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Resolution of finite fuzzy relation equations based on strong pseudo-t-norms☆

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Abstract

This work studies the problem of solving a sup-T composite finite fuzzy relation equation, where T is an infinitely distributive strong pseudo-t-norm. A criterion for the equation to have a solution is given. It is proved that if the equation is solvable then its solution set is determined by the greatest solution and a finite number of minimal solutions. A necessary and sufficient condition for the equation to have a unique solution is obtained. Also an algorithm for finding the solution set of the equation is presented.

Keywords: Fuzzy relation equation; Strong pseudo-t-norm; Infinitely distributive strong pseudo-t-norm

1. Introduction

The resolution of fuzzy relation equations is one of the most important and widely studied problems in the field of fuzzy sets and fuzzy systems. The majority of fuzzy inference systems can be implemented by using the fuzzy relation equations [11]. Fuzzy relation equations can also be used for processes of compression/decompression of images and videos [8].

The sup–inf composite fuzzy relation equation was first proposed by Sanchez in 1976, and since then different kinds of fuzzy relation equations have been studied by many researchers [2–5,8–13,18]. Recently, Wang and Yu [15] introduced the notion of pseudo-t-norms. Building on this, Dai and Wang [1,14] considered the fuzzy relation equations with pseudo-t-norms. Meanwhile, Han and Li [7] introduced the concept of a strong pseudo-t-norm to correct some incorrect main results in [1,14–16].

In this work, we study in detail the resolution problem of a sup-T composite finite fuzzy relation equation, where T is an infinitely distributive strong pseudo-t-norm.

Throughout this work, L denotes the real unit interval $[0, 1]$ and J always stands for any nonempty set of subscripts.

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2. Strong pseudo-t-norm

In this section, we give some definitions.

A binary operation T on L is called a pseudo-t-norm [15] if it satisfies the following conditions:

(T1) $T(1, a) = a$ and $T(0, a) = 0$ for all $a \in L$,
(T2) $a, b, c \in L$ and $b \leqslant c \Rightarrow T(a, b) \leqslant T(a, c)$.

A pseudo-t-norm T on L is said to be infinitely $\vee$-distributive [15] if it satisfies the following condition:

($T_\vee$) $a, b_j \in L (j \in J) \Rightarrow T(a, \vee_{j \in J} b_j) = \vee_{j \in J} T(a, b_j)$.

A pseudo-t-norm T on L is said to be infinitely $\wedge$-distributive [16] if it satisfies the following condition:

($T_\wedge$) $a, b_j \in L (j \in J) \Rightarrow T(a, \wedge_{j \in J} b_j) = \wedge_{j \in J} T(a, b_j)$.

A pseudo-t-norm T on L is said to be infinitely distributive [16] if it is both infinitely $\vee$-distributive and infinitely $\wedge$-distributive.

Let $A \in L^{L \times L}$. Define $I(A), T(A) \in L^{L \times L}$ as follows:

$$I(A)(a, b) := \vee\{u \in L \mid A(a, u) \leqslant b\},$$
$$T(A)(a, b) := \wedge\{u \in L \mid A(a, u) \geqslant b\},$$

where $a, b \in L$. It is tacitly assumed that $\vee\emptyset = 0$ and $\wedge\emptyset = 1$.

Theorem 2.1. *If T is an infinitely $\vee$-distributive pseudo-t-norm on L, then the following conditions are equivalent:*

(1) $T(a, c) \leqslant b \Leftrightarrow c \leqslant I(T)(a, b)$ *for all* $a, b, c \in L$;
(2) $T(a, 0) = 0$ *for all* $a \in L$.

Proof. (1) $\Rightarrow$ (2) For any $a \in L$, we have $0 \leqslant I(T)(a, 0)$. Using (1), we obtain $T(a, 0) \leqslant 0$, i.e., $T(a, 0) = 0$.

(2) $\Rightarrow$ (1) Let $a, b, c \in L$. If $T(a, c) \leqslant b$, then $I(T)(a, b) = \vee\{u \in L \mid T(a, u) \leqslant b\} \geqslant c$. Conversely, suppose $I(T)(a, b) \geqslant c$. Using (2), $0 \in \{u \in L \mid T(a, u) \leqslant b\} \neq \emptyset$. By (T2) and ($T_\vee$), $T(a, c) \leqslant T(a, I(T)(a, b)) = T(a, \vee\{u \in L \mid T(a, u) \leqslant b\}) = \vee\{T(a, u) \mid T(a, u) \leqslant b\} \leqslant b$. $\square$

A pseudo-t-norm T on L is said to be strong [7] if it satisfies the following condition:

(T3) $T(a, 0) = 0$ for all $a \in L$.

It is obvious that t-norms and weak t-norms [6] are the particular cases of strong pseudo-t-norms. And there exists an infinitely $\vee$-distributive pseudo-t-norm that is not strong (see the pseudo-t-norm T_M in [15], for instance).

Example 2.1. Put

$$T(a, b) = \begin{cases} b, & a = 1, \\ 0, & a = 0, \\ 0, & 0 < a < 1, b = 0, \\ 1, & 0 < a < 1, b > 0, \end{cases}$$

where $a, b \in L$. Then T is an infinitely $\vee$-distributive strong pseudo-t-norm on L. However, it is not infinitely $\wedge$-distributive on L, since $T(a, \wedge\{b \in L \mid b > 0\}) = T(a, 0) = 0 \neq 1 = \wedge\{T(a, b) \mid b > 0\}$ for $0 < a < 1$.

Example 2.2 (*Wang and Yu [15]*). Let

$$T_W(a, b) = \begin{cases} b, & a = 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b \in L$. Then T_W is an infinitely distributive strong pseudo-t-norm on L.

Example 2.3 (*Yager [17]*). Let

$$T_Y(a,b) = \begin{cases} b^{1/a}, & a \cdot b > 0, \\ 0, & a \cdot b = 0, \end{cases}$$

where $a, b \in L$. Then T_Y is also an infinitely distributive strong pseudo-t-norm on L. Moreover,

$$I(T_Y)(a,b) = \begin{cases} 1, & a = 0, \\ 0, & a > 0, b = 0, \\ b^a, & a > 0, b > 0, \end{cases}$$

$$T(T_Y)(a,b) = \begin{cases} 0, & b = 0, \\ b^a, & b > 0, \end{cases}$$

where $a, b \in L$. Here we notice that $T_Y(a, 1) \equiv a$ does not hold.

From now on, T denotes any given infinitely distributive strong pseudo-t-norm on L.

3. Solvability of equation

In this section, we first give a criterion for the existence of the solution.

We denote by $L^{m\times n}$ and L^n the set of all $m \times n$ matrices over L and the set of all column vectors of order n over L, respectively. For any positive integer n, $\underline{n}$ always indicates the set $\{1, 2, \ldots, n\}$.

Given two matrices $A = (a_{ik}) \in L^{m\times n}$ and $B = (b_{kj}) \in L^{n\times l}$, the sup-$T$ *composition* $A \circ_T B \in L^{m\times l}$ of A and B is defined by

$$A \circ_T B := (\vee_{k\in\underline{n}} T(a_{ik}, b_{kj})).$$

Lemma 3.1. *If $A \in L^{m\times n}$ and $C_1, C_2 \in L^{n\times l}$ with $C_1 \leqslant C_2$, then $A \circ_T C_1 \leqslant A \circ_T C_2$.*

Proof. It follows immediately from the definitions. □

The present work deals with the fuzzy relation equation $A \circ_T X = B$, i.e.,

$$\vee_{j\in\underline{n}} T(a_{ij}, x_j) = b_i, \qquad i \in \underline{m}, \tag{1}$$

where $A = (a_{ij}) \in L^{m\times n}$ and $B = (b_i) \in L^m$ are given, but $X = (x_j) \in L^n$ is unknown. We denote by $\mathcal{X}$ the set of all solutions of Eq. (1). Eq. (1) is said to be *solvable* in L when $\mathcal{X} \neq \emptyset$.

Put $\overline{X} = (\overline{x}_j) \in L^n$, where

$$\overline{x}_j = \wedge_{i\in\underline{m}} I(T)(a_{ij}, b_i), \qquad j \in \underline{n}. \tag{2}$$

Theorem 3.1. *If Eq. (1) is solvable in L, then $\overline{X}$ is the greatest solution of it.*

Proof. If $\mathcal{X} \neq \emptyset$ and $X = (x_j) \in \mathcal{X}$, then for any $i \in \underline{m}$ and $j \in \underline{n}$ we have $T(a_{ij}, x_j) \leqslant b_i$. Hence, $x_j \leqslant I(T)(a_{ij}, b_i)$, and so $x_j \leqslant \wedge_{i\in\underline{m}} I(T)(a_{ij}, b_i) = \overline{x}_j$. By Theorem 2.1, $T(a_{ij}, x_j) \leqslant T(a_{ij}, \overline{x}_j) \leqslant T(a_{ij}, I(T)(a_{ij}, b_i)) \leqslant b_i$. Thus $b_i = \vee_{j\in\underline{n}} T(a_{ij}, x_j) \leqslant \vee_{j\in\underline{n}} T(a_{ij}, \overline{x}_j) \leqslant b_i$, i.e., $\vee_{j\in\underline{n}} T(a_{ij}, \overline{x}_j) = b_i$. Therefore, $\overline{X}$ is the greatest element of $\mathcal{X}$. □

Theorem 3.1 gives a necessary and sufficient condition for the existence of the solution of Eq. (1). And Theorem 3.1 also holds for any infinitely $\vee$-distributive strong pseudo-t-norm T.

We next consider the solution set of Eq. (1) in the case that $m = n = 1$.

Theorem 3.2. *Let $a, b \in L$. If the equation $T(a, x) = b$ is solvable in L, then its solution set is the interval $[T(T)(a, b), I(T)(a, b)]$ in L.*

Proof. By Theorem 3.1, $I(T)(a, b)$ is the greatest solution. If $x \in L$ is a solution, then $x \in \{u \in L \mid T(a, u) \geqslant b\} \neq \emptyset$. Hence, $T(T)(a, b) \leqslant x$. By ($T_\wedge$), $b = T(a, x) \geqslant T(a, T(T)(a, b)) = T(a, \wedge\{u \in L \mid T(a, u) \geqslant b\}) = \wedge\{T(a, u) \in L \mid T(a, u) \geqslant b\} \geqslant b$, i.e., $T(a, T(T)(a, b)) = b$. Thus $T(T)(a, b)$ is the least solution. By Lemma 3.1, we can see that the interval $[T(T)(a, b), I(T)(a, b)]$ in L is the solution set of the equation. □

4. Solution set of equation

In this section, we assume that Eq. (1) is solvable, and show that the solution set is represented by the greatest solution and a finite number of minimal solutions.

Given a solution $X = (x_j) \in \mathcal{X}$ of Eq. (1), we put

$$J_i(X) := \{j \in \underline{n} \mid T(a_{ij}, x_j) = b_i\}, \qquad i \in \underline{m}, \tag{3}$$

$$J(X) := J_1(X) \times J_2(X) \times \cdots \times J_m(X). \tag{4}$$

Then $J_i(X) \neq \emptyset$ for all $i \in \underline{m}$, and so $J(X) \neq \emptyset$.

For every $f = (f_i) \in J(X)$ and for every $j \in \underline{n}$, we put

$$I_j^{(f)} := \{i \in \underline{m} \mid f_i = j\}, \tag{5}$$

$$x_j^{(f)} := \begin{cases} \vee_{i \in I_j^{(f)}} T(T)(a_{ij}, b_i), & I_j^{(f)} \neq \emptyset, \\ 0, & I_j^{(f)} = \emptyset. \end{cases} \tag{6}$$

Thus we can make a vector $X^{(f)} := (x_j^{(f)}) \in L^n$. Finally, we construct a finite poset

$$\mathcal{F}(X) := \{X^{(f)} \mid f \in J(X)\}. \tag{7}$$

Lemma 4.1. *If $X \in \mathcal{X}$ and $f \in J(X)$, then $X^{(f)} \leqslant X$.*

Proof. Suppose $X^{(f)} \nleqslant X$. Then there exists a $j \in \underline{n}$ such that $x_j^{(f)} \nleqslant x_j$. Hence, $x_j^{(f)} \neq 0$ and so $\vee_{i \in I_j^{(f)}} T(T)(a_{ij}, b_i) \nleqslant x_j$ with $I_j^{(f)} \neq \emptyset$. There is an $i \in I_j^{(f)}$ such that $T(T)(a_{ij}, b_i) \nleqslant x_j$. On the other hand, $i \in I_j^{(f)}$ implies that $j \in J_i(X)$ and $T(a_{ij}, x_j) = b_i$. Therefore, $T(T)(a_{ij}, b_i) \leqslant x_j$, which is a contradiction. □

Lemma 4.2. *If $X \in \mathcal{X}$, then $\mathcal{F}(X) \subseteq \mathcal{X}$.*

Proof. Let $f \in J(X)$. By Lemmas 3.1 and 4.1, $A \circ_T X^{(f)} \leqslant A \circ_T X = B$. It suffices to verify $A \circ_T X^{(f)} \geqslant B$. Assume $A \circ_T X^{(f)} \ngeqslant B$. Then there is a $k \in \underline{m}$ such that $\vee_{j \in \underline{n}} T(a_{kj}, x_j^{(f)}) \ngeqslant b_k$. On the other hand, $\vee_{j \in \underline{n}} T(a_{kj}, x_j) = b_k$ and $T(a_{kl}, x_l) = b_k$ for $l = f_k$. Hence, $k \in I_l^{(f)} \neq \emptyset$. Since $x_l^{(f)} = \vee_{i \in I_l^{(f)}} T(T)(a_{il}, b_i) \geqslant T(T)(a_{kl}, b_k)$, by Theorem 3.2 we have $T(a_{kl}, x_l^{(f)}) \geqslant T(a_{kl}, T(T)(a_{kl}, b_k)) = b_k$. Thus $\vee_{j \in \underline{n}} T(a_{kj}, x_j^{(f)}) \geqslant b_k$, which is a contradiction. □

Lemma 4.3. *If $X, Y \in \mathcal{X}$ with $X \leqslant Y$ and $X^{(f)} \in \mathcal{F}(X)$, then $X^{(f)} = Y^{(f)} \in \mathcal{F}(Y)$.*

Proof. Let $Y = (y_j)$. If $f = (f_i) \in J(X)$, then for any $i \in \underline{m}$, $b_i = T(a_{if_i}, x_{f_i}) \leqslant T(a_{if_i}, y_{f_i}) \leqslant b_i$, i.e., $T(a_{if_i}, y_{f_i}) = b_i$. Hence, $f_i \in J_i(Y)$ and $f = (f_i) \in J(Y)$. Therefore, $X^{(f)} = Y^{(f)} \in \mathcal{F}(Y)$. □

For any poset P, we denote by P_0 the set of all minimal elements in P.

Lemma 4.4 ([12, Lemma 2.2]). *Let Q be a nonempty subset of a poset P. If for every $p \in P$ there exists a $q \in Q$ with $q \leqslant p$, then $P_0 = Q_0$.*

Theorem 4.1. *The solution set $\mathcal{X}$ of Eq. (1) and the finite poset $\mathcal{F}(\overline{X})$ contain the same minimal elements, i.e., $\mathcal{X}_0 = \mathcal{F}(\overline{X})_0 \neq \emptyset$.*

Proof. By Lemma 4.2, $\mathcal{F}(\overline{X}) \subseteq \mathcal{X}$. If $X \in \mathcal{X}$, then $X \leqslant \overline{X}$ by Theorem 3.1. By Lemma 4.3, $X^{(f)} \in \mathcal{F}(X)$ implies $X^{(f)} = \overline{X}^{(f)} \in \mathcal{F}(\overline{X})$. Again by Lemma 4.1, $X^{(f)} \leqslant X$. Hence, $\mathcal{X}_0 = \mathcal{F}(\overline{X})_0$ by Lemma 4.4. Moreover, $\mathcal{F}(\overline{X})_0 \neq \emptyset$ since $\mathcal{F}(\overline{X})$ is a nonempty finite set. □

With respect to the solution set $\mathcal{X}$ of Eq. (1), we obtain the following representation theorem.

Theorem 4.2. *The solution set $\mathcal{X}$ of Eq.* (1) *is the union of a finite number of intervals in L^n:*

$$\mathcal{X} = \bigcup_{X_\wedge \in \mathcal{F}(\overline{X})_0} [X_\wedge, \overline{X}], \tag{8}$$

where $[X_\wedge, \overline{X}] = \{X \in L^n \mid X_\wedge \leqslant X \leqslant \overline{X}\}$ *for any* $X_\wedge \in \mathcal{F}(\overline{X})_0$.

Proof. If $X \in \mathcal{X}$, then it follows from the proof of Theorem 4.1 that there exists a $Y \in \mathcal{F}(\overline{X})$ with $Y \leqslant X$. Since $\mathcal{F}(\overline{X})$ is a finite poset, there is an $X_\wedge \in \mathcal{F}(\overline{X})_0$ with $X_\wedge \leqslant Y$. Hence, $X_\wedge \leqslant Y \leqslant X \leqslant \overline{X}$ by Theorem 3.1, and so $X \in [X_\wedge, \overline{X}]$. Therefore, $\mathcal{X} \subseteq \bigcup_{X_\wedge \in \mathcal{F}(\overline{X})_0} [X_\wedge, \overline{X}]$. The converse inclusion follows from Lemmas 4.2 and 3.1. □

Now we can derive a criterion for the uniqueness of the solution of Eq. (1).

Theorem 4.3. *Eq.* (1) *has a unique solution if and only if* $\mathcal{F}(\overline{X}) = \{\overline{X}\}$.

Proof. The necessity follows from Theorem 3.1 and Lemma 4.2. The sufficiency follows from Theorem 4.2. □

5. An algorithm for resolution

We first present an algorithm for solving Eq. (1) that is ensured by the preceding theorems:

Step 1. Compute $\overline{X}$ using (2) and check by the substitution whether $\overline{X}$ is a solution of Eq. (1). If no, then put $\mathcal{X} = \emptyset$ and stop. Otherwise, go to Step 2.
Step 2. Construct $J(\overline{X})$ using (3) and (4).
Step 3. Determine $\mathcal{F}(\overline{X})$ using (5)–(7).
Step 4. Find $\mathcal{F}(\overline{X})_0$ by pairwise comparison.
Step 5. Construct $\mathcal{X}$ using (8).

We next give a numerical example to illustrate the algorithm presented above.

Example 5.1. Let T_Y be the infinitely distributive strong pseudo-t-norm on L as in Example 2.3. We consider a fuzzy relation equation $A \circ_{T_Y} X = B$, where

$$A = \begin{pmatrix} 0.4 & 0.5 & 1 \\ 0.8 & 0.5 & 0.6 \\ 0 & 0.125 & 0.5 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.2 \\ 0.16 \\ 0.04 \end{pmatrix}.$$

We will use $I(T_Y)$ and $T(T_Y)$ in Example 2.3. Using (2), we compute $\overline{X} = (0.16^{0.8}, 0.4, 0.2)^{\mathsf{T}}$. Since $A \circ_{T_Y} \overline{X} = B$, the equation is solvable and $\overline{X}$ is the greatest solution. Using (3) and (4), we construct $J(\overline{X})$: $J_1(\overline{X}) = \{3\}$, $J_2(\overline{X}) = \{1, 2\}$, $J_3(\overline{X}) = \{3\}$; $J(\overline{X}) = \{(3, 1, 3), (3, 2, 3)\}$. Using (5)–(7), we determine $\mathcal{F}(\overline{X})$. For $f = (f_1, f_2, f_3) = (3, 1, 3)$, we have

$$\begin{aligned}
&I_1^{(f)} = \{2\}, && \overline{x}_1^{(f)} = T(T_Y)(0.8, 0.16) = 0.16^{0.8};\\
&I_2^{(f)} = \emptyset, && \overline{x}_2^{(f)} = 0;\\
&I_3^{(f)} = \{1, 3\}, && \overline{x}_3^{(f)} = T(T_Y)(1, 0.2) \vee T(T_Y)(0.5, 0.04) = 0.2^1 \vee 0.04^{0.5} = 0.2;\\
&\overline{X}^{(f)} = (0.16^{0.8}, 0, 0.2)^{\mathsf{T}}.
\end{aligned}$$

For $f = (f_1, f_2, f_3) = (3, 2, 3)$, we have

$$\begin{aligned}
&I_1^{(f)} = \emptyset, && \overline{x}_1^{(f)} = 0;\\
&I_2^{(f)} = \{2\}, && \overline{x}_2^{(f)} = T(T_Y)(0.5, 0.16) = 0.16^{0.5} = 0.4;\\
&I_3^{(f)} = \{1, 3\}, && \overline{x}_3^{(f)} = T(T_Y)(1, 0.2) \vee T(T_Y)(0.5, 0.04) = 0.2^1 \vee 0.04^{0.5} = 0.2;\\
&\overline{X}^{(f)} = (0, 0.4, 0.2)^{\mathsf{T}}.
\end{aligned}$$

Thus $\mathcal{F}(\overline{X}) = \{(0.16^{0.8}, 0, 0.2)^{\mathsf{T}}, (0, 0.4, 0.2)^{\mathsf{T}}\}$. The set of minimal solutions is $\mathcal{F}(\overline{X})_0 = \mathcal{F}(\overline{X})$. Using (8), we obtain the solution set $\mathcal{X} = \{(0.16^{0.8}, [0, 0.4], 0.2)^{\mathsf{T}}, ([0, 0.16^{0.8}], 0.4, 0.2)^{\mathsf{T}}\}$.

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