The Twelfth East Asia-Pacific Conference on Structural Engineering and Construction

An Unconditionally Stable Explicit Method for Structural Dynamics

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Abstract

A new unconditionally stable explicit time-integration method is proposed herein, which can inherit the numerical characteristics of any existing implicit Runge-Kutta algorithms. Thus, the proposed method not only holds computational efficiency but also possesses good numerical characteristics. Moreover, the paper presents an exact derivation of the increment of mechanical energy of an undamped system to investigate the stability condition and algorithmic damping of the proposed algorithm for linear problems, which is different from the analysis of spectral radii of integration algorithms in previous studies. The numerical characteristics of the proposed method are computationally examined from the viewpoints of energy conservation of conservative systems.

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Keywords: Unconditionally stable explicit method, time-integration method.

1. Introduction

Numerical integration of ordinary differential equations is the most important technique in continuous time dynamics. For solving structural dynamic problems, many methods have been proposed such as the Newmark-β method (Newmark 1959), Wilson-θ method (Wilson et al. 1973), HHT-α method (Hilber et al. 1977), WBZ-α method (Wood et al. 1981), HP-01 method (Hoff and Pahl 1988a, b), CH-α method (Chung and Hulbert 1993). These methods are implicit, second-order accurate and A-stable (i.e. unconditionally stable) in linear dynamics, and capable of controlling numerical dissipation in high-frequency modes when adequate parameters are used.

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In contrast to implicit methods, explicit methods do not involve any iterative procedure and thus need far less computational effort in each time step. However, explicit methods can only have conditional stability, which means that a very small time step is required to obtain a stable integration result. To improve this shortcoming, some explicit methods with unconditional stability have been proposed (Chang 2002, 2007).

It will be very favorable if an explicit method holds computational efficiency and also possesses the same numerical characteristics as the fourth order L-stable implicit Runge–Kutta methods (Hairer et al. 1993, 1996) for linear dynamic problems. In this study, the parameters used in the proposed explicit method are determined by equating its map (difference equation) derived from a differential equation to the corresponding one to the existing implicit Runge–Kutta algorithm such that the proposed explicit method can inherit the numerical characteristics of the emulated algorithm.

Moreover, the paper presents an exact derivation of the increment of mechanical energy of an undamped system from the nth to the (n+1)th time step for the Runge–Kutta family to investigate their stability condition and algorithmic damping for linear problems, which is different from the analysis of spectral radii of integration algorithms in previous studies. In addition, the numerical characteristics of the proposed method are computationally examined based on energy conservation of conservative systems.

2. Proposed Explicit Method

After the mathematical discretization of a linear structural system, its equation of motion can be written as

$$\mathbf{M}\ddot{x} + \mathbf{C}\dot{x} + \mathbf{K}x = f,$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, viscous damping, and stiffness matrices, $\mathbf{x}$, $\dot{\mathbf{x}}$, and $\ddot{\mathbf{x}}$ are the displacement, velocity, and acceleration vectors, and $\mathbf{f}$ is the external force vector. The general formulations of the proposed explicit method can be expressed as

$$x_{i+1} = x_i + \Delta t \beta_1 \dot{x}_i + \Delta t^2 \beta_2 \ddot{x}_i,$$

$$\dot{x}_{i+1} = \dot{x}_i + \Delta t \gamma_1 \ddot{x}_i + \Delta t \gamma_2 \dddot{x}_{i+1},$$

where the subscript $i$ denotes the $i$th time step, $\Delta t$ is the time increment, and $\beta_1$, $\beta_2$, $\gamma_1$, and $\gamma_2$ are the coefficient matrices determined as the following procedure.

First, for convenience, we assume that the system is undamped and free of external force. Eq. (1) can be reduced to a system of first-order ordinary differential equations by introducing new variables which are usually made to be derivatives of the original variables as follows

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t) = \mathbf{A}\mathbf{y},$$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\mathbf{M}^{-1}\mathbf{K} & 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix}$.

The general description of an $s$-stage Runge–Kutta method for solving Eq. (3) can be given by the formulas
\[ y_{i+1} = y_i + \Delta t \sum_{m=1}^{s} b_m k_m, \]
\[ k_m = F(t_i + c_m \Delta t, y_i + \Delta t \sum_{l=1}^{s} a_{ml} k_l) = A(y_i + \Delta t \sum_{l=1}^{s} a_m k_l) \]

where the coefficients \( b_m, c_m, \) and \( a_{ml} \) in the Butcher tableau determine the particular method and its accuracy and stability properties.

Then,
\[ y_{i+1} = y_i + \Delta t b_1 k_1 + \Delta t b_2 k_2 + \cdots + \Delta t b_s k_s \]
\[ = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} y_i = R y_i \quad (4) \]

Eq. (4) is the so-called map (difference equation) of the Runge–Kutta method in which \( R \) is known as the amplification matrix.

In addition, in order to construct the map of the proposed scheme, Eq. (2) is rearranged by substituting \( x_{i+1} = -M^{-1} K x_{i+1} \) and \( \ddot{x}_{i+1} = -M^{-1} K x_{i+1} \) into Eq. (2) as
\[ \begin{bmatrix} x_{i+1} \\ \ddot{x}_{i+1} \end{bmatrix} = \begin{bmatrix} I - \Delta t^2 \beta_2 M^{-1} K & \Delta t \beta_1 \\ -\Delta t \gamma_1 M^{-1} K - \Delta t \gamma_2 M^{-1} K (I - \Delta t^2 \beta_2 M^{-1} K) & I - \Delta t \gamma_2 M^{-1} K \beta_1 \end{bmatrix} \begin{bmatrix} x_i \\ \dot{x}_i \end{bmatrix} \quad (5) \]

To make the proposed explicit method possess the same numerical characteristics as the emulated Runge–Kutta methods, we equate the amplification matrix of the proposed method to the one of the Runge–Kutta algorithm. Then, the coefficient matrices \( \beta_1, \beta_2, \gamma_1 \) and \( \gamma_2 \) in Eq. (2) can be obtained as
\[ \begin{align*}
\beta_1 &= \frac{1}{\Delta t} R_2 \\
\beta_2 &= \frac{1}{\Delta t^2} (I - R_1) K^{-1} M \\
\gamma_1 &= \frac{-1}{\Delta t} (R_3 + (I - R_4) R_2^{-1} R_1) K^{-1} M \\
\gamma_2 &= \frac{1}{\Delta t} (I - R_4) R_2^{-1} K^{-1} M
\end{align*} \quad (6) \]

Note that, unlike a general time integration method, it is interesting to find that the coefficient matrices \( \beta_1, \beta_2, \gamma_1 \) and \( \gamma_2 \) depend on the structural properties and the size of the integration time step, rather than some constants. It should be mentioned that, once \( \beta_1, \beta_2, \gamma_1 \) and \( \gamma_2 \) are determined before time marching, they are unchanged in a complete step-by-step integration procedure for solving linear dynamics.

In this study, three Implicit Runge–Kutta methods are emulated, including the implicit midpoint method (Newmark average acceleration method), the modified extended backward differentiation formulae (ME-BDF) method, and the fourth order L-stable singly-diagonally-implicit Runge Kutta (SDIRK) method. To clearly demonstrate the implementation of the proposed method, a single degree-of-freedom system is considered here. The matrix \( A \) in Eq. (3) can be expressed as
Thus, the amplification matrix \( R \) is reduced to be a \( 2 \times 2 \) matrix in which \( R_1, R_2, R_3, \) and \( R_4 \) are reduced as scalars. The coefficient matrices \( \beta_1, \beta_2, \gamma_1, \) and \( \gamma_2 \) used in the proposed explicit method are also reduced as scalars and obtained using Eq. (6) as in Table 1. Note that the above procedure can be applied to any kind of Runge–Kutta algorithms and produces the equivalent explicit method.

### Table 1: The coefficient matrices \( \beta_1, \beta_2, \gamma_1, \) and \( \gamma_2 \) used in the proposed explicit method

<table>
<thead>
<tr>
<th>Method</th>
<th>The implicit midpoint method</th>
<th>The ME-BDF method</th>
<th>The 4th order L-stable SDIRK method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1 )</td>
<td>( \frac{4}{4 + \omega^2 \Delta t^2} )</td>
<td>( \frac{2 + 7 \omega^2 \Delta t^2 + \omega^4 \Delta t^4}{2(1 + \omega^2 \Delta t^2)^3} )</td>
<td>( \frac{4(786432 + 114688 \omega^2 \Delta t^2 - 4352 \omega^4 \Delta t^4)}{3(16 + \omega^2 \Delta t^2)^3} )</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>( \frac{2}{4 + \omega^2 \Delta t^2} )</td>
<td>( \frac{1 + 7 \omega^2 \Delta t^2 + 2 \omega^4 \Delta t^4}{2(1 + \omega^2 \Delta t^2)^3} )</td>
<td>( \frac{1572864 + 360448 \omega^2 \Delta t^2 + 22272 \omega^4 \Delta t^4}{3(16 + \omega^2 \Delta t^2)^3} )</td>
</tr>
<tr>
<td>( \gamma_1 )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{2 + 3 \omega^2 \Delta t^2}{2(2 + 7 \omega^2 \Delta t^2 + \omega^4 \Delta t^4)} )</td>
<td>( \frac{8 \times (147456 + 33792 \omega^2 \Delta t^2)}{3(786432 + 114688 \omega^2 \Delta t^2 - 4352 \omega^4 \Delta t^4)} )</td>
</tr>
<tr>
<td>( \gamma_2 )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1 + 7 \omega^2 \Delta t^2 + 2 \omega^4 \Delta t^4}{2 + 7 \omega^2 \Delta t^2 + \omega^4 \Delta t^2} )</td>
<td>( \frac{(1572864 + 360448 \omega^2 \Delta t^2 + 22272 \omega^4 \Delta t^4)}{4(786432 + 114688 \omega^2 \Delta t^2 - 4352 \omega^4 \Delta t^4)} )</td>
</tr>
</tbody>
</table>

3. **Numerical stability from viewpoints of Energy conservation**

For linear or nonlinear conservative systems, energy conservation should be satisfied. Theoretically, the dynamics of the map resulting from different step-by-step integration schemes for solving such systems should closely obey this law. In other words, the law can be employed to investigate stability condition and algorithmic damping of these schemes. To achieve this aim, first, the equation of motion of a single degree-of-freedom undamped system is considered as

\[
m\ddot{x} + kx = 0, \quad (7)
\]

and the energy conservation relation can be obtained by multiplying Eq. (7) with the velocity \( \dot{x} \) as

\[
\frac{d}{dt} \left( \frac{1}{2} mx^2 + \frac{1}{2} kx^2 \right) = 0.
\]
The increment of mechanical energy of an conservative system from the \(i\)th to the \((i+1)\)th time step can be expressed as

\[
\frac{1}{2} \left( m \dot{x}_{i+1}^2 + \frac{1}{2} kx_{i+1}^2 \right) - \frac{1}{2} m \dot{x}_i^2 - \frac{1}{2} kx_i^2 = \frac{1}{2} m (\dot{x}_{i+1} - \dot{x}_i)(\dot{x}_{i+1} + \dot{x}_i) + \frac{1}{2} k (x_{i+1} - x_i)(x_{i+1} - x_i)
\]

(8)

Recall the map of Runge–Kutta methods for a single degree-of-freedom system as

\[
\begin{bmatrix}
  x_{i+1} \\
  \dot{x}_{i+1}
\end{bmatrix} =
\begin{bmatrix}
  R_1 & R_2 \\
  R_3 & R_4
\end{bmatrix}
\begin{bmatrix}
  x_i \\
  \dot{x}_i
\end{bmatrix}
= \begin{bmatrix}
  R_1 & R_2 \\
  -\omega^2 R_2 & R_1
\end{bmatrix}
\begin{bmatrix}
  x_i \\
  \dot{x}_i
\end{bmatrix}.
\]

(9)

By substituting Eq. (9) into Eq.(8), Eq.(8) is written as

\[
\frac{1}{2} m \left[ \omega^4 R_2^2 x_i^2 - 2 R_1 R_2 \omega^2 x_i \dot{x}_i + (R_1^2 - 1) \dot{x}_i^2 \right] + \frac{1}{2} k [(R_1^2 - 1)x_i^2 + 2 R_1 R_2 x_i \dot{x}_i + R_2^2 x_i^2]
\]

\[
= (R_1^2 + \omega^2 R_2^2 - 1)(\frac{1}{2} kx_i^2 + \frac{1}{2} \dot{x}_i^2)
\]

(10)

Finally, we can obtain the incremental rate of mechanical energy of the system as

\[
\frac{(E_{i+1} - E_i)}{E_i} = (R_1^2 + \omega^2 R_2^2 - 1),
\]

(11)

Theoretically, the incremental rate should be zero because energy conservation should be satisfied. However, due to the temporal discretization, this rate may be positive, equal to zero, or negative, depending on \(R_1\) and \(R_2\) for different Runge–Kutta algorithms.

Figure 1 shows the incremental rate of mechanical energy versus \(\omega \Delta t\) in which \(\omega\) is the natural frequency of the system and \(\Delta t\) is the integration time step. Based on the observations on these two plots, there are some discussions. For the 4th order explicit Runge–Kutta method, the condition of stability of the method can be obtained by equating \((R_1^2 + \omega^2 R_2^2 - 1)\) to zero. The critical value of \(\omega \Delta t\) is calculated as \(2\sqrt{2}\) in the case where \(R_1 = 1 - \omega^2 \Delta t^2 / 2 + \omega^4 \Delta t^4 / 24\) and \(R_2 = \Delta t - \omega^2 \Delta t^3 / 6\). When \(\omega \Delta t\) is more than \(2\sqrt{2}\), \(\Delta E_i / E_i\) is more than zero. Thus, the energy grows with time, and the method becomes unstable. In contrast, for the implicit midpoint method (Newmark average acceleration method), \(\Delta E_i / E_i\) is equal to zero and the energy can be conserved for all \(\omega \Delta t\). It is a well-known unconditional stable (A-stable) algorithm with no algorithmic damping. The ME-BDF method and the 4th order SDIRK method have two features in common. \(\Delta E_i / E_i\) is less than zero for all \(\omega \Delta t\) more than zero and is equal to -1 when \(\omega \Delta t\) goes to infinity. The former means that the two methods are not only unconditional stable but also provide algorithmic damping, and the latter implies that the energy \(E_{i+1}\) is completely damped out (i.e. the two methods are so-called L-stable). In particular, the properties of algorithmic damping of the two methods are compared. The ME-BDF method is seen to damp the low modes too strongly since \(\Delta E_i / E_i\) drops drastically when \(\omega \Delta t\) is more than 0.2. As a result, the ME-BDF method may not only dissipate spurious high-frequency response due to the spatial discretization but also suppress real and dominant low-frequency response. In contrast, the 4th order SDIRK method performs...
better as a “numerical low-pass filter”, suppressing high-frequency response and leaving low-frequency response intact. Therefore, from this viewpoint, it is valuable to produce the equivalent explicit method that possesses the same numerical characteristics as the 4th order SDIRK method. Note that the stability criteria of time integration scheme \((i.e. -1 \leq \Delta E_t / E_t \leq 0)\) derived in this study is equivalent to \(0 \leq \rho(\omega \Delta t) \leq 1\) in which \(\rho\) is the spectral radius based on eigenvalue analysis of the corresponding amplification matrix and is equal to \(\sqrt{(R_1^2 + \omega^2 R_2^2)}\).

![Figure 1: The incremental rate of mechanical energy of a linear SDOF system for different time integration algorithms including the 4th order explicit Runge–Kutta method, the implicit midpoint method, the ME-BDF method, and the 4th order SDIRK method.](image1)

**4. Application to nonlinear systems**

For linear dynamics, a rigorous stability analysis of time-stepping algorithms based on energy conservation is feasible. However, for nonlinear dynamics, it is difficult to analytically obtain the expression of \(\Delta E_t / E_t\) of these algorithms. Consequently, the numerical stability of these algorithms in dynamics of an undamped nonlinear system needs to be computationally investigated by observing the time variation of the mechanical energy of the system. Here, we consider a governing equation of an undamped nonlinear system as \(m\ddot{x} + kx + ax^3 = 0\), and solve for it by using the Newmark explicit method and the proposed explicit methods which respectively inherit the Newmark average acceleration method, the ME-BDF method, and the 4th order SDIRK method. For convenience, these three proposed explicit methods are named the NAA explicit method, the ME-BDF explicit method, and the 4th order SDIRK explicit method.

In this example, \(m, k,\) and \(a\) are chosen as 1 kg, 1 N/m, and 0.1N/m\(^3\), the integration time step is used as 1.0 sec, and the initial conditions are considered as \(x(0) = 0.5\) and \(\dot{x}(0) = 0\). Ideally, the mechanical energy \((m\ddot{x}^2 / 2 + kx^2 / 2 + ax^4 / 4)\) should be kept a constant as 0.1266 for this nonlinear conservative system at any time. However, due to the temporal discretization and numerical characteristics of different algorithms, the mechanical energy may oscillate or decay with time. Figure 2 shows the time series of mechanical energy of the system solved by different time integration algorithms including the Newmark explicit method, the NAA explicit method, the ME-BDF explicit method, and the 4th order SDIRK explicit method. One may find that the NAA explicit method and the 4th order SDIRK explicit method perform better than the other methods since the corresponding calculated mechanical energy oscillates around 0.1266 with very small amplitude. It is not reasonable that the mechanical energy computed by the ME-BDF explicit method decays very fast due to its too strong algorithmic damping.
Figure 2: The time series of mechanical energy of an nonlinear system with cubic stiffness (a=0.1) for different time integration algorithms including the Newmark explicit method, the NAA explicit method, the ME-BDF explicit method, and the 4th order SDIRK explicit method.

5. Conclusions

In this paper, we propose a new unconditionally stable explicit time-integration method which can inherit the numerical characteristics of any existing implicit Runge-Kutta algorithms. In addition, the paper presents an exact derivation of the increment of mechanical energy of an undamped system to investigate the stability condition and algorithmic damping of the proposed algorithm for linear problems, which is different from the analysis of spectral radii of integration algorithms in previous studies. Finally, the numerical characteristics of the proposed method for nonlinear systems are computationally examined from the viewpoints of energy conservation of conservative systems.

Acknowledgments

The author wishes to acknowledge the National Science Council, Taiwan for the generous support of this work.

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