129.
[10] T. Enright, Analogues of Kostant's $\mathfrak{u}$-cohomology formulas for unitary highest weight modules, J. Reine Angew. Math. 392 (1988) 27–36.
[11] T. Enright, R. Howe, N. Wallach, A classification of unitary highest weight modules, in: Representation Theory of Reductive Groups, Park City, Utah, 1982, in: Progr. Math., vol. 40, Birkhäuser, Boston, MA, 1983, pp. 97–143.
[12] R. Goodman, N. Wallach, Representations and Invariants of the Classical Groups, Cambridge University Press, Cambridge, 1998.
[13] M.D. Gould, R.B. Zhang, Classification of all star irreps of $gl(m|n)$, J. Math. Phys. 31 (1990) 2552–2559.
[14] R. Howe, Remarks on classical invariant theory, Trans. Amer. Math. Soc. 313 (1989) 539–570.
[15] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, in: The Schur Lectures, Tel Aviv, in: Israel Math. Conf. Proc., vol. 8, 1992, pp. 1–182.
[16] H.P. Jakobsen, The full set of unitarizable highest weight modules of basic classical Lie superalgebras, Mem. Amer. Math. Soc. 111 (532) (1994), pp. vi+116.
[17] V. Kac, Lie superalgebras, Adv. Math. 16 (1977) 8–96.
[18] M. Kashiwara, M. Vergne, On the Segal–Shale–Weil representations and harmonic polynomials, Invent. Math. 44 (1978) 1–47.
[19] I.G. Macdonald, Symmetric Functions and Hall Polynomials, in: Oxford Math. Monogr., Clarendon, Oxford, 1995.
[20] M. Nazarov, Quantum Berezinian and the classical Capelli identity, Lett. Math. Phys. 21 (2) (1991) 123–131.
[21] G. Ol'shanskii, M. Prati, Extremal weights of finite-dimensional representations of the Lie superalgebra $\mathfrak{gl}_{n|m}$, Nuovo Cimento 85 A (1985) 1–18.

[22] A. Sergeev, An analog of the classical invariant theory for Lie superalgebras, I, II, Michigan Math. J. 49 (2001) 147–168.
[23] P.K. Townsend, M-theory from its superalgebra, in: Strings, Branes and Dualities, Cargse, 1997, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 520, Kluwer Academic, Dordrecht, 1999, pp. 141–177.