Bounds on the base of primitive nearly reducible sign pattern matrices

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Abstract

In [J.Y. Shao, L.H. You, Bound on the base of irreducible generalized sign pattern matrices, Discrete Math., in press], Shao and You extended the concept of the base from powerful sign pattern matrices to non-powerful (and generalized) sign pattern matrices. In this paper, we study the base for primitive non-powerful nearly reducible sign pattern (and generalized sign pattern) matrices. We obtain sharp upper bounds, together with complete characterization of the equality cases of the base for primitive nearly reducible sign pattern (and generalized sign pattern) matrices. We also show that there exist “gaps” in the base set of the classes of such matrices.

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1. Introduction

The sign of a real number $a$, denoted by sgn $a$, is defined to be 1, $-1$ or 0, according to $a > 0$, $a < 0$ or $a = 0$. The sign pattern of a real matrix $A$, denoted by sgn $A$, is the $(0, 1, -1)$-matrix obtained from $A$ by replacing each entry by its sign.
The powers (especially the sign patterns of the powers) of a square sign pattern matrix $A$ have recently been studied to some extent (see [5,9–11]). Notice that in the computations of (the signs of) the entries of the power $A^k$, an “ambiguous sign” may arise when we add a positive sign to a negative sign. So a new symbol “#” has been introduced to denote the ambiguous sign in [5]. For convenience, we call the set $\Gamma = \{0, 1, -1, #\}$ generalized sign set and define addition and multiplication involving the symbol # as follows (addition and multiplication which do not involve # are obvious):

$$(-1) + 1 = 1 + (-1) = #; \quad a + # = # + a = # \quad (\text{for all } a \in \Gamma)$$

$$0 \cdot # = # \cdot 0 = 0; \quad b \cdot # = # \cdot b = # \quad (\text{for all } b \in \Gamma \setminus \{0\})$$

It is straightforward to check that addition and multiplication in $\Gamma$ defined in this way are commutative and associative, and that multiplication is distributive with respect to addition.

In [5] and [10], matrices with entries in the set $\Gamma$ are called generalized sign pattern matrices. Addition and multiplication of generalized sign pattern matrices are defined in the usual way, so that the sum and product (including powers) of generalized sign pattern matrices are still generalized sign pattern matrices.

From now on we assume that all matrix operations considered in this paper are operations on matrices over the set $\Gamma$.

We now introduce some graph theoretical concepts (see [3,6]).

**Definition 1.1.** A signed digraph $S$ is a digraph where each arc of $S$ is assigned a sign 1 or $-1$. A generalized signed digraph $S$ is a digraph where each arc of $S$ is assigned a sign 1, $-1$ or #.

**Definition 1.2.** A walk $W$ in a digraph is a sequence of arcs: $e_1, e_2, \ldots, e_k$ such that the terminal vertex of $e_i$ is the same as the initial vertex of $e_{i+1}$ for $i = 1, \ldots, k - 1$. The number $k$ of edges is called the length of the walk $W$, denoted by $l(W)$. The sign of the walk $W$ (in a signed digraph), denoted by $\text{sgn} W$, is defined to be $\prod_{i=1}^{k} \text{sgn}(e_i)$.

**Definition 1.3 ([9]).** Two walks $W_1$ and $W_2$ in a signed digraph is called a pair of SSSD walks, if they have the same initial vertex, same terminal vertex and same length, but they have different signs.

Let $A = (a_{ij})$ be a square sign pattern matrix of order $n$. The associated digraph $D(A)$ of $A$ (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j) | a_{ij} \neq 0\}$. The associated signed digraph $S(A)$ of $A$ is obtained from $D(A)$ by assigning the sign of $a_{ij}$ to each arc $(i, j)$ in $D(A)$.

**Definition 1.4 ([5]).** A square generalized sign pattern matrix $A$ is called powerful if each power of $A$ contains no # entry.

It is easy to see from the above relation between matrices and signed digraphs that a sign pattern matrix $A$ is powerful if and only if the associated signed digraph $S(A)$ contains no pairs of SSSD walks.

In [9], Shao and You extended the concepts of the base and period from (powerful) sign pattern matrices (see [5]) to (square) generalized sign pattern matrices as follows.
Definition 1.5 ([9]). Let $A$ be a square generalized sign pattern matrix of order $n$ and $A, A^2, A^3, \ldots$ be the sequence of powers of $A$. (Since there are only $4^n^2$ different generalized sign patterns of order $n$, there must be repetitions in the sequence.) Suppose $A^l$ is the first power that is repeated in the sequence. Namely, suppose $l$ is the least positive integer such that there is a positive integer $p$ such that

$$A^l = A^{l+p},$$

(1.1)
then $l$ is called the generalized base (or simply base) of $A$, and is denoted by $l(A)$. The least positive integer $p$ such that (1.1) holds for $l = l(A)$ is called the generalized period (or simply period) of $A$, and is denoted by $p(A)$.

For convenience, we will also define the corresponding concepts for signed digraphs.

Definition 1.6. Let $S$ be a signed digraph of order $n$. Then there is a sign pattern matrix $A$ of order $n$ whose signed associated digraph $S(A)$ is $S$. We say that $S$ is powerful if $A$ is powerful (i.e., $S$ contains no pairs of $SSSD$ walks). Also we define $l(S) = l(A)$ and $p(S) = p(A)$.

As we know, a square matrix $A$ of order $n$ is reducible if there exists a permutation matrix $P$ of order $n$ such that

$$PAP^T = \begin{pmatrix} B & 0 \\ D & C \end{pmatrix},$$

where $B$ and $C$ are square non-vacuous matrices. The matrix $A$ is irreducible if it is not reducible, and is nearly reducible (or simply NR) if it is irreducible and each matrix obtained from $A$ by replacing a nonzero entry by 0 is reducible.

For a generalized sign pattern matrix $A$, we use $|A|$ to denote the $(0, 1)$-matrix obtained from $A$ by replacing each nonzero entry by 1. Clearly $|A|$ completely determines the zero pattern of $A$. Notice that for the operations defined for the generalized sign set $\Gamma = \{0, 1, -1, \#\}$, we have $a + b = 0$ if and only if both $a$ and $b$ are zero (and $a \cdot b = 0$ if and only if one of $a$ and $b$ is zero). So we have $|AB| = ||A||B||$ for generalized sign pattern matrices $A$ and $B$. In particular, we have $|A^k| = ||A||^k$.

Definition 1.7. A nonnegative square matrix $A$ is primitive if some power $A^k > 0$ ($A^k$ is entrywise positive). The least such $k$ is called the primitive exponent of $A$, denoted by $\exp(A)$. A square generalized sign pattern matrix $A$ is called primitive if $|A|$ is primitive, and in this case we define $\exp(A) = \exp(|A|)$.

A square generalized sign pattern matrix $A$ is called NR if $|A|$ is NR.

Definition 1.8. A digraph $D$ is called a primitive digraph, if there is a positive integer $k$ such that for each vertex $x$ and vertex $y$ (not necessarily distinct) in $D$, there exists a walk of length $k$ from $x$ to $y$. The least such $k$ is called the primitive exponent of $D$, denoted by $\exp(D)$.

As we know, a digraph $D$ is primitive if and only if $D$ is strongly connected (or simply strong) and the greatest common divisor (or simply g.c.d.) of the lengths of all the cycles of $D$ is 1 (see [2,6]).

It is well known from the basic relations between matrices and digraphs that a square matrix $A$ is irreducible if and only if $D(A)$ is strong, $A$ is NR if and only if $D(A)$ is a minimally strong
digraph (or simply a NR digraph), and A is primitive if and only if $D(A)$ is primitive, and in this case we have $\exp(A) = \exp(D(A))$.

In this paper, we study the (generalized) base of the primitive nearly reducible (or simply primitive NR) sign pattern (and generalized sign pattern) matrices. In Section 3 we consider the powerful cases and some non-powerful cases, then in Section 4 we consider the non-powerful cases (of sign pattern matrices) and general cases (including sign pattern and generalized sign pattern matrices). We obtain sharp upper bounds, together with complete characterization of the equality cases of the base for primitive NR sign pattern (and generalized sign pattern) matrices. We also show that there exist “gaps” in the base set of the classes of such matrices.

2. Some preliminaries

In this section, we introduce some definitions, notation and properties which we need to use in the presentations and proofs of our main results in next sections.

In [9], Shao and You obtained an important characterization for non-powerful irreducible sign pattern matrices from the characterization of powerful irreducible sign pattern matrices (see [5]). The following Theorem 2.A is the graph theoretical version of this characterization.

Theorem 2.A ([9]). If $S$ is a primitive signed digraph, then $S$ is non-powerful if and only if $S$ contains a pair of cycles $C_1$ and $C_2$ (say, with lengths $p_1$ and $p_2$, respectively) satisfying one of the following two conditions:

(A1) $p_1$ is odd and $p_2$ is even and $\text{sgn } C_2 = -1$;
(A2) Both $p_1$ and $p_2$ are odd and $\text{sgn } C_1 = -\text{sgn } C_2$.

A pair of cycles $C_1$ and $C_2$ satisfying (A1) or (A2) is a “distinguished cycle pair”. It is easy to see that if $C_1$ and $C_2$ is a distinguished cycle pair with lengths $p_1$ and $p_2$, respectively, then the closed walks $W_1 = p_2C_1$ (walk around $C_1$ $p_2$ times) and $W_2 = p_1C_2$ have the same length $p_1 p_2$ and the different signs:

$$(\text{sgn } C_1)^{p_2} = -(\text{sgn } C_2)^{p_1}$$

We will need the following well-known upper bound on the exponent of a primitive NR digraph (see [7]):

Theorem 2.B ([7]). Let $D$ be a primitive NR digraph of order $n$, and let $s$ be the length of the shortest cycle of $D$. Then

$$\exp(D) \leq n + s(n - 3),$$

where equality holds if and only if $D$ is isomorphic to the digraph $D_{n-1,s}$ (see Fig. 1). In particular, if $\text{g.c.d.}(s, n - 1) \neq 1$, then $\exp(D) < n + s(n - 3)$; and if $\text{g.c.d.}(s, n - 1) = 1$, then $D_{n-1,s}$ is a primitive NR digraph of order $n$ with exponent $n + s(n - 3)$.

Also, the Frobenius numbers can be used to estimated the exponent of a primitive NR digraph. Let $a_1, \ldots, a_k$ be positive integers. Define the Frobenius set $S(a_1, \ldots, a_k)$ as:

$$S(a_1, \ldots, a_k) = \{r_1a_1 + \ldots + r_ka_k | r_1, \ldots, r_k \text{ are nonnegative integers}\}$$
It is well known that if \( \gcd(a_1, \ldots, a_k) = 1 \), then \( S(a_1, \ldots, a_k) \) contains all the sufficiently large positive integers. In this case we define the Frobenius number \( \phi(a_1, \ldots, a_k) \) to be the least integer \( \phi \) such that \( m \in S(a_1, \ldots, a_k) \) for all integers \( m \geq \phi \).

Clearly, \( \phi(a_1, \ldots, a_k) - 1 \) is not in \( S(a_1, \ldots, a_k) \). It is also well known that if \( \gcd(a, b) = 1 \), then \( \phi(a, b) = (a - 1)(b - 1) \).

Let \( k \geq 3, a_1, a_2, \ldots, a_k \) are integers with \( a_1 > a_2 > \ldots > a_k > 0 \) and \( \gcd(a_1, a_2, \ldots, a_k) = 1 \), then (see [4]):

\[
\phi(a_1, a_2, \ldots, a_k) \leq \left\lfloor \frac{(a_1 - 2)(a_2 - 1)}{2} \right\rfloor. \tag{2.3}
\]

Let \( v \) be a vertex of a primitive digraph \( D \). The vertex exponent of \( v \), denoted by \( \exp_D(v) \), is defined to be the least positive integer \( k \) such that for each vertex \( u \) in \( D \), there is a walk of length \( k \) from \( v \) to \( u \).

Let \( R = \{l_1, \ldots, l_r\} \) be a set of cycle lengths in a primitive digraph \( D \) such that \( \gcd(l_1, \ldots, l_r) = 1 \). For each vertex \( x \) and vertex \( y \) in \( D \), let \( d(x, y) \) be the distance from \( x \) to \( y \) and let \( d_R(x, y) \) be the length of the shortest walk from \( x \) to \( y \) which meets at least one vertex of cycles of length \( l_i \) for each \( i = 1, \ldots, r \). Let \( \phi_R = \phi(l_1, \ldots, l_r) \) be the Frobenius number. We have the following known upper bounds (see [8]):

\[
\exp(D) \leq \phi_R + \max_{x, y \in V(D)} d_R(x, y); \tag{2.4}
\]

\[
\exp_D(v) \leq \phi_R + \max_{u \in V(D)} d_R(v, u). \tag{2.5}
\]

From [9], we know that for a primitive non-powerful signed digraph \( S \), \( l(S) \) is the least positive integer such that there is a pair of \( \text{SSSD} \) walks of length \( l(S) \) between any two vertices in \( S \). And the following definitions and properties, which were established in [9], will be used in next sections.

**Definition 2.1** ([9]). Let \( S \) be a non-powerful signed digraph. Then the ambiguous index of \( S \), denoted by \( r(S) \), is defined to be the least integer \( r \) such that there is a pair of \( \text{SSSD} \) walks of length \( r \) in \( S \).
Theorem 2.C ([9]). Let $S$ be a primitive non-powerful signed digraph, $W_1$ and $W_2$ be a pair of SSSD walks of length $r_{u,v}$ from vertex $u$ to vertex $v$, $d(S)$ is the diameter of the digraph $S$. Then we have
\[
\begin{align*}
  l(S) &\leq d(S) + r_{u,v} + \exp_S(v); \\
  l(S) &\leq d(S) + r(S) + \exp(S).
\end{align*}
\]

3. The powerful cases and some non-powerful cases

We begin by studying NR signed digraphs. Throughout the remainder of the paper, let $D_{n-1,s}$ and $H_n$ be the primitive NR digraphs of order $n$ given in Fig. 1, respectively.

It was shown in [5, Theorem 4.3] that if an irreducible sign pattern matrix $A$ is powerful, then $l(A) = l(|A|)$. This means that the study of the base $l(A)$ for primitive powerful NR sign pattern matrices is essentially the study of the base (i.e., exponent) for primitive NR nonnegative matrices. Thus we have the following theorem from the results in [1] and [7].

Theorem 3.1. Let $S$ be a primitive powerful NR signed digraph of order $n \geq 6$. Then (1)
\[ l(S) \leq n^2 - 4n + 6 \]
with equality if and only if the underlying digraph is isomorphic to $D_{n-1,n-2}$.

(2) For each integer $k$ with $n^2 - 5n + 9 < k < n^2 - 4n + 6$ or $n^2 - 6n + 12 < k < n^2 - 5n + 9$, there is no primitive powerful NR signed digraph $S$ of order $n$ with $l(S) = k$.

(3) Up to isomorphism, there exists zero or one ($D_{n-1,n-3}$) primitive NR digraph on $n$ vertices as the underlying digraph of $S$ such that $l(S) = n^2 - 5n + 9$, according to whether $n$ is odd or even. Furthermore, there exist either one ($H_n$) or two non-isomorphic primitive NR digraphs ($H_n$, $D_{n-1,n-4}$) on $n$ vertices as the underlying digraphs of $S$ such that $l(S) = n^2 - 6n + 12$, according to whether $n - 1$ is or is not a multiple of three.

We now consider non-powerful signed digraphs. We begin with the following useful (and obvious) result.

Let $S$ be a primitive, non-powerful signed digraph of order $n \geq s + 2$ with $D_{n-1,s}$ as its underlying digraph, we will obtain a bound on the base of $S$ in the following Lemma 3.1.

Lemma 3.1. Let $S$ be a primitive, non-powerful signed digraph of order $n \geq 4$ with $D_{n-1,s}$ as its underlying digraph. Let $C_{n-1}$ and $C_s$ be the only two cycles of lengths $n - 1$ and $s$ in $S$. Then
\[ l(S) \geq 2ns + n - 4s. \]

Proof. We show that there are no pairs of SSSD walks of length $k = 2ns + n - 4s - 1$ from vertex $s + 1$ to vertex $n$. Suppose that $W_1$ and $W_2$ are walks of length $k$ from vertex $s + 1$ to vertex $n$. Clearly $k > n - s - 1$ and each $W_i$ ($i = 1, 2$) is the “union” of the path $P$ from $s + 1$ to $n$ and cycles. Since $k > n - s - 1$, each union contains at least one cycle. Furthermore, since $s + 1$ and $n$ (and thus all vertices on $P$) are only on the cycle $C_{n-1}$, each union contains at least one cycle $C_{n-1}$. That is to say, take $W_i = P + a_iC_{n-1} + b_iC_s, a_i \geq 1, b_i \geq 0, (i = 1, 2)$, we have
\[ k = l(W_i) = a_i(n - 1) + b_is + (n - s - 1), \quad a_i \geq 1, b_i \geq 0, (i = 1, 2). \]
So \((a_2 - a_1)(n - 1) = (b_1 - b_2)s\). Write \(b_1 - b_2 = (n - 1)x\), then \(a_2 - a_1 = sx\). We claim that \(s = 0\).

If \(x \geq 1\), then \(a_2 = sx + a_1 \geq s + 1\) (since \(a_1 \geq 1\), so

\[
    k = a_2(n - 1) + b_2s + (n - s - 1) \\
    = (a_2 - (s + 1))(n - 1) + b_2s + (s + 1)(n - 1) + (n - s - 1),
\]

which implies (Note that g.c.d. \((n - 1, s) = 1\) since \(S\) is primitive, \(C_{n-1}\) and \(C_s\) be the only two cycles of lengths \(n - 1\) and \(s\) in \(S\).)

\[
    \phi(n - 1, s) - 1 = (n - 2)(s - 1) - 1 = k - ((s + 1)(n - 1) + (n - s - 1)) \\
    = (a_2 - (s + 1))(n - 1) + b_2s \in S(n - 1, s),
\]

contradicting the definition of the Frobenius number \(\phi(n - 1, s)\). Similarly we can get a contradiction if \(x \leq -1\). Thus we have \(x = 0\). So \(a_1 = a_2, b_1 = b_2\) and thus \(\text{sgn}(W_1) = \text{sgn}(W_2)\). This argument shows that

\[
l(S) \geq 2ns + n - 4s. \quad \Box
\]

Let \(S_1, S_2, S_3, S_4\) be primitive non-powerful NR signed digraphs of order \(n\) with \(D_{n-1,n-2}, D_{n-1,n-3}, H_n, D_{n-1,n-4}\) as their underlying digraphs, respectively. Then Lemma 3.1 implies the following inequalities:

\[
l(S_1) \geq 2n^2 - 7n + 8 \quad (\text{for } s = n - 2); \quad (3.3) \\
l(S_2) \geq 2n^2 - 9n + 12 \quad (\text{for } s = n - 3). \quad (3.4)
\]

We now show that equality holds in (3.3) and (3.4).

**Lemma 3.2.** Let \(S_1\) be a primitive, non-powerful signed digraph of order \(n \geq 5\) with \(D_{n-1,n-2}\) as its underlying digraph. Then

\[
l(S_1) = 2n^2 - 7n + 8. \quad (3.5)
\]

**Proof.** We only need to show that \(l(S_1) \leq 2n^2 - 7n + 8\) by (3.3).

Let \(u\) and \(v\) be any two (not necessarily distinct) vertices of \(S_1\). We will show that there is a pair of \(SSSD\) walks of length \(2n^2 - 7n + 8\) from vertex \(u\) to vertex \(v\). For this purpose, let \(P\) be the path of length \(l = l(P)\) from vertex \(u\) to vertex \(v\), then \(0 \leq l = l(P) \leq n - 1\). Let \(C_{n-2}\) and \(C_{n-1}\) be the only two cycles of lengths \(n - 2\) and \(n - 1\) in \(S_1\). Take

\[
W = P + (n - l)C_{n-1} + (n + l - 4)C_{n-2}.
\]

Then \(n - l \geq 1, n + l - 4 \geq 1\) and

\[
l(W) = l + (n - l)(n - 1) + (n + l - 4)(n - 2) = 2n^2 - 7n + 8.
\]

**Case 1:** \(0 \leq l \leq 2\).

Then \(n - l \geq n - 2\). Take

\[
W_1 = (n - 2)C_{n-1} + [(2 - l)C_{n-1} + (n + l - 4)C_{n-2} + P]
\]

and

\[
W_2 = (n - 1)C_{n-2} + [(2 - l)C_{n-1} + (n + l - 4)C_{n-2} + P].
\]
Case 2: $3 \leq l \leq n - 1.$

Then $n + l - 4 \geq n - 1.$ Take

$$W_1 = [P + (n - l)C_{n-1} + (l - 3)C_{n-2}] + (n - 1)C_{n-2}$$

and

$$W_2 = [P + (n - l)C_{n-1} + (l - 3)C_{n-2}] + (n - 2)C_{n-1}.$$  

Clearly, $W_1$ (or $W_2$) is a "union" of $P$ and several cycles of $S_1$. We now show $W_1$ and $W_2$ are two different walks from vertex $u$ to vertex $v$ in $S_1$.

If there exists a vertex $w$ on $P$ which is belong to $\{2, 3, \ldots, n - 2\}$, then $w$ is on both $C_{n-1}$ and $C_{n-2}$ (note that $C_{n-1}$ and $C_{n-2}$ have a common path which is from 2 to $n - 2$), and thus $W_1$ and $W_2$ are two different walks from vertex $u$ to vertex $v$ in $S_1$.

Otherwise, each vertex on $P$ is not belong to $\{2, 3, \ldots, n - 2\}$, then one of the following two situations will occur:

(i) $u = v = 1$;
(ii) $u = v \in \{n - 1, n\}$ or $u = n - 1, v = n$.

We know (i) and (ii) are belong to Case 1 (for $0 \leq l \leq 1$) and thus $2 - l \geq 1, n + l - 4 \geq 1$. For convenience, let $P(x \rightarrow y)$ be a path from vertex $x$ to $y$ in the following.

If $u = v = 1$, then $W_0 = P(1 \rightarrow 2) + (n + l - 5)C_{n-2} + (2 - l)C_{n-1} + P(2 \rightarrow 1)$ is a walk from $u$ to $v$ (note that the vertex 2 is on both $C_{n-1}$ and $C_{n-2}$), so $(2 - l)C_{n-1} + (n + l - 4)C_{n-2} + P$ is a walk from $u$ to $v$, and thus $W_1$ and $W_2$ are two different walks from $u$ to $v$.

Otherwise, $W_0 = P(u \rightarrow 2) + (1 - l)C_{n-1} + (n + l - 4)C_{n-2} + P(2 \rightarrow v)$ is a walk from $u$ to $v$, so $(2 - l)C_{n-1} + (n + l - 4)C_{n-2} + P$ is a walk from $u$ to $v$, and thus $W_1$ and $W_2$ are two different walks from $u$ to $v$.

Combing the above, we see that $W_1$ and $W_2$ are two different walks of length $2n^2 - 7n + 8$ from vertex $u$ to vertex $v$ in $S_1$.

Since $S_1$ is non-powerful, and $C_{n-2}$ and $C_{n-1}$ are the only two cycles of $S_1$, $C_{n-2}$ and $C_{n-1}$ must be a distinguished cycle pair by Theorem 2.A. So $(n - 1)C_{n-2}$ and $(n - 2)C_{n-1}$ have different signs by (2.1).

Hence $W_1$ and $W_2$ also have different signs (since $W_1$ and $W_2$ have difference only from the closed walks (with the same length) $(n - 1)C_{n-2}$ and $(n - 2)C_{n-1}$), and so is a pair of $SSSD$ walks of length $2n^2 - 7n + 8$. Thus we have

$$l(A) \leq 2n^2 - 7n + 8. \quad (3.6)$$

Combining the above two inequalities (3.3) and (3.6), we obtain $l(S_1) = 2n^2 - 7n + 8$. □

Lemma 3.3. Let $n \geq 7, n \equiv 0 (\text{mod } 2)$. Let $S_2$ be a primitive, non-powerful signed digraph of order $n$ with $D_{n-1,n-3}$ as its underlying digraph. Then

$$l(S_2) = 2n^2 - 9n + 12. \quad (3.7)$$

Proof. We only need to show that $l(S_2) \leq 2n^2 - 9n + 12$ by (3.4).

Let $u$ and $v$ be any two (not necessarily distinct) vertices of $S_2$. First we show that there is a pair of $SSSD$ walks of length $2n^2 - 9n + 12$ from vertex $u$ to vertex $v$. For this purpose, let $P$ be the path of length $l = l(P)$ from vertex $u$ to vertex $v$, then $0 \leq l = l(P) \leq n - 1$. Let $C_{n-3}$ and $C_{n-1}$ be the only two cycles of lengths $n - 3$ and $n - 1$ in $S_2$. 
Case 1: $l = l(P)$ is odd.
Then $l \in \{1, 3, \ldots, n - 1\}$. Write $l = 2m + 1$, then $0 \leq m \leq \frac{n - 2}{2}$. Take

$$W = P + (n - 2 - m)C_{n-1} + (n + m - 3)C_{n-3}.$$  

Then $n - 2 - m > 0$, $n + m - 3 > 0$ and

$$l(W) = (2m + 1) + (n - 2 - m)(n - 1) + (n + m - 3)(n - 3) = 2n^2 - 9n + 12.$$

Subcase 1.1: $0 \leq m \leq 1$ (That is, $l \in \{1, 3\}$).
Then $n - 2 - m \geq n - 3$. Take

$$W_1 = (n - 3)C_{n-1} + [(1 - m)C_{n-1} + (n + m - 3)C_{n-3} + P]$$

and

$$W_2 = (n - 1)C_{n-3} + [(1 - m)C_{n-1} + (n + m - 3)C_{n-3} + P].$$

Subcase 1.2: $2 \leq m \leq \frac{n - 2}{2}$ (That is, $l \in \{5, \ldots, n - 1\}$).
Then $n + m - 3 \geq n - 1$. Take

$$W_1 = [P + (n - 2 - m)C_{n-1} + (m - 2)C_{n-3}] + (n - 1)C_{n-3}$$

and

$$W_2 = [P + (n - 2 - m)C_{n-1} + (m - 2)C_{n-3}] + (n - 3)C_{n-1}.$$

Case 2: $l = l(P)$ is even.
Then $l \in \{0, 2, \ldots, n - 2\}$. Write $l = n - 2m$, then $1 \leq m \leq \frac{n}{2}$. Take

$$W = P + mC_{n-1} + (2n - m - 4)C_{n-3}.$$  

Then $m \geq 1$, $2n - m - 4 \geq 2n - 4 - \frac{n}{2} > n - 1$ and

$$l(W) = n - 2m + m(n - 1) + (2n - m - 4)(n - 3) = 2n^2 - 9n + 12.$$

Take

$$W_1 = [P + mC_{n-1} + (n - m - 3)C_{n-3}] + (n - 1)C_{n-3}$$

and

$$W_2 = [P + mC_{n-1} + (n - m - 3)C_{n-3}] + (n - 3)C_{n-1}.$$  

Clearly, $W_1$ (or $W_2$) is a “union” of $P$ and several cycles of $S_2$. We now show $W_1$ and $W_2$ are two different walks from $u$ to $v$ in $S_2$.

If there exists a vertex $w$ on $P$ which is belong to $\{2, 3, \ldots, n - 3\}$, then $w$ is on both $C_{n-1}$ and $C_{n-3}$ (note that $C_{n-1}$ and $C_{n-3}$ have a common path which is from 2 to $n - 3$), and thus $W_1$ and $W_2$ are two different walks from $u$ to $v$ in $S_2$.

Otherwise, each vertex on $P$ is not belong to $\{2, 3, \ldots, n - 3\}$, then one of the following three situations will occur:

(i) $u = v = 1$ (belong to Case 2 for $l = l(P) = 0$);
(ii) $u = v \in \{n - 2, n - 1, n\}$ (belong to Case 2 for $l = l(P) = 0$) or $u = n - 2, v = n$ (belong to Case 2 for $l = l(P) = 2$);
(iii) $u = n - 2, v = n - 1$ or $u = n - 1, v = n$ (belong to Subcase 1.1 for $l = l(P) = 1$).

If $u = v = 1$, then $W_0 = P(1 \rightarrow 2) + (n - m - 4)C_{n-3} + mC_{n-1} + P(2 \rightarrow 1)$ is a walk from $u$ to $v$ (note that the vertex 2 is on both $C_{n-1}$ and $C_{n-3}$), so $mC_{n-1} + (n - m - 3)C_{n-3} + P$ is a walk from $u$ to $v$, and thus $W_1$ and $W_2$ are two different walks from $u$ to $v$. 

If $u = v \in \{n - 2, n - 1, n\}$ or $u = n - 2, v = n$, then $W_0 = P(u \rightarrow v) + (m - 1)C_{n-1} + (n - m - 3)C_{n-3} + P(2 \rightarrow v)$ is a walk from $u$ to $v$, so $mC_{n-1} + (n - m - 3)C_{n-3} + P$ is a walk from $u$ to $v$, and thus $W_1$ and $W_2$ are two different walks from $u$ to $v$.

Otherwise, if $u = n - 2, v = n - 1$ or $u = n - 1, v = n$, then $l = l(P) = 1$ and $m = 0$, and $W_0 = P(u \rightarrow v) + (n + m - 3)C_{n-3} + P(2 \rightarrow v)$ is a walk from $u$ to $v$, so $l(S_{n-1} + (n + m - 3)C_{n-3} + P$ is a walk from $u$ to $v$, and thus $W_1$ and $W_2$ are two different walks from $u$ to $v$.

Combining the above, we see that $W_1$ and $W_2$ are two different walks of length $2n^2 - 9n + 12$ from $u$ to $v$ in $S_2$.

Since $S_2$ is non-powerful, and $C_{n-3}$ and $C_{n-1}$ are the only two cycles of $S_2$, $C_{n-3}$ and $C_{n-1}$ must be a distinguished cycle pair by Theorem 2.A. So $(n - 1)C_{n-3}$ and $(n - 3)C_{n-1}$ have different signs by (2.1). Hence $W_1$ and $W_2$ also have different signs, and so is a pair of $SSSD$ walks of length $2n^2 - 9n + 12$. Thus we have

$$l(A) \leq 2n^2 - 9n + 12.$$  \hfill (3.8)

Combining the above two inequalities (3.4) and (3.8), we obtain $l(S_2) = 2n^2 - 9n + 12$. \hfill □

**Lemma 3.4.** Let $S_3$ be a primitive, non-powerful signed digraph of order $n \geq 7$ with $H_n$ as its underlying digraph. Then

$$l(S_3) = 2n^2 - 11n + 18.$$  \hfill (3.9)

**Proof.** Let $u$ and $v$ be any two (not necessarily distinct) vertices of $S_3$. First we show that there is a pair of $SSSD$ walks of length $2n^2 - 11n + 18$ from vertex $u$ to vertex $v$. For this purpose, let $P$ be the path of length $l = l(P)$ from vertex $u$ to vertex $v$, then $0 \leq l = l(P) \leq n - 1$. Let $C_{n-3}$ and $C_{n-2}$ be the cycles of lengths $n - 3$ and $n - 2$ in $S_3$. Take

$$W = P + (n - l)C_{n-2} + (n + l - 6)C_{n-3}.$$  

Then $n - l \geq 1, n + l - 6 \geq 1$ and

$$l(W) = l + (n - l)(n - 2) + (n + l - 6)(n - 3) = 2n^2 - 11n + 18.$$  

**Case 1:** $0 \leq l \leq 3$.

Then $n - l \geq n - 3$. Take

$$W_1 = (n - 3)C_{n-2} + [(3 - l)C_{n-2} + (n + l - 6)C_{n-3} + P]$$  

and

$$W_2 = (n - 2)C_{n-3} + [(3 - l)C_{n-2} + (n + l - 6)C_{n-3} + P].$$  

**Case 2:** $4 \leq l \leq n - 1$, then $n + l - 6 \geq n - 2$. Take

$$W_1 = [P + (n - l)C_{n-2} + (l - 4)C_{n-3}] + (n - 2)C_{n-3}$$  

and

$$W_2 = [P + (n - l)C_{n-2} + (l - 4)C_{n-3}] + (n - 3)C_{n-2}.$$  

Similar to the proof of Lemmas 3.2 and 3.3, we can show $W_1$ and $W_2$ are two different walks from vertex $u$ to vertex $v$ in $S_3$.

Since $S_3$ is non-powerful, and $C_{n-2}$ and $C_{n-3}$ are the only two cycles of $S_3$, $C_{n-2}$ and $C_{n-3}$ must be a distinguished cycle pair by Theorem 2.A. So $(n - 3)C_{n-2}$ and $(n - 2)C_{n-3}$ have
different signs by (2.1). Hence $W_1$ and $W_2$ also have different signs, and so is a pair of $SSSD$ walks of length $2n^2 - 11n + 18.$

Thus we have

$$l(A) \leq 2n^2 - 11n + 18. \quad (3.10)$$

Next we show that there is no pair of $SSSD$ walks of length $k = 2n^2 - 11n + 17$ from vertex $n - 2$ to vertex $n.$ Let $W_1$ and $W_2$ be any two walks of length $k$ from vertex $n - 2$ to vertex $n.$ Then each $W_i$ is a “union” of the unique path $P$ from vertex $n - 2$ to vertex $n$ (of length 2) and several cycles $C_{n-2}$ and several (at least one because all vertices on $P$ are only on the cycle $C_{n-2}$ and $k > 2$) cycles $C_{n-2} (i = 1, 2).$ Thus we have

$$k = l(W_i) = a_i(n - 2) + b_i(n - 3) + 2, \quad a_i \geq 1, \ b_i \geq 0 \ (i = 1, 2).$$

It is similar to the proof of Lemma 3.1 we can obtain $a_1 = a_2, b_1 = b_2$ and thus $\text{sgn}(W_1) = \text{sgn}(W_2).$ This argument shows that

$$l(S) \geq 2n^2 - 11n + 18. \quad (3.11)$$

Combining the above two inequalities (3.10) and (3.11), we obtain $l(S_3) = 2n^2 - 11n + 18.$ \hfill \Box

**Lemma 3.5.** Let $n \geq 6,$ $n - 1 \neq 0 (\text{mod} \ 3).$ Let $S_4$ be a primitive, non-powerful signed digraph of order $n$ with $D_{n-1,n-4}$ as its underlying digraph. Then

$$l(S_4) \leq 2n^2 - 11n + 17. \quad (3.12)$$

**Proof.** Let $C_{n-1}$ and $C_{n-4}$ be the only two cycles of lengths $n - 4$ and $n - 1$ in $S_4.$ Let $Q_1 = (n - 4, 1) + (1, 2)$ be the path of length 2 from vertex $n - 4$ to vertex 2, and $Q_2 = (n - 3, n - 2) + (n - 2, n - 1) + (n - 1, n) + (n, 2)$ be the path of length 5 from vertex $n - 4$ to vertex 2. Let $P$ be the unique path from vertex 2 to $n - 4,$ and let

$$W_1 = Q_1 + (n - 2)C_{n-4}, \quad W_2 = Q_2 + (n - 5)C_{n-1}.$$

Then $l(W_1) = l(W_2) = n^2 - 6n + 10,$ and

$$W_1 + P = (n - 1)C_{n-4}, \quad W_2 + P = (n - 4)C_{n-1}.$$

Since $S_4$ is non-powerful, and $C_{n-1}$ and $C_{n-4}$ are the only cycles of $S_4,$ $C_{n-1}$ and $C_{n-4}$ must be a distinguished cycle pair by Theorem 2.A. So $(n - 4)C_{n-1}$ and $(n - 1)C_{n-4}$ have different signs by (2.1). Hence $W_1$ and $W_2$ also have different signs, and so is a pair of $SSSD$ walks of length $n^2 - 6n + 10.$ So

$$r_{n-4,2} \leq n^2 - 6n + 10,$$

and by (2.5) we have

$$\exp_{S_4}(2) = \exp_{D_{n-1,n-4}}(2) \leq \phi(n - 4, n - 1) + (n - 2) = n^2 - 6n + 8.$$

Thus by (2.6) we have

$$l(S_4) \leq d(S_4) + r_{n-4,2} + \exp_{S_4}(2)$$

$$\leq (n - 1) + (n^2 - 6n + 10) + (n^2 - 6n + 8)$$

$$= 2n^2 - 11n + 17. \quad \Box$$
4. The non-powerful cases and general cases

Lemma 4.1. Let $D$ be a primitive NR digraph and $C$ be a cycle of length $n - 1$ in $D$, then there only exists a unique cycle of length $l$ ($1 < n - 1$) satisfying $\text{g.c.d.} (n - 1, l) = 1$ in $D$.

Proof. Without loss of generality, we may assume that $V(C) = \{1, 2, \ldots, n - 1\}$ and $E(C) = \{(1, 2), (2, 3), \ldots, (n - 2, n - 1), (n - 1, 1)\}$.

Since $D$ is strong, $D$ must contain a cycle $C'$ (of length $l$) such that $n \in V(C')$. Let $E(C') = \{(n, l_1), (l_1, l_2), \ldots, (l_{l-1}, n)\}$, where $l_1, l_2, \ldots, l_{l-1} \in \{1, 2, \ldots, n - 1\}$.

First we show that $l_1 < l_2 < \cdots < l_{l-1}$. Otherwise, there exists $j \in \{1, 2, \ldots, l - 2\}$ such that $l_j > l_{j+1}$. However, there exists a path $P$ from vertex $i_j$ to vertex $i_{j+1} + 1$ in $D \setminus (i_j, i_{j+1})$: $(i_j, i_{j+1} + 1) + \cdots + (n - 1, 1) + (1, 2) + \cdots + (i_{j+1} - 1, i_{j+1})$. So $D \setminus (i_j, i_{j+1})$ is strong, it is a contradiction because $D$ is a NR digraph.

For each $j \in \{1, 2, \ldots, l - 2\}$, we claim that $i_{j+1} - i_j = 1$. Otherwise, there exists some integer $k$ such that $i_{j+1} - i_j = k > 1$, but $Q = (i_j, i_{j+1} + 1) + \cdots + (i_j + (k - 1), i_{j+1})$ is a path from vertex $i_j$ to vertex $i_{j+1}$ of length $k$ in $D \setminus (i_j, i_{j+1})$. Thus $D \setminus (i_j, i_{j+1})$ is also strong, it is a contradiction because $D$ is a NR digraph.

Combining the above, we see that the arcs $(i_1, i_2), (i_2, i_3), \ldots, (i_{l-2}, i_{l-1})$ of $C'$ are also the arcs of $C$. Thus $C$ and $C'$ have a common path $P$ from vertex $i_1$ to $i_{l-1}$, and $(C \cup C') \setminus P$ contains only the two paths from $i_{l-1}$ to $i_1$. So $C \cup C'$ contains $n$ vertices and the only two cycles $C$ and $C'$.

If $D$ still contains a cycle $C''$, then there exists a arc $e \in E(C'')$ such that $e \notin E(C)$ and $e \notin E(C')$. Thus $C \cup C' \subseteq D \setminus \{e\}$ and $D \setminus \{e\}$ is strong. This contradicts that $D$ is a NR digraph.

So $D$ contains the only two cycles $C$ (of length $n - 1$) and $C'$ (of length $l$). Because $D$ is primitive, $\text{g.c.d.} (n - 1, l) = 1$. □

Lemma 4.1 implies that $D_{n-1,n-2}$ is the only primitive NR digraph with the set of cycle lengths $R = \{n - 2, n - 1\}$, $D_{n-1,n-3}$ is the only primitive NR digraph with $R = \{n - 3, n - 1\}$ (if $n$ is even), and $D_{n-1,n-4}$ is the only primitive NR digraph with $R = \{n - 4, n - 1\}$ (if $n \equiv 1 \pmod{3}$). Similarly, Lemma 4.1 implies that if $S$ contains two cycles (with different length) of length less than $n - 1$, then $S$ contains no cycle of length $n - 1$.

Lemma 4.2. Let $R = \{l_1, \ldots, l_r\}$ be a set of cycle lengths in a primitive digraph $D$ with $\frac{n}{2} < l_1 < l_2 < \cdots < l_r$. Then for each vertex $x$ and vertex $y$ in $D$, we have
\[
\text{d}_R(x, y) \leq n + \max\{|l_{i+1} - l_i| : i \in \{1, \ldots, r - 1\}\} - 1.
\]

Proof. Let $d = \max\{|l_{i+1} - l_i| : i \in \{1, \ldots, r - 1\}\} > 0$. Since $l_i + l_j > n$ for each $i$ and $j$, each pair of cycles in $D$ have at least one vertex in common. Thus, for each $j \in \{1, \ldots, r\}$, the cycle $C_{l_j}$ (of length $l_j$) will meets at least one vertex of cycles of length $l_i$ for each $i = 1, \ldots, r$. Let $P$ be the shortest path (of length $d(x, y)$) from $x$ to $y$, we consider the following three cases:

Case 1: $d(x, y) \geq n - l_1$.

Then the path $P$ (with $d(x, y) + 1$ vertices) will meet at least one vertex of cycles of length $l_i$ for each $i = 1, \ldots, r$. So $\text{d}_R(x, y) = d(x, y) < n \leq n + d - 1$. 

\[\text{d}_R(x, y) \leq n + \max\{|l_{i+1} - l_i| : i \in \{1, \ldots, r - 1\}\} - 1.\]
Case 2: \( n - l_{i+1} \leq d(x, y) < n - l_i \) for any \( i \in \{1, \ldots, r - 1\} \).

Then \( d_R(x, y) \leq d(x, y) + l_{i+1} \leq n - l_i - 1 + l_{i+1} = n + (l_{i+1} - l_i) - 1 \leq n + d - 1 \).

Case 3: \( d(x, y) < n - l_r \).

Then there exists some \( j \in \{1, \ldots, r\} \) such that some vertex \( z \) which is on both \( P \) and \( C_{l_j} \) (of length \( l_j \leq l_r \)) because \( D \) is strong, so \( d_R(x, y) \leq d(x, y) + l_j \leq n - l_r - 1 + l_j < n \leq n + d - 1 \).

Then \( d_R(x, y) \leq n + \max\{l_{i+1} - l_i|i \in \{1, \ldots, r - 1\}\} - 1 \) follows directly from the above cases. \( \square \)

In the remainder of this paper, let \( H_n^{(i)} (i = 1, 2, 3, 4, 5) \) be the primitive NR digraph of order \( n \geq 6 \) given in Fig. 2, respectively.

According to the results in [7], we know that all primitive NR digraphs on \( n \) vertices with the set of cycle lengths \( R = \{n - 2, n - 3\} \) are \( H_n^{(i)} (i = 1, 2, 3, 4, 5) \) and \( H_n \). Then it is well-known from the theory of nonnegative primitive matrices (see [7]) that:

\[
\exp(H_n^{(i)}) = n^2 - 6n + 11 \quad (i = 1, 2, 3); \quad \exp(H_n^{(i)}) = n^2 - 6n + 10 \quad (i = 4, 5).
\]

And for all other primitive NR digraphs of order \( n \) except \( D_{n-1,n-2}, D_{n-1,n-3} \) (\( n \) is even), \( D_{n-1,n-4} (n \not\equiv 1(\text{mod } 3)) \) and \( H_n \), we have

\[
\exp(D) \leq n^2 - 6n + 11. \tag{4.2}
\]

Let \( S_3^{(i)} \) be a primitive, non-powerful signed digraph of order \( n \geq 6 \) with \( H_n^{(i)} \) as its underlying digraph \( (i = 1, 2, 3, 4, 5) \), respectively. We will study the base of \( S_3^{(i)} (i = 1, 2, 3, 4, 5) \) in the following Lemmas 4.3 and 4.4.

**Lemma 4.3.** Let \( S_3^{(i)} \) be a primitive, non-powerful signed digraph of order \( n \geq 6 \) with \( H_n^{(i)} \) \((i = 1, 2, 3)\) as its underlying digraph. Then

1. If the (only) two cycles of length \( n - 2 \) of \( S_3^{(i)} \) have different signs, then
   \[
   l(S_3^{(i)}) \leq n^2 - 5n + 11 \quad (i = 1, 2, 3).
   \]
2. If the (only) two cycles of length \( n - 2 \) of \( S_3^{(i)} \) have same sign, then
   \[
   l(S_3^{(i)}) \leq 2n^2 - 11n + 17 \quad (i = 1, 2, 3).
   \]

**Proof.** We only show the case \( i = 1 \), and the proof of \( i = 2, 3 \) is similar to \( i = 1 \).

1. In (a) of Fig. 2, let \( Q_1 = (n - 4, n - 3) + (n - 3, n - 2) \) and \( Q_2 = (n - 4, n) + (n, n - 2) \) be two paths of length 2 from vertex \( n - 4 \) to vertex \( n - 2 \). If the two cycles of length \( n - 2 \) of \( S_3^{(1)} \) have different signs, then we must have \( \text{sgn } Q_1 = - \text{sgn } Q_2 \), so clearly \( r(S_3^{(1)}) \leq 2 \). Thus we have
   \[
   l(S_3^{(1)}) \leq d(S_3^{(1)}) + r(S_3^{(1)}) + \exp(S_3^{(1)}) \leq (n - 2) + 2 + (n^2 - 6n + 11) = n^2 - 5n + 11.
   \]

2. Let \( u \) and \( v \) be any two (not necessarily distinct) vertices of \( S_3^{(1)} \). We will show that there is a pair of SSSD walks of length \( 2n^2 - 11n + 17 \) from vertex \( u \) to vertex \( v \). For this purpose, let \( P \) be a path of length \( l = l(P) \) from vertex \( u \) to vertex \( v \), then \( 0 \leq l = l(P) \leq n - 2 \). Let \( C_{n-2} \) and \( C_{n-3} \) be the cycles of lengths \( n - 2 \) and \( n - 3 \) in \( S_3^{(1)} \). Take
   \[
   W = P + (n - l - 1)C_{n-2} + (n + l - 5)C_{n-3}.
   \]
Then $n - l - 1 \geq 1$, $n + l - 5 \geq 1$ and

$$l(W) = l + (n - l - 1)(n - 2) + (n + l - 5)(n - 3) = 2n^2 - 11n + 17.$$  

**Case 1**: $0 \leq l \leq 2$, then $n - l - 1 \geq n - 3$. Take

$$W_1 = (n - 3)C_{n-2} + [(2 - l)C_{n-2} + (n + l - 5)C_{n-3} + P]$$

and

$$W_2 = (n - 2)C_{n-3} + [(2 - l)C_{n-2} + (n + l - 5)C_{n-3} + P].$$
Case 2: $3 \leq l \leq n - 1$, then $n + l - 5 \geq n - 2$. Take
\[ W_1 = [P + (n - l - 1)C_{n-2} + (l - 3)C_{n-3}] + (n - 2)C_{n-3} \]
and
\[ W_2 = [P + (n - l - 1)C_{n-2} + (l - 3)C_{n-3}] + (n - 3)C_{n-2}. \]

Similar to the proof of Lemmas 3.2–3.4, it is sure that $W_1$ and $W_2$ are two different walks from vertex $u$ to vertex $v$ in the above two cases.

If the only two cycles of length $n - 2$ in $S_3(i)$ have the same sign, then sgn $Q_1 = \text{sgn} Q_2$. Also each cycle of length $n - 2$ and the cycle of length $n - 3$ will form a distinguished cycle pair by Theorem 2.A, since $S_3(i)$ is non-powerful and the only three cycles of $S_3(i)$ are the two cycles of length $n - 2$ and one cycle of length $n - 3$. So $(n - 2)C_{n-3}$ and $(n - 3)C_{n-2}$ will have different signs by (2.1). Hence $W_1$ and $W_2$ also have different signs, and so is a pair of $SSSD$ walks of length $2n^2 - 11n + 17$. Thus
\[ l(S_3(i)) \leq 2n^2 - 11n + 17. \] 
\[ \square \]

For $i = 4, 5$, using the method similar to Lemma 4.3 (Note that $r(S_3(i)) \leq 3$ and $\exp(S_3(i)) = n^2 - 6n + 10$ if the (only) two cycles of length $n - 3$ of $S_3(i)$ have different signs), we have

**Lemma 4.4.** Let $S_3(i)$ be a primitive non-powerful signed digraph of order $n \geq 6$ with $H_n(i)$ ($i = 4, 5$) as its underlying digraph. Then we have:

1. If the (only) two cycles of length $n - 3$ of $S_3(i)$ have different signs, then
   \[ l(S_3(i)) \leq n^2 - 5n + 11 \quad (i = 4, 5). \]
2. If the (only) two cycles of length $n - 3$ of $S_3(i)$ have same sign, then
   \[ l(S_3(i)) \leq 2n^2 - 11n + 17. \quad (i = 4, 5). \]

Combining the Lemmas 3.2–3.5, we can discuss the base of primitive, non-powerful signed digraphs and sign pattern matrices of order $n \geq 7$.

**Theorem 4.1.** Let $S$ be a primitive, non-powerful, NR signed digraph of order $n \geq 7$. Then (1)
\[ l(S) \leq 2n^2 - 7n + 8, \] 
with equality if and only if the underlying digraph is isomorphic to $D_{n-1,n-2}$.
(2) For each integer $k$ with $2n^2 - 9n + 12 < k < 2n^2 - 7n + 8$ or $2n^2 - 11n + 18 < k < 2n^2 - 9n + 12$, there is no primitive non-powerful NR signed digraph $S$ of order $n$ with $l(S) = k$.
(3) $l(S) = 2n^2 - 9n + 12$ if and only if $n$ is even and the underlying digraph of $S$ is isomorphic to $D_{n-1,n-3}$; and there is no primitive, non-powerful, NR signed digraph $S$ of order $n$ with $l(S) = 2n^2 - 9n + 12$ if $n$ is odd.
(4) $l(S) = 2n^2 - 11n + 18$ if and only if the underlying digraph of $S$ is isomorphic to $H_n$.

**Proof.** Since $S$ is primitive non-powerful, there is a distinguished cycle pair $C_1$ and $C_2$ (with lengths, say, $p_1$ and $p_2$, respectively) by Theorem 2.A, where $p_1C_2$ and $p_2C_1$ have different signs by (2.1). Let $D$ be the underlying digraph of $S$. 
Case 1: \(C_1\) and \(C_2\) have no common vertices.

Then \(p_1 + p_2 \leq n\) (Note that \(D\) is not isomorphic to \(D_{n-1,n-2}\), \(D_{n-1,n-3}\), \(H_n\) and \(D_{n-1,n-4}\), so \(\exp(S) = \exp(D) \leq n^2 - 6n + 11\)).

Let \(Q\) be a shortest path from \(C_1\) to \(C_2\) with length \(q\). Then \(q \leq n - p_1 - p_2 + 1\), \(p_2 C_1 + Q\) and \(Q + p_1 C_2\) is a pair of SSSD walks with length \(p_1 p_2 + q\). So we have

\[
\exp(S) \leq (p_1 - 1)(p_2 - 1) + n
\]

Then by Theorem 2.1 we have

\[
l(S) \leq d(S) + r(S) + \exp(S) \leq (n - 1) + \left(\frac{n^2}{4} + 1\right) + (n^2 - 6n + 11)
\]

\[
= \frac{5}{4}n^2 - 5n + 11 < 2n^2 - 11n + 18(n \geq 7).
\]

Case 2: \(C_1\) and \(C_2\) have some common vertices.

Subcase 2.1: \(p_1 = p_2\).

Then \(C_1\) and \(C_2\) is also a pair of SSSD walks (since \(C_1\) and \(C_2\) have common vertices) of length \(p_1\). Thus \(r(S) \leq p_1 \leq n - 2\) by Lemma 4.1. So we have

\[
l(S) \leq d(S) + r(S) + \exp(S) \leq (n - 1) + (n - 2) + (n^2 - 6n + 11)
\]

\[
= n^2 - 4n + 8 < 2n^2 - 11n + 18(n \geq 7).
\]

In the following cases, we will consider the situation \(p_1 \neq p_2\). By Lemma 4.1 we know the length of the longest cycle of \(S\) is not exceeding \(n - 1\). So we only need to consider the four cases:

\[
\begin{align*}
\max\{p_1, p_2\} &= n - 1, & \min\{p_1, p_2\} &\leq n - 2; \\
\max\{p_1, p_2\} &= n - 2, & \min\{p_1, p_2\} &= n - 3; \\
\max\{p_1, p_2\} &= n - 2, & \min\{p_1, p_2\} &= n - 4; \\
\max\{p_1, p_2\} &\leq n - 2, & \min\{p_1, p_2\} &\leq n - 5.
\end{align*}
\]

Clearly, if \(\max\{p_1, p_2\} \leq n - 2\) (and \(p_1 \neq p_2\)), then \(S\) contains no cycle of length \(n - 1\) by Lemma 4.1.

First we have

\[
r(S) \leq p_1 p_2
\]

(4.4)

because \(p_2 C_1\) and \(p_1 C_2\) is a pair of SSSD walks (since \(C_1\) and \(C_2\) have common vertices) of length \(p_1 p_2\).

Subcase 2.2: \(\max\{p_1, p_2\} = n - 1, \min\{p_1, p_2\} \leq n - 2\).

It is clearly that the set of cycle lengths \(R = \{p_1, p_2\}\) by Lemmas 4.1 and 4.1. So we need to consider the situation \(\min\{p_1, p_2\} \leq n - 5\) since by Lemma 4.1 we know that the situation \(\min\{p_1, p_2\} \in \{n - 2, n - 3, n - 4\}\) has been studied in Lemmas 3.2 and 3.3 and Lemma 3.5 in Section 3.
Then by (4.4) we have
\[
l(S) \leq d(S) + r(S) + \exp(S) \leq n - 1 + (n - 1)(n - 5) + (n^2 - 6n + 11) \\
= 2n^2 - 11n + 15 < 2n^2 - 11n + 18.
\]

**Subcase 2.3**: \(\max\{p_1, p_2\} = n - 2, \min\{p_1, p_2\} = n - 3.

**Subcase 2.3.1**: \(s = n - 3\).

If \(D\) is isomorphic to \(H_n\), then \(l(S) = l(S_3) = 2n^2 - 11n + 18\) by Lemma 3.4;

If \(D\) is not isomorphic to \(H_n\), then \(D\) is isomorphic to one of \(H_n^{(i)}\) \((i = 1, 2, 3, 4, 5)\) according to the results in [7]. Then \(l(S) < 2n^2 - 11n + 18\) by Lemmas 4.3 and 4.4.

**Subcase 2.3.2**: \(s = n - 4\) or \(n - 5\).

First we have \(d_R(x, y) \leq n + 1\) for each vertex \(x\) and each vertex \(y\) in \(D\) by Lemma 4.2. Thus by (2.3) and (2.4) we have
\[
\exp(D) \leq \phi_R + \max_{x, y \in V(D)} d_R(x, y) \leq \left\lfloor \frac{(n - 4)^2}{2} \right\rfloor + n + 1.
\]
So by (4.4) we have
\[
l(S) \leq d(S) + r(S) + \exp(S) \leq (n - 1) + (n - 2)(n - 3) + \left\lfloor \frac{(n - 4)^2}{2} \right\rfloor + n + 1 \\
= n^2 - 7n + 14 + \left\lfloor \frac{n^2}{2} \right\rfloor < 2n^2 - 11n + 18(n \geq 7).
\]

**Subcase 2.3.3**: \(s \leq n - 6\).

Then \(\exp(S) \leq n + s(n - 3) - 1 \leq n + (n - 6)(n - 3) - 1 = n^2 - 8n + 17\) by Theorem 2.B. Thus by (4.4) we have
\[
l(S) \leq d(S) + r(S) + \exp(S) \leq (n - 1) + (n - 2)(n - 3) + (n^2 - 8n + 17) \\
= 2n^2 - 12n + 22 < 2n^2 - 11n + 18.
\]

**Subcase 2.4**: \(\max\{p_1, p_2\} = n - 2, \min\{p_1, p_2\} = n - 4\).

**Subcase 2.4.1**: \(s = n - 4\).

If the set of cycle lengths \(R = \{n - 2, n - 4\}\), then \(n \equiv 1\) (mod 2). So by (2.4) and Lemma 4.2 we have
\[
\exp(D) \leq \phi_R + \max_{x, y \in V(D)} d_R(x, y) \leq (n - 3)(n - 5) + (n + 1) = n^2 - 7n + 16.
\]
Thus by (4.4) we have
\[
l(S) \leq d(S) + r(S) + \exp(S) \leq (n - 1) + (n - 2)(n - 4) + (n^2 - 7n + 16) \\
= 2n^2 - 12n + 23 < 2n^2 - 11n + 18.
\]

If the set of cycle lengths \(R = \{n - 2, n - 3, n - 4\}\), the proof is similar to Subcase 2.3.2.

**Subcase 2.4.2**: \(s \leq n - 5\).

Then \(\exp(S) \leq n + s(n - 3) - 1 \leq n + (n - 5)(n - 3) - 1 = n^2 - 7n + 14\) by Theorem 2.B. Thus by (4.4) we have...
\[ l(S) \leq d(S) + r(S) + \exp(S) \leq (n - 1) + (n - 2)(n - 4) + (n^2 - 7n + 14) \\
= 2n^2 - 12n + 21 < 2n^2 - 11n + 18. \]

**Subcase 2.5:** $\max\{p_1, p_2\} \leq n - 2, \min\{p_1, p_2\} \leq n - 5.$

First we have

\[ r(S) \leq p_1 p_2 \leq (n - 2)(n - 5) \]

and

\[ \exp(S) \leq n + (n - 5)(n - 3) - 1 = n^2 - 7n + 14. \]

Thus

\[ l(S) \leq d(S) + r(S) + \exp(S) \leq (n - 1) + (n - 2)(n - 5) + (n^2 - 7n + 14) \\
= 2n^2 - 13n + 23 < 2n^2 - 11n + 18. \]

Combining the above results and Lemmas 3.2–3.5 in Section 3, we complete the proof of Theorem 4.1. \(\square\)

Now by combining our above results in Theorem 4.1 for the non-powerful cases with the results in Theorem 3.1 for the powerful cases on the estimations of the (generalized) base of primitive NR signed digraphs, we obtain the following theorem.

**Theorem 4.2.** Let $A$ be a primitive NR sign pattern matrix of order $n \geq 7$. Then

(1) $l(A) \leq 2n^2 - 7n + 8,$ \hspace{1cm} (4.5)

with equality if and only if $A$ is non-powerful and the associated digraph $D(A)$ of $A$ is isomorphic to $D_{n-1,n-2}$.

(2) For each integer $k$ with $2n^2 - 9n + 12 < k < 2n^2 - 7n + 8$ or $2n^2 - 11n + 18 < k < 2n^2 - 9n + 12$, there is no primitive NR sign pattern matrix $A$ of order $n$ with $l(A) = k$.

(3) $l(A) = 2n^2 - 9n + 12$ if and only if $n$ is even and $A$ is non-powerful and the associated digraph $D(A)$ of $A$ is isomorphic to $D_{n-1,n-3}$; and there is no primitive NR sign pattern matrix $A$ of order $n$ with $l(A) = 2n^2 - 9n + 12$ if $n$ is odd.

(4) $l(A) = 2n^2 - 11n + 18$ if and only if $A$ is non-powerful and the associated digraph $D(A)$ of $A$ is isomorphic to $H_n$.

**Proof.** We consider two cases.

**Case 1:** $A$ is powerful.

Then by the results in Theorem 3.1 we have

\[ l(A) \leq n^2 - 4n + 6 < 2n^2 - 11n + 18. \]

**Case 2:** $A$ is non-powerful.

Then the results follow directly from Theorem 4.1. \(\square\)

The result (3) of Theorem 4.2 actually means that there exist “gaps” in the base set of the class of primitive NR sign pattern matrices of order $n$.

Finally, we would like to point out that if $A$ itself contains a # entry, then also $l(A) < 2n^2 - 11n + 18$. To see this, we only need to consider Case 2 of the proof of Theorem 4.2 (Case 1 does
not occur). Now in this case, the “ambiguous index” \( r(S) = 1 \), since \( A \) itself contains a \# entry (where \( S \) is the associated generalized signed digraph of \( A \)), thus by (2.7), Theorem 2.3 and Lemma 4.1, we have (for \( n \geq 7 \)):

\[
\begin{align*}
l(A) &= l(S) \leq d(S) + r(S) + \exp(S) \\
&= n^2 - 3n + 6 < 2n^2 - 11n + 18.
\end{align*}
\]

This comment suggests that the results of Theorem 4.2 can be extended to generalized sign pattern matrices as follows.

**Theorem 4.3.** Let \( A \) be a primitive NR generalized sign pattern matrix of order \( n \geq 7 \). Then (1)

\[
l(A) \leq 2n^2 - 7n + 8,
\]

with equality if and only if \( A \) is non-powerful sign pattern matrix and the associated digraph \( D(A) \) of \( A \) is isomorphic to \( D_{n-1,n-2} \).

(2) For each integer \( k \) with \( 2n^2 - 9n + 12 < k < 2n^2 - 7n + 8 \) or \( 2n^2 - 11n + 18 < k < 2n^2 - 9n + 12 \), there is no primitive NR generalized sign pattern matrix \( A \) of order \( n \) with \( l(A) = k \).

(3) \( l(A) = 2n^2 - 9n + 12 \) if and only if \( n \) is even and \( A \) is non-powerful sign pattern matrix and the associated digraph \( D(A) \) of \( A \) is isomorphic to \( D_{n-1,n-3} \); and there is no primitive NR generalized sign pattern matrix \( A \) of order \( n \) with \( l(A) = 2n^2 - 9n + 12 \) if \( n \) is odd.

(4) \( l(A) = 2n^2 - 11n + 18 \) if and only if \( A \) is non-powerful sign pattern matrix and the associated digraph \( D(A) \) of \( A \) is isomorphic to \( H_n \).

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**References**