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Coarse version of the Banach–Stone theorem

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ABSTRACT

We show that if there exists a Lipschitz homeomorphism T between the nets in the Banach spaces $C(X)$ and $C(Y)$ of continuous real valued functions on compact spaces X and Y, then the spaces X and Y are homeomorphic provided $l(T) \times l(T^{-1}) < \frac{6}{5}$. By $l(T)$ and $l(T^{-1})$ we denote the Lipschitz constants of the maps T and T^{-1}. This improves the classical result of Jarosz and the recent result of Dutrieux and Kalton where the constant obtained is $\frac{17}{16}$. We also estimate the distance of the map T from the isometry of the spaces $C(X)$ and $C(Y)$.

1. Introduction

This paper deals with Banach spaces and some equivalences arising from the concepts of large scale geometry (coarse geometry) and the geometry of Banach spaces. Let us first recall the very well-known notion of a Lipschitz map and the related notion of a bi-Lipschitz map:

Definition 1.1. Let E and F be metric spaces and $T: E \mapsto F$ be a map between these spaces. Then T is

- a Lipschitz map if for some constant $M > 0$ it satisfies the inequality $d_F(Tx, Ty) \leqslant M d_E(x, y)$ for all $x, y \in E$;
- an M – bi-Lipschitz map if it is an onto map and the inequality

$$\frac{1}{M} d_E(x, y) \leqslant d_F(Tx, Ty) \leqslant M d_E(x, y)$$

holds for all $x, y \in E$.

For a Lipschitz map T we denote by $l(T) = \sup\{\frac{d_F(Tx, Ty)}{d_E(x, y)}; x \neq y\}$.

Let us now introduce two notions that are generalizations of the above in the large scale direction:

Definition 1.2. Let E and F be metric spaces and $T\colon E \mapsto F$ be a map between these spaces. Then T is a coarse Lipschitz map (Lipschitz for large distances) if for some constants $M > 0$ and $L \geqslant 0$ the inequality $d_F(Tx, Ty) \leqslant M d_E(x, y) + L$ is satisfied for all $x, y \in E$. For a coarse Lipschitz map T we denote by $l_\infty(T)$ the number $\inf_{\theta > 0} \sup_{\|x-y\| \geqslant \theta} \frac{\|Tx - Ty\|}{\|x - y\|}$ (a Lipschitz constant at ∞).

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It is worth mentioning that if T is a uniformly continuous map between Banach spaces then it is also a coarse Lipschitz map. Although this is a basic fact it is very important for a nonlinear classification of Banach spaces. For more information please see the book [1]. Another definition is a natural generalization of the notion of bi-Lipschitz map.

Definition 1.3. Let E and F be metric spaces and $T: E \mapsto F$ be a map between these spaces. T is a coarse (M, L)-quasi isometry (or just coarse quasi isometry) if it satisfies the following conditions:

- $\frac{1}{M} d_E(x, y) - L \leqslant d_F(Tx, Ty) \leqslant M d_E(x, y) + L$ for all $x, y \in E$;
- there exists $\xi > 0$ such that for every $y \in F$ there exists $x \in E$ such that $d_F(y, Tx) \leqslant \xi$. In other words $T(E)$ is ξ dense in F.

The class of maps defined above is sometimes called just quasi isometries (see [2]) however the "coarse" is added since in some contexts quasi isometries are those coarse quasi isometries for which $L = 0$ (see [9]). We shall focus on coarse quasi isometries between Banach spaces.

In our considerations we find a condition under which the existence of a coarse quasi isometry between spaces $C(X)$ and $C(Y)$ implies the existence of a homeomorphism of topological spaces X and Y. The last section of this paper is devoted to stability problems i.e. we estimate the distance of a coarse (M, L)-quasi isometry of the Banach spaces $C(X)$ and $C(Y)$ to an isometry of these spaces as $M \to 1$. In paper [4] Dutrieux and Kalton consider different kinds of nonlinear distances between Banach spaces. Among the others they consider the uniform distance between two Banach spaces E and F as well as the net distance. Let us recall the definition of both:

Definition 1.4. Let E and F be Banach spaces. By $d_u(E, F)$, $d_N(E, F)$ we denote the uniform and the net distance, respectively:

- $d_u(E, F) = \inf l_\infty(u) \times l_\infty(u^{-1})$, where the infimum is taken over all uniform homeomorphisms u between E and F;
- $d_N(E, F) = \inf l(T) \times l(T^{-1})$, where the infimum is taken over all bi-Lipschitz maps T between the nets N_E and N_F in the Banach spaces E, F, respectively.

Let us recall that a subset $N_E \subset E$ of a metric space E is called an (ε, δ) – net if every element of E is of a distance less then ε to some element of N_E. Moreover every two elements of N_E are of the distance at least δ. It is easy to observe that $d_u(E, F) \geqslant d_N(E, F)$. In the mentioned paper of Dutrieux and Kalton they work with the Gromov–Hausdorff distance $d_{GH}(E, F)$ between Banach spaces E and F. They show that $d_N(E, F) \geqslant d_{GH}(E, F) + 1$. This fact and their result that the inequality $d_{GH}(C(X), C(Y)) < \frac{1}{16}$ implies the existence of a homeomorphism of compact spaces X and Y give us that X and Y are also homeomorphic when $d_N(C(X), C(Y)) < \frac{17}{16}$. In our paper we improve the constant to $\frac{6}{5}$. Please keep in mind that we consider only the net distance.

Let us now discuss the connection between the net distance and the notion of a coarse quasi isometry. We shall start with the following fact:

Fact 1.5. *Let us consider a coarse (M, L)-quasi isometry $T: A \mapsto B$ from a ξ_E dense set in Banach spaces E onto a ξ_F dense set in F. Then there exists a map $\widetilde{T}: E \mapsto F$, which is a bijective coarse $(M, (4M^2 + 3)L + 4\xi_F + 2M\xi_E)$-quasi isometry and $\|\widetilde{T}x - Tx\| \leqslant (2M^2 + 2)L + 2\xi_F + M\xi_E$ for all $x \in A$.*

Proof. Consider N_E a maximal $ML + M\eta$ separated set in A where $\eta > 0$ is arbitrary. Let us first notice that N_E is an $(\varepsilon_E, \delta_E)$ net in E where $\varepsilon_E = ML + M\eta + \xi_E$ and $\delta_E = ML + M\eta$. Obviously $T|N_E$ is a bijection between the net N_E and $N_F = T(N_E)$ which is also an $(\varepsilon_F, \delta_F)$ net in F where $\varepsilon_F = M^2 L + M^2 \eta + L + \xi_F$ (since N_E is $ML + M\eta$ dense in A and T is onto) and $\delta_F = \eta$. Let us enumerate the elements of N_E that is $N_E = \{x_\alpha\}_{\alpha \in \tau}$. Modifying balls around the points of N_E and N_F we obtain families $(E_\alpha)_{\alpha \in \tau}$, $(F_\alpha)_{\alpha \in \tau}$ of subsets of E and F, respectively such that:

(i) $B(x_\alpha, \frac{\delta_E}{2}) \subset E_\alpha$, $B(Tx_\alpha, \frac{\delta_E}{2}) \subset F_\alpha$;
(ii) $E_\alpha \subset B(x_\alpha, \varepsilon_E)$ and $F_\alpha \subset B(Tx_\alpha, \varepsilon_F)$;
(iii) $E_\alpha \cap E_\beta = \emptyset$, $F_\alpha \cap F_\beta = \emptyset$ for $\beta \neq \alpha$;
(iv) $\bigcup_{\alpha \in \tau} E_\alpha = E$ and $\bigcup_{\alpha \in \tau} F_\alpha = F$.

Since N_E and N_F are of the same cardinality then also all open sets in E and F have the same cardinalities. Hence for every $\alpha \in \tau$ we can find a bijection $\widetilde{T}_\alpha: E_\alpha \mapsto F_\alpha$ such that $\widetilde{T}_\alpha(x_\alpha) = Tx_\alpha$. Setting $\widetilde{T} = \bigcup_{\alpha \in \tau} \widetilde{T}_\alpha$ we obtain the desired map. Indeed let us take $x, y \in E$. Choose E_α and E_β such that $x \in E_\alpha$ and $y \in E_\beta$. We have then

$$\frac{1}{M} \|x_\alpha - x_\beta\| - L \leqslant \|\widetilde{T} x_\alpha - \widetilde{T} x_\beta\| \leqslant M \|x_\alpha - x_\beta\| + L.$$

Since the partitions satisfy conditions (i) and (ii) we obtain:

$$\|\widetilde{T}x_\alpha - \widetilde{T}x\| \leqslant \varepsilon_F,$$
$$\|\widetilde{T}x_\beta - \widetilde{T}y\| \leqslant \varepsilon_F,$$
$$\|x_\alpha - x\| \leqslant \varepsilon_E,$$
$$\|x_\beta - y\| \leqslant \varepsilon_E.$$

Combining all the above inequalities we show that $\widetilde{T}$ is a bijective coarse $(M, L + 2M\varepsilon_E + 2\varepsilon_F)$-quasi isometry. Consider $x \in A$. Obviously there exists $\alpha \in \tau$ such that $x \in E_\alpha$. From the definition of $\widetilde{T}$ we know that $\widetilde{T}x \in F_\alpha \subset B(Tx_\alpha, \varepsilon_F)$ and $\widetilde{T}x_\alpha = Tx_\alpha$. This way we obtain the inequalities:

$$\|\widetilde{T}x - \widetilde{T}x_\alpha\| \leqslant \varepsilon_F,$$
$$\|Tx - Tx_\alpha\| \leqslant M\varepsilon_E + L.$$

Hence $\|\widetilde{T}x - Tx\| \leqslant M\varepsilon_E + \varepsilon_F + L$. Setting $\eta = \frac{\xi_F}{2M^2}$ the proof is finished. □

At this moment it is worth mentioning that from the definition of $d_N(E,F)$ it follows that for every $M > 0$ such that $M^2 > d_N(E,F)$ there are nets N_E and N_F in E and F, respectively and a bi-Lipschitz map T between them such that $M^2 = l(T) \times l(T^{-1}) > d_N(E,F)$. However we cannot be sure that $l(T) = l(T^{-1})$. In order to obtain that consider the map $\widetilde{T}x = \sqrt{\frac{l(T^{-1})}{l(T)}}Tx$. It is a bi-Lipschitz map of nets (different ones) in E and F such that $l(\widetilde{T}) = l(\widetilde{T}^{-1}) = M$. From this remark and Fact 1.5 we can easily deduce

Fact 1.6. *Let E and F be Banach spaces. If $d_N(E,F) < M^2$ then there exists a bijective coarse (M,L)-quasi isometry between E and F for some constant $L \geqslant 0$.*

Proof. Indeed if $d_N(E,F) < M^2$ then there exists a bijective map $T : N_E \mapsto N_F$ between nets in Banach spaces E and F, respectively such that

$$\frac{1}{M}\|x - y\| \leqslant \|Tx - Ty\| \leqslant M\|x - y\|$$

for all $x, y \in N_E$. From Fact 1.5 there exists $\widetilde{T}$ which is the desired coarse (M,L)-quasi isometry. □

On the other hand it is not difficult to show that if there exists a coarse (M,L)-quasi isometry between spaces E and F then $d_N(E,F) \leqslant M^2$.

The following fact is another conclusion from Fact 1.5:

Fact 1.7. *Let us consider a coarse (M,L)-quasi isometry $T : E \mapsto F$ of Banach spaces E and F where ξ is such that TE is ξ dense in F. Then there exists a bijective coarse $(M, (4M^2 + 3)L + 4\xi)$-quasi isometry $\widetilde{T} : E \mapsto F$ such that $\|\widetilde{T}x - Tx\| \leqslant (2M^2 + 2)L + 2\xi$ for all $x \in E$.*

From now on we only consider bijective coarse (M,L)-quasi isometries. Indeed from the above fact we know that whenever we have a coarse (M,L)-quasi isometry we can define a new bijective coarse quasi isometry changing only the constant L. Moreover it is not more than $10L + 2\xi$ from the original coarse (M,L)-quasi isometry if $M < 2$ which is going to be our case in further considerations. Obviously ξ is as in Definition 1.3. Since we are concerned mainly with large distances then both maps, the original one and the corrected bijective version are no different from the large scale perspective. The only thing that matters is the constant M which is unchanged.

2. The coarse version of the Banach–Stone theorem

As it was mentioned in Section 1 we investigate Banach spaces of continuous real valued functions defined on compact spaces. For a compact spaces X we denote such a Banach space by $C(X)$. As usual $C(X)$ is endowed with the *sup* norm. Our main goal is to consider maps between such spaces that are small perturbations of isometries (from the large scale perspective) i.e. we consider coarse (M,L)-quasi isometries where $M \to 1$.

Theorem 2.1. *Let X and Y be compact spaces and $C(X)$, $C(Y)$ Banach spaces of continuous real valued functions on X and Y, respectively. Let $T : C(X) \mapsto C(Y)$ be a bijective coarse (M,L)-quasi isometry such that $T(0) = 0$. Then for every $M < \sqrt{1.2}$ there is a homeomorphism $\varphi : X \mapsto Y$ such that for every $f \in C(X)$, $x \in X$*

$$\big|\big|Tf\big(\varphi(x)\big)\big| - \big|f(x)\big|\big| \leqslant 5\big(M^2 - M\big)\|f\| + \Delta.$$

The constant Δ depends only on M and L. Moreover, for $L = 0$, we have $\Delta = 0$.

The proof of this theorem is a modified technique of K. Jarosz from [6]. Without loss of generality we can assume that L is so that both T and T^{-1} are coarse (M, L)-quasi isometries. We also use the notion of Moore–Smith convergence when dealing with topology of general topological spaces. Σ will always denote a directed set and whenever we write $a_\sigma \to a$ we always mean $\lim_{\sigma\in\Sigma} a_\sigma = a$. We give two special definitions, and then state and prove several facts which will be used to prove the theorem.

Definition 2.2. $(f_\sigma^m)_{\sigma\in\Sigma} \subset C(X)$ is the m-peak sequence at $x \in X$, for some directed set Σ if

- $\|f_\sigma^m\| = |f_\sigma^m(x)| = m$ for all $\sigma \in \Sigma$,
- $\lim_{\sigma\in\Sigma} f_\sigma^m|(X \setminus U) \equiv 0$ uniformly for all open neighborhoods U of x.

The set of m-peak sequences at x we denote by $P_m^X(x)$.

Definition 2.3. Let $D > 0$ and $m > 0$. We define the following:

$$S_m^D(x) = \left\{y \in Y;\ \exists\left(f_\sigma^m\right)_{\sigma\in\Sigma} \in P_m^X(x)\ \exists y_\sigma \to y\ \forall\sigma \in \Sigma\ Tf_\sigma^m\left(y_\sigma^m\right) \geqslant Dm \text{ and } T\left(-f_\sigma^m\right)\left(y_\sigma^m\right) \leqslant -Dm\right\}$$

and analogically

$$S_m^{-D}(y) = \left\{x \in X;\ \exists\left(g_\sigma^m\right)_{\sigma\in\Sigma} \in P_m^Y(y)\ \exists x_\sigma^m \to x\ \forall\sigma \in \Sigma\ T^{-1}g_\sigma^m\left(x_\sigma^m\right) \geqslant Dm \text{ and } T^{-1}\left(-g_\sigma^m\right)\left(x_\sigma^m\right) \leqslant -Dm\right\}.$$

Fact 2.4. *Let us consider D such that $D < \frac{2}{M} - M$. There exists m_0 (depending on M, L and D) such that for all $m > m_0$ we have $S_m^D(x) \neq \emptyset$ for all $x \in X$. Moreover if $L = 0$ then $m_0 = 0$.*

Proof. Let us take any $(\widetilde{f}_\sigma^m)_{\sigma\in\Sigma} \in P_m^X(x)$ such that for all $\sigma \in \Sigma$ $\widetilde{f}_\sigma^m(x) = m$. We have

$$\forall\sigma \in \Sigma \quad \left\|T\widetilde{f}_\sigma^m - T\left(-\widetilde{f}_\sigma^m\right)\right\| \geqslant \frac{2}{M}m - L.$$

Hence $\forall\sigma \in \Sigma$ there exists $y_\sigma^m \in Y$ such that $|T\widetilde{f}_\sigma^m(y_\sigma^m) - T(-\widetilde{f}_\sigma^m)(y_\sigma^m)| \geqslant \frac{2}{M}m - L$. Let us observe that numbers $T\widetilde{f}_\sigma^m(y_\sigma^m)$ and $T(-\widetilde{f}_\sigma^m)(y_\sigma^m)$ must be of different signs. Assume the contrary. Since $\|T(\pm\widetilde{f}_\sigma^m)\| \leqslant Mm + L$ we have $Mm + L \geqslant \frac{2}{M}m - L$ which is impossible for m large enough provided $\frac{2}{M} > M$ (that is if $M < \sqrt{2}$). We can and we do assume that $\forall\sigma \in \Sigma$ $T\widetilde{f}_\sigma^m(y_\sigma^m) \geqslant 0$ or $\forall\sigma \in \Sigma$ $T\widetilde{f}_\sigma^m(y_\sigma^m) \leqslant 0$. We define $\forall\sigma \in \Sigma$ $f_\sigma^m = \widetilde{f}_\sigma^m$ if $T\widetilde{f}_\sigma^m(y_\sigma^m) \geqslant 0$ or $\forall\sigma \in \Sigma$ $f_\sigma^m = -\widetilde{f}_\sigma$ otherwise. We have $Tf_\sigma^m(y_\sigma^m) - T(-f_\sigma^m)(y_\sigma^m) \geqslant \frac{2}{M}m - L$. Because $\|T(\pm f_\sigma^m)\| \leqslant Mm + L$ then

$$Tf_\sigma^m\left(y_\sigma^m\right) \geqslant \left(\frac{2}{M} - M\right)m - 2L,$$

$$T\left(-f_\sigma^m\right)\left(y_\sigma^m\right) \leqslant -\left(\frac{2}{M} - M\right)m + 2L.$$

By compactness of Y we can assume that $y_\sigma^m \to y \in Y$. Therefore for every $D < \frac{2}{M} - M$ there exists such m_0 (depending on D, M and L) that $S_m^D(x) \neq \emptyset$ for $m > m_0$. Let us notice that for $L = 0$ we have $m_0 = 0$. □

Fact 2.5. *For every $m > m_0$ and f such that $m \geqslant \|f\|$ we have*

$$\left|Tf(y)\right| \leqslant \left|f(x)\right| + \varepsilon(M)m + L,$$

where $\varepsilon(M) = 2M - 1 - D$ and y is any element of $S_m^D(x)$.

Proof. Consider $(f_\sigma^m)_{\sigma\in\Sigma} \in P_m^X(x)$ and the corresponding sequence $y_\sigma^m \to y$ as in the definition of $S_m^D(x)$. We have that $\liminf_{\sigma\in\Sigma} \||f_\sigma^m| + |f|\| = m + |f(x)|$. Since $\max\{\|f + f_\sigma^m\|, \|f - f_\sigma^m\|\} = \||f_\sigma^m| + |f|\|$ we get:

$$\liminf_{\sigma\in\Sigma} \max\left\{\left\|Tf - T\left(-f_\sigma^m\right)\right\|, \left\|Tf - Tf_\sigma^m\right\|\right\} \leqslant Mm + (M-1)\left|f(x)\right| + \left|f(x)\right| + L.$$

Using the facts that $|Tf(y_\sigma^m)| + Dm \leqslant |Tf(y_\sigma^m) - T(\lambda f_\sigma^m)(y_\sigma^m)|$ for some $\lambda \in \{-1, 1\}$, $\|f\| \leqslant m$ and $y_\sigma^m \to y$ we obtain:

$$\begin{aligned} Dm + \left|Tf(y)\right| &\leqslant \liminf_{\sigma\in\Sigma} \max\left\{\left|Tf\left(y_\sigma^m\right) - T\left(-f_\sigma^m\right)\left(y_\sigma^m\right)\right|, \left|Tf\left(y_\sigma^m\right) - Tf_\sigma^m\left(y_\sigma^m\right)\right|\right\} \\ &\leqslant \liminf_{\sigma\in\Sigma} \max\left\{\left\|Tf - T\left(-f_\sigma^m\right)\right\|, \left\|Tf - Tf_\sigma^m\right\|\right\} \\ &\leqslant Mm + (M-1)\left|f(x)\right| + \left|f(x)\right| + L \leqslant (2M-1)m + \left|f(x)\right| + L. \qquad \square \end{aligned}$$

Fact 2.6. *Assume that $1-\varepsilon(M)M-\varepsilon(M)>0$ where $\varepsilon(M)=2M-1-D$. Then for every $\delta>0$ there exists $m_1\geqslant m_0$ such that if $x_0\in X$, $m>m_1$ and $Mm+L+\delta>k\geqslant Mm+L$ then $S_k^{-D}(y)=\{x_0\}$ provided $y\in S_m^D(x_0)$. Moreover the choice of m_1 depends on M, L, D and δ only and if $L=0$ then m_1 can be as close to 0 as we wish.*

Proof. Let us assume the contrary that $x_1\in S_k^{-D}(y)$ and $x_1\neq x_0$. Consider f such that $f(x_0)=0$ and $f(x_1)=m=\|f\|$. Applying Fact 2.5 to T and T^{-1} we obtain

$$m-\varepsilon(M)k-L=\bigl|f(x_1)\bigr|-\varepsilon(M)k-L\leqslant\bigl|Tf(y)\bigr|\leqslant\varepsilon(M)m+\bigl|f(x_0)\bigr|+L=\varepsilon(M)m+L$$

if only $k\geqslant Mm+L\geqslant\|Tf\|$. After rearranging we have

$$m-\varepsilon(M)(m+Mm+L+\delta)\leqslant m-\varepsilon(M)(m+k)\leqslant 2L.$$

This is equivalent to the condition

$$\bigl(1-\varepsilon(M)M-\varepsilon(M)\bigr)m\leqslant 2L+\varepsilon(M)L+\varepsilon(M)\delta.$$

Therefore if $1-\varepsilon(M)M-\varepsilon(M)>0$ we get a contradiction for sufficiently large m. If $L=0$ then if we want to obtain m_1 as small as we wish it is enough to consider δ small enough. $\square$

Let us assume from now on that M and D are such that D satisfies the condition from Fact 2.4 and the inequality $1-\varepsilon(M)M-\varepsilon(M)>0$ holds. Consider $m>Mm_1+L$. Since the construction of S_m^{-D} and S_m^D is symmetric and involves exactly the same constants we can conclude from the above fact that $\bigcup_{x\in X}S_m^D(x)=Y$ and $\bigcup_{y\in Y}S_m^{-D}(y)=X$. Moreover for every $x\in X$ and $y\in Y$ we have that $S_m^D(x)$ and $S_m^{-D}(y)$ are singletons. Hence we can clearly see that $S_k^D(x)=S_l^D(x)$ provided $k,l\geqslant Mm_1+L$ and $|k-l|<\delta$. However locally constant functions are just constant functions hence $S_k^D(x)=S_l^D(x)$ for $k,l>Mm_1+L$. Similarly $S_k^{-D}(y)=S_l^{-D}(y)$ for $k,l\geqslant Mm_1+L$ and all $y\in Y$. Therefore we can define $\varphi(x)=S_m^D(x)$ and $\psi(y)=S_m^{-D}(y)$ where $m>Mm_1+L$. Another obvious conclusion from Fact 2.6 is that $\varphi\circ\psi\equiv id_Y$ and $\psi\circ\varphi\equiv id_X$. If $L=0$ then $S_k^D(x)=S_l^D(x)$ for all $k,l>0$.

Fact 2.7. *There exists $m_2\geqslant 0$ such that for every $\|f\|\geqslant m_2$ we have*

$$\bigl|f(x)\bigr|-\varepsilon(M)M\|f\|-\bigl(\varepsilon(M)+1\bigr)L\leqslant\bigl|Tf\bigl(\varphi(x)\bigr)\bigr|\leqslant\bigl|f(x)\bigr|+\varepsilon(M)\|f\|+L.$$

If $L=0$ then we can take $m_2=0$.

Proof. Let us notice that the inequality

$$\bigl|Tf\bigl(\varphi(x)\bigr)\bigr|\leqslant\bigl|f(x)\bigr|+\varepsilon(M)\|f\|+L$$

holds for f such that $\|f\|\geqslant Mm_1+L$. Indeed it follows from Fact 2.5 if we take $m=\|f\|$. In this case $S_m^D(x)=\{\varphi(x)\}$. Similarly, applying Fact 2.5 to T^{-1} and $m=\|Tf\|$ we get

$$\begin{aligned}\bigl|f(x)\bigr|=\bigl|T^{-1}(Tf)\bigl(\psi\bigl(\varphi(x)\bigr)\bigr)\bigr|&\leqslant\bigl|Tf\bigl(\varphi(x)\bigr)\bigr|+\varepsilon(M)\|Tf\|+L\\&\leqslant\bigl|Tf\bigl(\varphi(x)\bigr)\bigr|+\varepsilon(M)M\|f\|+\bigl(\varepsilon(M)+1\bigr)L.\end{aligned}$$

It holds if $\|Tf\|\geqslant Mm_1+L$ which is true if $\|f\|\geqslant M(Mm_1+L)+L$. In order to finish the proof put $m_2=M(Mm_1+L)+L$. $\square$

Fact 2.8. *φ is a continuous function from X to Y.*

Proof. Assume the contrary that there is a sequence $(x_\sigma)_{\sigma\in\Sigma}$ converging to $x_0\in X$ such that $\lim_{\sigma\in\Sigma}\varphi(x_\sigma)\neq\varphi(x_0)$. By compactness of Y we can assume that $\lim_{\sigma\in\Sigma}\varphi(x_\sigma)=y\neq\varphi(x_0)$. Consider f such that $\|f\|=f(\psi(y))=m$ and $f(x_0)=0$. It can be done since $\psi(y)\neq x_0=\psi(\varphi(x_0))$. Let us assume that $m>m_2$. Then Fact 2.7 applied to T and x_σ gives us

$$\bigl|Tf(y)\bigr|\leqslant\bigl|f(x_0)\bigr|+\varepsilon(M)\|f\|+L.$$

On the other hand by applying Fact 2.7 once again to T^{-1} we obtain

$$\bigl|f\bigl(\psi(y)\bigr)\bigr|-\varepsilon(M)\|Tf\|-L\leqslant\bigl|Tf(y)\bigr|.$$

Hence combining the above inequalities and the fact that $\|Tf\|\leqslant Mm+L$ we get

$$m-\varepsilon(M)m-\varepsilon(M)Mm\leqslant 2L+\varepsilon(M)L.$$

It is obviously impossible if $1-\varepsilon(M)-\varepsilon(M)M>0$ provided m is large enough. $\square$

Proof of Theorem 2.1. We are ready to prove Theorem 2.1. Assume that $M<\sqrt{1.2}$. Let us consider $D=\frac{2}{M}-M-(M-1)^2<\frac{2}{M}-M$ (D is positive). Therefore spaces X and Y are homeomorphic if $1-\varepsilon(M)M-\varepsilon(M)>0$ where $\varepsilon(M)=2M-1-D$.

The simple analysis gives us that $D \geqslant 4 - 3M$ for $1 \leqslant M \leqslant \sqrt{1.2}$ hence $\varepsilon(M) \leqslant 5(M-1)$. Using that estimation we see that the inequality

$$6 - 5M^2 \geqslant 1 - \varepsilon(M)M - \varepsilon(M) > 0$$

holds for $M^2 < 1.2$. In order to complete the proof of Theorem 2.1 let us set $\Delta = \max\{(5M-4)L, \max\{||Tf(\varphi(x))| - |f(x)||;\ \|f\| \leqslant m_2\}\} \leqslant (M+1)m_2 + L$.

Obviously for $L = 0$ also $m_2 = 0$ hence $\Delta = 0$. $\square$

This way we proved that whenever $d_N(C(X), C(Y)) < \frac{6}{5}$ then the compact spaces X and Y are homeomorphic which improves the constant that follows from the paper of Dutrieux and Kalton. Since $d_N(C(X), C(Y)) \leqslant d_{BM}(C(X), C(Y))$ (the Banach–Mazur distance) it is well known (see [3]) that the constant obtained cannot be more than 2. However it is unknown if 2 is the best we can obtain in the nonlinear case (even for bi-Lipschitz maps). We discuss this problem in the last section.

3. Stability of coarse quasi isometries between function spaces

In this section we estimate the distance of a coarse (M, L)-quasi isometry between function spaces to the isometry between these spaces as $M \to 1$. At this point it is worth mentioning that for $M = 1$ the problem is a classical Hyers–Ulam problem. The result of Gevirtz [5] shows that if there exists a bijective coarse $(1, L)$-quasi isometry (L-isometry in short) between general Banach spaces then there exists isometry of these spaces which is no more than $5L$ from the original map (the optimal distance from the isometry is $2L$ which was shown by Omladič and Šemrl [7]). Let us notice that these results and the classical Banach–Stone theorem give us Theorem 2.1 for $M = 1$. Moreover the results of this section easily follows. For more information see the survey paper of Rassias [8].

In all the further considerations of this Section we assume that $1 < M < \sqrt{1.2}$ and that $D = \frac{2}{M} - M - (M-1)^2$. In order to define the isometry between function spaces which is close to the original coarse quasi isometry we need to define a continuous function $\lambda_m : X \mapsto \{-1, 1\}$ for all $m > m_2$. The definition requires the analysis of the proof of Fact 2.4. More precisely we show that there exists such a sequence $\{f_\sigma^m\}_{\sigma\in\Sigma} \in P_m^X(x)$ for every $m > m_2 \geqslant m_0$ and $x \in X$ that shows that $S_m^D(x) \neq \emptyset$. From now we assume that for a given x and m we have chosen such a sequence which is going to be denoted by $\{f_\sigma^m\}_{\sigma\in\Sigma}$ and the corresponding sequence $\{y_\sigma^m\}_{\sigma\in\Sigma}$ such that $y_\sigma^m \to y \in S_m^D(x) = \{\varphi(x)\}$. We define $\lambda_m(x) = \operatorname{sign} f_\sigma^m(x)$. Let us notice that the function is well defined since from the construction of $\{f_\sigma^m\}_{\sigma\in\Sigma}$ it follows that the value of $\operatorname{sign} f_\sigma^m(x)$ does not depend on the choice of $\sigma \in \Sigma$. It also follows from the construction that $\forall \sigma \in \Sigma\ Tf_\sigma^m(y_\sigma^m) > 0$ and $T(-f_\sigma^m)(y_\sigma^m) < 0$. Hence the sign of $T\lambda f_\sigma^m(y_\sigma^m)$ is the same as the sign of $\lambda_m(x)\lambda f_\sigma^m(x)$ where $\lambda \in \{-1, +1\}$. The following fact shows that the similar property holds for all functions $f \in C(X)$ not only for f_σ^m.

Fact 3.1. *Assume that $|f(x)| > 10(M-1)\|f\|$ and let $\|f\| = m$. Then for sufficiently large $m > m_3$ (which depends on M and L only), the sign of $Tf(\varphi(x))$ is the same as the sign of $\lambda_m(x)f(x)$. If $L = 0$ then $m_3 = 0$.*

Proof. It is enough to show that if the sign of $f(x)$ is the same as the sign of $\lambda f_\sigma^m(x)$ ($\lambda \in \{-1, 1\}$) then the signs of $Tf(\varphi(x))$ and $T(\lambda f_\sigma^m)(y_\sigma^m)$ are the same as well (for some $\sigma \in \Sigma$). Assume the contrary. Then for all $\sigma \in \Sigma$ we have $|Tf(\varphi(x)) - T(\lambda f_\sigma^m)(y_\sigma^m)| = |Tf(\varphi(x))| + |T(\lambda f_\sigma^m)(y_\sigma^m)|$. Therefore

$$\begin{aligned}\left|Tf\left(\varphi(x)\right)\right| + Dm &\leqslant \liminf_{\sigma\in\Sigma}\left|Tf\left(y_\sigma^m\right) - T\left(\lambda f_\sigma^m\right)\left(y_\sigma^m\right)\right| \\ &\leqslant \liminf_{\sigma\in\Sigma} M\left\|f - \lambda f_\sigma^m\right\| + L \leqslant Mm + L.\end{aligned}$$

Hence $|Tf(\varphi(x))| \leqslant (M-D)m + L$. Because $D \geqslant -3M + 4$ for $M < \sqrt{1.2}$ then $|Tf(\varphi(x))| \leqslant 4(M-1)m + L$. By Theorem 2.1 we obtain

$$\left|f(x)\right| - 5M(M-1)m - \Delta \leqslant \left|Tf\left(\varphi(x)\right)\right| \leqslant 4(M-1)m + L.$$

Hence $10(M-1)\|f\| < |f(x)| \leqslant (5M+4)(M-1)m + L + \Delta$. After rearranging we get $(6-5M)(M-1)m < L + \Delta$. Obviously if $1 < M < \sqrt{1.2}$ for m large enough we get a contradiction. If $L = 0$ the contradiction is obtained for all $m > 0$. $\square$

Fact 3.2. *There exists such $m_4 \geqslant 0$ that for all $m, l > m_4$ and $x \in X$ $\lambda_m(x) = \lambda_l(x)$.*

Proof. Let us set the sequence $\{g_\sigma\}_{\sigma\in\Sigma} \in P_1^X(x)$. Assume that $\lambda_m(x) = -\lambda_{m+\delta}(x)$ where $m > m_3$ and $\delta > 0$. Consider $h_\sigma^k(a) = \lambda_k(x)kg_\sigma(a)$ for all $a \in X$. We get that $\|h_\sigma^m + h_\sigma^{m+\delta}\| \leqslant \delta$. Therefore

$$\left|T\left(h_\sigma^m\right)\left(\varphi(x)\right) - T\left(-h_\sigma^{m+\delta}\right)\left(\varphi(x)\right)\right| \leqslant M\delta + L.$$

Using Fact 3.1 we have that $\operatorname{sign} T(-h_\sigma^{m+\delta})(\varphi(x)) = -\operatorname{sign} T(h_\sigma^m)(\varphi(x))$ hence $|T(h_\sigma^m)(\varphi(x))| + |T(-h_\sigma^{m+\delta})(\varphi(x))| \leqslant M\delta + L$ and finally from Theorem 2.1

$$2m + \delta - 5\left(M^2 - M\right)(2m + \delta) - 2\Delta \leqslant M\delta + L.$$

Rearranging the above we get

$$\left(-10M^2 + 10M + 2\right)m < \left(5M^2 - 4M - 1\right)\delta + 2\Delta + L. \tag{1}$$

We can clearly see that if $-10M^2 + 10M + 2 > 0$ (this is the case if $M < \sqrt{1.2}$) then for every $\delta_0 > 0$ there exists $m_4 > 0$ (depending on M, L and $\delta_0 > 0$) such that if $m > m_4$ the inequality (1) does not hold.

Hence $\lambda_m(x) = \lambda_{m+\delta}(x)$ for $m > m_4$ and $\delta < \delta_0$. However locally constant functions are constant thus $\lambda_m(x) = \lambda_l(x)$ where $m, l > m_4$. If $L = 0$ then we can set $m_4 = 0$ by taking $\delta_0 \to 0$. □

From now on we denote by $\lambda(x) = \lambda_m(x)$ for some $m > m_4$. The following theorem is a kind of summary of all the above facts:

Theorem 3.3. *Let X and Y be compact spaces and $C(X)$, $C(Y)$ Banach spaces of continuous real valued functions on X and Y, respectively. Let $T : C(X) \mapsto C(Y)$ be a bijective coarse (M, L)-quasi isometry such that $T(0) = 0$. Then for every $M < \sqrt{1.2}$ there is a homeomorphism $\varphi : X \mapsto Y$, a constant m and a continuous function $\lambda : X \mapsto \{-1, +1\}$ such that whenever $|f(x)| > 10(M-1)\|f\|$ then:*

$$\left|Tf\left(\varphi(x)\right) - \lambda(x)f(x)\right| \leqslant 5\left(M^2 - M\right)\|f\| + \Delta$$

for every $x \in X$ and $f \in C(X)$. Δ depends on M and L only and $\Delta = 0$ whenever $L = 0$.

Proof. Because of Theorem 2.1, Fact 3.1 and Fact 3.2 it is enough to prove that λ is a continuous function. Let us consider $x_\sigma \to x_0$ such that $\lambda_m(x_\sigma) \to 1$ and $\lambda_m(x_0) = -1$ where $m > m_4$. Then take f so that $f(x_0) = m = \|f\|$. Therefore $|Tf(\varphi(x_0)) + m| \leqslant 5(M^2 - M)m + \Delta$ and $|Tf(\varphi(x_\sigma)) - m| \leqslant 5(M^2 - M)m + \Delta$. Hence $Tf(\varphi(x_\sigma)) \to Tf(\varphi(x_0))$ we obtain that $m \leqslant 5(M^2 - M)m + \Delta$. This is impossible if $1 > 5(M^2 - M)$ (in particular if $M < \sqrt{1.2}$) for sufficiently large m. □

Now we are ready to formulate the answer to the main problem of this section that is we estimate the distance of a coarse (M, L)-quasi isometry of function spaces from the isometry as $M \to 1$.

Corollary 3.4. *Let us assume that T and λ are as in the theorem above. Then*

$$\left|Tf\left(\varphi(x)\right) - \lambda(x)f(x)\right| \leqslant 26(M-1)\|f\| + \Delta$$

for every $x \in X$ and $f \in C(X)$. Δ depends on M and L only and $\Delta = 0$ whenever $L = 0$.

Proof. Indeed if $|f(x)| \leqslant 10(M-1)\|f\|$ then

$$\left|Tf\left(\varphi(x)\right)\right| \leqslant \left|f(x)\right| + 5\left(M^2 - M\right)\|f\| + \Delta \leqslant 16(M-1)\|f\| + \Delta.$$

Hence

$$\left|Tf\left(\varphi(x)\right) - \lambda(x)f(x)\right| \leqslant 26(M-1)\|f\| + \Delta \tag{2}$$

as the value of $\lambda(x)$ does not matter if $|f(x)| \leqslant 10(M-1)\|f\|$. In the case $|f(x)| > 10(M-1)\|f\|$ the inequality (2) follows from Theorem 3.3 and the fact that for $M < \sqrt{1.2}$ we have $5(M^2 - M) \leqslant 26(M-1)$. □

Let us notice that the above results work also in the Lipschitz case that is when $L = 0$ hence also $\Delta = 0$. This way we improve the estimations obtained by Jarosz in [6]. More precisely, from Theorem 2 in [6] it follows that

$$\left|Tf\left(\varphi(x)\right) - \lambda(x)f(x)\right| \leqslant c'(M-1)\|f\|,$$

where $c'(M-1) \to 0$ as $M \to 1$ and φ is a homeomorphism – however defined differently than in our paper. The analysis of the Jarosz's paper gives us that $400(M-1)^{\frac{1}{10}} \leqslant c'(M-1)$. It is clear then that $c'(M-1) > 26(M-1)$ for $M < \sqrt{1.2}$ which shows that Corollary 3.4 gives us an improvement. It is also worth remarking that although all the above results hold for bijective coarse quasi isometries the analogous estimation can be obtained for the general case using results of this section and Fact 1.7. The only thing that is going to change is the term Δ.

4. Final remarks

In this paper we deal with the problem of finding the optimal constant C for which from the inequality $d_N(C(X), C(Y)) < C$ it follows the existence of a homeomorphism between compact spaces X and Y. The result obtained is $C = \frac{6}{5}$. At the end of Section 2 we have mentioned that from the linear theory it follows that $C \leqslant 2$. Let us formulate the following open problem:

Problem 4.1. Is it true that for all compact spaces X and Y the inequality $d_N(C(X), C(Y)) < 2$ implies that X and Y are homeomorphic?

When investigating the proof of Theorem 2.1 one can notice that if $M^2 < 2$ (which is the case if $d_N(C(X), C(Y)) < 2$) then there exists a positive number D such that $D < \frac{2}{M} - M$ hence from Fact 2.4 we obtain a multifunction $S_m^D : X \mapsto 2^Y$ for suitably chosen m. In the author's opinion there is a hope that the existence of such a multifunction S_m^D may help to answer the above problem in positive. Perhaps the first approach could be to look at some special classes of compact spaces such as all countable compacta which have very well understood structure. More precisely the following problems should be easier but still interesting

Problem 4.2. What is the answer to Problem 4.1 if we consider only countable compacta X and Y? What if X is a convergent sequence i.e. $\{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\}$?

In papers of Jarosz [6] and Dutrieux and Kalton [4] the results are obtained not only for $C(X)$ spaces where X is compact but also for $C_0(X)$ spaces where X is locally compact and $C_0(X)$ is the space of all continuous real valued functions vanishing at ∞. Although the author was able to obtain a slight improvement on the constant $\frac{17}{16}$ in the locally compact case he decided not to include it in the paper since it would make the proof more technical without any reasonable gain.

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