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Existence of solutions for a $p(x)$-Kirchhoff-type equation ☆

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ABSTRACT

This paper is concerned with the existence and multiplicity of solutions to a class of $p(x)$-Kirchhoff-type problem with Dirichlet boundary data. By means of a direct variational approach and the theory of the variable exponent Sobolev spaces, we establish conditions ensuring the existence and multiplicity of solutions for the problem.

1. Introduction

In this paper we study the following problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p = p(x) \in C(\overline{\Omega})$ with $1 < p^- := \inf_\Omega p(x) \leqslant p^+ := \sup_\Omega p(x) < N$, $M(t)$ is a continuous function and $f(x,u): \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition.

The operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to problem of this type is the model of motion of electrorheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field [26,31]. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium [1,2]. Another field of application of equations with variable exponent growth conditions is image processing [7]. The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the reader to [13,22,27,29,30] for an overview of and references on this subject, and to [8–12,15–21] for the study of the $p(x)$-Laplacian equations and the corresponding variational problems.

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The problem (1.1) is a generalization of a model introduced by Kirchhoff [24]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. The equation

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 \,\mathrm{d}x)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

is related to the stationary analogue of Eq. (1.2). Eq. (1.3) received much attention only after Lions [25] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [3–6,14,23].

In [6], the authors present several sufficient conditions for the existence of positive solutions to a class of nonlocal boundary value problems of the p-Kirchhoff type equation. Fan and Zhang in [18] studied $p(x)$-Laplacian equation with $f(x,u)$ satisfying Ambrosetti–Rabinowitz condition. Motivated by above, we consider the case of the $p(x)$-Laplacian, instead of the Laplacian and p-Laplacian. The $p(x)$-Laplacian possesses more complex nonlinearities, which raises some of the essential difficulties. We establish conditions ensuring the existence and multiplicity of solutions for the problem. To the best of our knowledge, this is the first or second paper that discusses the $p(x)$-Kirchhoff-type equation via variational method.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, we give some existence results of weak solutions of problem (1.1). In Section 4, we give some corollaries for a special problem, i.e., taking $M(t) = a + bt$ in (1.1).

2. Preliminaries

In order to discuss problem (1.1), we need some theories on $W_0^{1,p(x)}(\Omega)$ which we call variable exponent Sobolev space. Firstly we state some basic properties of spaces $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [20]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere.

Write

$$C_+(\overline{\Omega}) = \left\{h\colon\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for any } x \in \overline{\Omega}\right\},$$
$$h^+ = \max_{\overline{\Omega}} h(x), \qquad h^- = \min_{\overline{\Omega}} h(x) \quad \text{for any } h \in C(\overline{\Omega})$$

and

$$L^{p(x)}(\Omega) = \left\{u \in \mathbf{S}(\Omega)\colon\ \int_\Omega \left|u(x)\right|^{p(x)} \mathrm{d}x < +\infty\right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\lambda > 0\colon\ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\right\},$$

and

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega)\colon\ |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

with the norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1. *(See [20].) The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2. *(See [20].) Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x$. For any $u \in L^{p(x)}(\Omega)$, then*

(1) *For $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;*
(2) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;

(3) *If* $|u|_{p(x)} > 1$, *then* $|u|_{p(x)}^{p^-} \leqslant \rho(u) \leqslant |u|_{p(x)}^{p^+}$;
(4) *If* $|u|_{p(x)} < 1$, *then* $|u|_{p(x)}^{p^+} \leqslant \rho(u) \leqslant |u|_{p(x)}^{p^-}$;
(5) $\lim_{k\to+\infty} |u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k\to+\infty} \rho(u_k) = 0$;
(6) $\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty \Leftrightarrow \lim_{k\to+\infty} \rho(u_k) = +\infty$.

Proposition 2.3. *(See [20].) In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds, that is, there exists a positive constant C such that*

$$|u|_{L^{p(x)}(\Omega)} \leqslant C|\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

So, $|\nabla u|_{L^{p(x)}(\Omega)}$ is a norm equivalent to the norm $\|u\|$ in the space $W_0^{1,p(x)}(\Omega)$. We will use the equivalent norm in the following discussion and write $\|u\| = |\nabla u|_{L^{p(x)}(\Omega)}$ for simplicity.

Proposition 2.4. *(See [17,20].) Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p(x) < N$. If $q \in C(\overline{\Omega})$ and $1 \leqslant q(x) \leqslant p^*(x)$ $(1 \leqslant q(x) < p^*(x))$ for $x \in \Omega$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where $p^* = \frac{Np}{N-p}$.*

We write

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x.$$

Proposition 2.5. *(See [18].) The functional $J : X \to \mathbb{R}$ is convex. The mapping $J' : X \to X^*$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type, namely*

$$u_n \rightharpoonup u \quad \text{and} \quad \overline{\lim}_{n\to\infty} J'(u_n)(u_n - u) \leqslant 0 \quad \text{implies } u_n \to u,$$

where $X = W_0^{1,p(x)}(\Omega)$.

Proposition 2.6. *(See [18,20].) The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\left| \int_\Omega uv\,\mathrm{d}x \right| \leqslant \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}.$$

3. Existence of solutions

In this section we will discuss the existence of weak solutions of (1.1). For simplicity we write $X = W_0^{1,p(x)}(\Omega)$, denote by c the general positive constant (the exact value may change from line to line).

Definition 3.1. We say that $u \in X$ is a weak solution of (1.1), if

$$M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x \right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi\,\mathrm{d}x = \int_\Omega f(x,u)\varphi\,\mathrm{d}x, \tag{3.1}$$

where $\varphi \in X$.

Define

$$\Phi(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x \right), \qquad \Psi(u) = \int_\Omega F(x,u)\,\mathrm{d}x, \tag{3.2}$$

where $\widehat{M}(t) = \int_0^t M(s)\,\mathrm{d}s$, $F(x,u) = \int_0^u f(x,t)\,\mathrm{d}t$. The energy functional $I = \Phi(u) - \Psi(u) : X \to \mathbb{R}$ associated with problem (1.1) is well defined. Then it is easy to see that $I \in C^1(X,\mathbb{R})$ is weakly lower semi-continuous and $u \in X$ is a weak solution of (1.1) if and only if u is a critical point of I. Moreover, we have

$$\begin{aligned} I'(u)v &= M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x \right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\,\mathrm{d}x - \int_\Omega f(x,u)v\,\mathrm{d}x \\ &= \Phi'(u)v - \Psi'(u), \quad \text{for all } v \in X. \end{aligned}$$

Hereafter, $f(x,t)$ and $M(t)$ are always supposed to verify the following assumption:

(f_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition and

$$\left|f(x,t)\right| \leqslant c\left(1 + |t|^{\alpha(x)-1}\right), \tag{3.3}$$

where $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for all $x \in \Omega$.

(M_0) $\exists m_0 > 0$ such that

$$M(t) \geqslant m_0. \tag{3.4}$$

(M_1) $\exists 0 < \mu < 1$ such that

$$\widehat{M}(t) \geqslant (1-\mu)M(t)t. \tag{3.5}$$

Remark 3.1. Even under the constant exponent case, our condition (M_1) is weaker than condition (2.2) in [6]. It forces us to deal with more potential functions than [6]; for example, we can deal with increasing function $M(t) = a + bt$. For details, see Section 4.

Remark 3.2. From (M_0) and Proposition 2.5 we can easily see that Φ' is of (S_+) type. It is clear that Ψ' is weak-strong continuous. So I' is of (S_+) type.

Theorem 3.1. *If M satisfies (M_0) and*

$$\left|f(x,t)\right| \leqslant c\left(1 + |t|^{\beta-1}\right), \tag{3.6}$$

where $1 \leqslant \beta < p^-$, then (1.1) has a weak solution.

Proof. From (3.4), (3.6) we have $|F(x,t)| \leqslant c(|t| + |t|^\beta)$ and $\widehat{M}(t) \geqslant m_0 t$. We have

$$\begin{aligned} I(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\right) - \int_\Omega F(x,u)\,\mathrm{d}x \\ &\geqslant m_0 \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x - c\int_\Omega |u|^\beta\,\mathrm{d}x - c\int_\Omega |u|\,\mathrm{d}x \\ &\geqslant \frac{m_0}{p^+}\|u\|^{p^-} - c\|u\|^\beta - c\|u\| \to +\infty, \quad \text{as } \|u\| \to +\infty. \end{aligned} \tag{3.7}$$

Since I is weakly lower semi-continuous, I has a minimum point u in X, and u is a weak solution of (1.1). The proof is completed. □

Remark 3.3. We do not need condition (M_1) in the proof of Theorem 3.1.

Definition 3.2. We say that I satisfies (PS) condition in X, if any sequence $\{u_n\} \subset X$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence, where (PS) means Palais–Smale.

Lemma 3.1. *If $M(t)$ satisfies (M_0) and (M_1), f satisfies (f_0) and A.R. condition, i.e.,*

(f_1) $\exists T > 0, \theta > \frac{p^+}{1-\mu}$ *such that*

$$0 < \theta F(x,t) \leqslant t f(x,t), \quad |t| \geqslant T, \text{ a.e. } x \in \Omega, \tag{3.8}$$

then I satisfies (PS) condition.

Proof. Suppose that $\{u_n\} \subset X$, $|I(u_n)| \leqslant c$ and $I'(u_n) \to 0$. Then

$$\begin{aligned} c + \|u_n\| &\geqslant I(u_n) - \frac{1}{\theta}I'(u_n)u_n \\ &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\right) - \int_\Omega F(x,u_n)\,\mathrm{d}x \\ &\quad - \frac{1}{\theta}M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x + \int_\Omega \frac{1}{\theta}f(x,u_n)u_n\,\mathrm{d}x \\ &\geqslant \left(\frac{1-\mu}{p^+} - \frac{1}{\theta}\right)M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x + \int_\Omega \left[\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right]\mathrm{d}x \end{aligned}$$

$$\geqslant \left(\frac{1-\mu}{p^+}-\frac{1}{\theta}\right) m_0 \|u_n\|^{p^-}-c. \tag{3.9}$$

Hence, $\{\|u_n\|\}$ is bounded. Without loss of generality, we assume that $u_n \rightharpoonup u$, then $I'(u_n)(u_n-u)\to 0$. Thus, we have

$$I'(u_n)(u_n-u) = M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,\mathrm{d}x-\int_\Omega f(x,u_n)(u_n-u)\,\mathrm{d}x$$
$$\to 0. \tag{3.10}$$

From (f_0), Propositions 2.4 and 2.6, we can easily get that $\int_\Omega f(x,u_n)(u_n-u)\,\mathrm{d}x\to 0$. Therefore, we have

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,\mathrm{d}x\to 0.$$

In view of (M_0), we have

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,\mathrm{d}x\to 0.$$

Using Proposition 2.5, we have $u_n\to u$. $\square$

Theorem 3.2. *If M satisfies (M_0), (M_1), and f satisfies (f_0), (f_1) and the following condition*

(f_2) $f(x,t)=o(|t|^{p^+-1})$, $t\to 0$, for $x\in\Omega$ uniformly,

where $\alpha^- > p^+$, then (1.1) has a nontrivial solution.

Proof. Let us show that I satisfies the conditions of Mountain Pass Theorem (see Theorem 2.10 of [28]). By Lemma 3.1, I satisfies (PS) condition in X. Since $p^+<\alpha^-\leqslant\alpha(x)<p^*(x)$, $X\hookrightarrow L^{p^+}(\Omega)$, then there exists $c>0$ such that

$$|u|_{p^+}\leqslant c\|u\|, \quad u\in X. \tag{3.11}$$

Let $\varepsilon>0$ be small enough such that $\varepsilon c^{p^+}\leqslant \frac{m_0}{2p^+}$. By the assumptions (f_0) and (f_2), we have

$$F(x,t)\leqslant \varepsilon|t|^{p^+}+c|t|^{\alpha(x)} \quad (x,t)\in\Omega\times\mathbb{R}. \tag{3.12}$$

In view of (M_0) and (3.12), we have

$$\begin{aligned} I(u) &\geqslant \frac{m_0}{p^+}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x-\varepsilon\int_\Omega |u|^{p^+}\,\mathrm{d}x-c\int_\Omega |u|^{\alpha(x)}\,\mathrm{d}x \\ &\geqslant \frac{m_0}{p^+}\|u\|^{p^+}-\varepsilon c^{p^+}\|u\|^{p^+}-c\|u\|^{\alpha^-} \\ &\geqslant \frac{m_0}{2p^+}\|u\|^{p^+}-c\|u\|^{\alpha^-} \quad \text{when } \|u\|\leqslant 1. \end{aligned} \tag{3.13}$$

Therefore, there exist $r>0$, $\delta>0$ such that $I(u)\geqslant\delta>0$ for every $\|u\|=r$.

From (f_1) it follows that

$$F(x,t)\geqslant c|t|^\theta \quad x\in\Omega,\ |t|\geqslant T. \tag{3.14}$$

When $t>t_0$, from (M_1) we can easily obtain that

$$\widehat{M}(t)\leqslant \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{1-\mu}}}t^{\frac{1}{1-\mu}}=ct^{\frac{1}{1-\mu}}, \tag{3.15}$$

where t_0 is an arbitrarily positive constant.

For $w \in X \setminus \{0\}$ and $t > 1$, we have

$$\begin{aligned} I(tw) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|t\nabla w|^{p(x)}\,dx\right) - \int_\Omega F(x,tw)\,dx \\ &\leqslant c\left(\int_\Omega |t\nabla w|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - ct^\theta \int_\Omega |w|^\theta\,dx - c \\ &\leqslant ct^{\frac{p^+}{1-\mu}}\left(\int_\Omega |\nabla w|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - ct^\theta \int_\Omega |w|^\theta\,dx - c \\ &\to -\infty, \quad \text{as } t \to +\infty, \end{aligned} \tag{3.16}$$

due to $\theta > \frac{p^+}{1-\mu}$. Since $I(0) = 0$, I satisfies the conditions of Mountain Pass Theorem. So I admits at least one nontrivial critical point. □

Theorem 3.3. *Assume that the conditions (M_0), (M_1), (f_0) and (f_1) hold, and f satisfies:*

(f_3) $f(x,-t) = -f(x,t)$, $x \in \Omega$, $t \in \mathbb{R}$.

Then Eq. (1.1) *has a sequence of solutions $\{\pm u_k\}_{k=1}^\infty$ such that $I(\pm u_k) \to +\infty$ as $k \to +\infty$.*

Theorem 3.4. *Assume that the conditions (M_0), (M_1), (f_0), (f_2), (f_3) hold, and f satisfies:*

(f_4) $f(x,t) \geqslant c|t|^{\gamma(x)-1}$, $t \to 0$, *where* $\gamma \in C_+(\overline{\Omega})$, $p^+ < \gamma^- \leqslant \gamma^+ < \frac{p^-}{1-\mu}$ *for a.e.* $x \in \Omega$.

Then Eq. (1.1) *has a sequence of solutions $\{\pm v_k\}_{k=1}^\infty$ such that $I(\pm v_k) < 0$, $I(\pm v_k) \to +\infty$ as $k \to 0$.*

Remark 3.4. Conditions (f_4) and (f_2) can be compatible with each other if and only if $p^+ < \frac{p^-}{1-\mu}$, which is rather harsh. It does not hold when $\mu \equiv 0$. Therefore, condition (M_1) is necessary in guaranteeing our main results.

We will use the following “Fountain theorem” and the “Dual fountain theorem” to prove Theorem 3.3 and Theorem 3.4, respectively.

Because X is a reflexive and separable Banach space, there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j:\ j = 1,2,\dots\}}, \qquad X^* = \overline{\operatorname{span}\{e_j^*:\ j = 1,2,\dots\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we write $X_j = \operatorname{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$.

Lemma 3.2. *(See [18].) If $\alpha \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$ for any $x \in \overline{\Omega}$, denote*

$$\beta_k = \sup\{|u|_{\alpha(x)}:\ \|u\| = 1,\ u \in Z_k\}, \tag{3.17}$$

then $\lim_{k\to+\infty} \beta_k = 0$.

Lemma 3.3 *(Fountain theorem). (See [28].) Assume*

(A_1) *X is a Banach space, $I \in C^1(X,\mathbb{R})$ is an even functional.*

If for each $k = 1,2,\dots$ there exist $\rho_k > r_k > 0$ such that

(A_2) $\inf_{u\in Z_k,\,\|u\|=r_k} I(u) \to +\infty$ *as* $k \to +\infty$,
(A_3) $\max_{u\in Y_k,\,\|u\|=\rho_k} I(u) \leqslant 0$,
(A_4) *I satisfies (PS) condition for every $c > 0$,*

then I has a sequence of critical values tending to $+\infty$.

Lemma 3.4 *(Dual fountain theorem). (See [28].) Assume (A_1) is satisfied and there is $k_0 > 0$ so that, for each $k \geqslant k_0$, there exist $\rho_k > \gamma_k > 0$ such that*

(B_1) $\inf_{u\in Z_k,\, \|u\|=\rho_k} I(u) \geqslant 0$.
(B_2) $b_k := \max_{u\in Y_k,\, \|u\|=r_k} I(u) < 0$.
(B_3) $d_k := \inf_{u\in Z_k,\, \|u\|\leqslant\rho_k} I(u) \to 0$ *as* $k \to +\infty$.
(B_4) *I satisfies $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0)$.*

Then I has a sequence of negative critical values converging to 0.

Definition 3.3. We say that I satisfies the $(\mathrm{PS})_c^*$ condition (with respect to (Y_n)), if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $I(u_{n_j}) \to c$ and $(I|_{Y_{n_j}})'(u_{n_j}) \to 0$, contains a subsequence converging to a critical point of I.

Lemma 3.5. *Assume that the conditions (M_0), (M_1), (f_0) and (f_1) hold, then I satisfies the $(PS)_c^*$ condition.*

Proof. Suppose $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $I(u_{n_j}) \to c$ and $(I|_{Y_{n_j}})'(u_{n_j}) \to 0$. Similar to the process of verifying the (PS) condition in the proof of Lemma 3.1, we can get the boundedness of $\|u_{n_j}\|$. Going, if necessary, to a subsequence, we can assume that $u_{n_j} \rightharpoonup u$ in X. As $X = \overline{\bigcup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned}\lim_{n_j\to+\infty} I'(u_{n_j})(u_{n_j} - u) &= \lim_{n_j\to+\infty} I'(u_{n_j})(u_{n_j} - v_{n_j}) + \lim_{n_j\to+\infty} I'(u_{n_j})(v_{n_j} - u)\\ &= \lim_{n_j\to+\infty} (I|_{Y_{n_j}})'(u_{n_j})(u_{n_j} - v_{n_j})\\ &= 0.\end{aligned} \tag{3.18}$$

As I' is of (S_+) type, we can conclude $u_{n_j} \to u$, furthermore we have $I'(u_{n_j}) \to I'(u)$.

Let us prove $I'(u) = 0$ below. Taking arbitrarily $w_k \in Y_k$, notice that when $n_j \geqslant k$ we have

$$\begin{aligned}I'(u)w_k &= \big(I'(u) - I'(u_{n_j})\big)w_k + I'(u_{n_j})w_k\\ &= \big(I'(u) - I'(u_{n_j})\big)w_k + (I|_{Y_{n_j}})'(u_{n_j})w_k.\end{aligned}$$

Going to limit in the right side of above equation reaches

$$I'(u)w_k = 0, \quad \forall w_k \in Y_k, \tag{3.19}$$

so $I'(u) = 0$, this shows I satisfies the $(\mathrm{PS})_c^*$ condition for every $c \in \mathbb{R}$. $\square$

Proof of Theorem 3.3. According to (f_3) and Lemma 3.1, I is an even functional and satisfies (PS) condition. We will prove that if k is large enough, then there exist $\rho_k > \gamma_k > 0$ such that (A_2) and (A_3) hold. Thus, the assertion of conclusion can be obtained from Fountain theorem.

(A_2) For any $u \in Z_k$, $\|u\| = \gamma_k = (c\alpha^+\beta_k^{\alpha^+} m_0^{-1})^{\frac{1}{p^- - \alpha^+}}$, we have

$$\begin{aligned}I(u) &= \widehat{M}\bigg(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\bigg) - \int_\Omega F(x,u)\,\mathrm{d}x\\ &\geqslant m_0 \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x\\ &\geqslant \frac{m_0}{p^+}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - c\int_\Omega |u|^{\alpha(x)}\,\mathrm{d}x - c\int_\Omega |u|\,\mathrm{d}x\\ &\geqslant \frac{m_0}{p^+}\|u\|^{p^-} - c|u|_{\alpha(x)}^{\alpha(\xi)} - c\|u\|, \quad \text{where } \xi \in \Omega\\ &\geqslant \begin{cases} \frac{m_0}{p^+}\|u\|^{p^-} - c - c\|u\| & \text{if } |u|_{\alpha(x)} \leqslant 1,\\ \frac{m_0}{p^+}\|u\|^{p^-} - c\beta_k^{\alpha^+}\|u\|^{\alpha^+} - c\|u\| & \text{if } |u|_{\alpha(x)} > 1\end{cases}\\ &\geqslant \frac{m_0}{p^+}\|u\|^{p^-} - c\beta_k^{\alpha^+}\|u\|^{\alpha^+} - c\|u\| - c\\ &= \frac{m_0}{p^+}\big(c\alpha^+\beta_k^{\alpha^+}m_0^{-1}\big)^{\frac{p^-}{p^- - \alpha^+}} - c\beta_k^{\alpha^+}\big(c\alpha^+\beta_k^{\alpha^+}m_0^{-1}\big)^{\frac{\alpha^+}{p^- - \alpha^+}} - c\|u\| - c\end{aligned}$$

$$= m_0\left(\frac{1}{p^+} - \frac{1}{\alpha^+}\right)\left(c\alpha^+\beta_k^{\alpha^+} m_0^{-1}\right)^{\frac{p^-}{p^- - \alpha^+}} - c\left(c\alpha^+\beta_k^{\alpha^+} m_0^{-1}\right)^{\frac{1}{p^- - \alpha^+}} - c$$
$$\to +\infty, \quad \text{as } k \to +\infty,$$

because of $p^+ < \alpha^+$, $p^- > 1$ and $\beta_k \to 0$.

(A_3) From (f_1), we have $F(x,t) \geqslant c|t|^\theta - c$. Therefore, for any $w \in Y_k$ with $\|w\| = 1$ and $1 < t = \rho_k$, we have

$$\begin{aligned} I(tw) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|t\nabla w|^{p(x)}\,\mathrm{d}x\right) - \int_\Omega F(x,tw)\,\mathrm{d}x \\ &\leqslant c\left(\int_\Omega |t\nabla w|^{p(x)}\,\mathrm{d}x\right)^{\frac{1}{1-\mu}} - ct^\theta \int_\Omega |w|^\theta\,\mathrm{d}x - c \\ &\leqslant c\rho_k^{\frac{p^+}{1-\mu}}\left(\int_\Omega |\nabla w|^{p(x)}\,\mathrm{d}x\right)^{\frac{1}{1-\mu}} - c\rho_k^\theta \int_\Omega |w|^\theta\,\mathrm{d}x - c. \end{aligned} \tag{3.20}$$

By $\theta > \frac{p^+}{1-\mu}$ and $\dim Y_k = k$, it is easy to see that $I(u) \to -\infty$ as $\|u\| \to +\infty$ for $u \in Y_k$.

Conclusion of Theorem 3.3 is reached by the Fountain theorem. □

Proof of Theorem 3.4. From (f_3) and Lemma 3.5, we know that I satisfies (A_1) and (B_4), the assertion of conclusion can be obtained from Dual fountain theorem.

(B_1): For any $v \in Z_k$, $\|v\| = 1$ and $0 < t < 1$, we have

$$\begin{aligned} I(tv) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla tv|^{p(x)}\,\mathrm{d}x\right) - \int_\Omega F(x,tv)\,\mathrm{d}x \\ &\geqslant m_0 \int_\Omega \frac{1}{p(x)}|\nabla tv|^{p(x)}\,\mathrm{d}x - \int_\Omega F(x,tv)\,\mathrm{d}x \\ &\geqslant \frac{m_0}{p^+}t^{p^+}\int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x - \varepsilon t^{p^+}\int_\Omega |v|^{p^+}\,\mathrm{d}x - ct^{\alpha^-}\int_\Omega |v|^{\alpha(x)}\,\mathrm{d}x \\ &\geqslant \left(\frac{m_0}{p^+} - \varepsilon c^{p^+}\right)\|v\|^{p^+}t^{p^+} - \begin{cases} c\beta_k^{\alpha^-}t^{\alpha^-}\|v\|^{\alpha^-} & \text{if } |u|_{\alpha(x)} \leqslant 1, \\ c\beta_k^{\alpha^+}t^{\alpha^-}\|v\|^{\alpha^+} & \text{if } |u|_{\alpha(x)} > 1 \end{cases} \\ &\geqslant \frac{m_0}{2p^+}t^{p^+} - \begin{cases} c\beta_k^{\alpha^-}t^{\alpha^-} & \text{if } |u|_{\alpha(x)} \leqslant 1, \\ c\beta_k^{\alpha^+}t^{\alpha^-} & \text{if } |u|_{\alpha(x)} > 1. \end{cases} \end{aligned} \tag{3.21}$$

Since $\alpha^- > p^+$, taking $\rho_k = t$ small enough and sufficiently large k, for $v \in Z_k$ with $\|v\| = 1$, we have $I(tv) \geqslant 0$. So for sufficiently large k,

$$\inf_{u \in Z_k,\, \|u\| = \rho_k} I(u) \geqslant 0, \tag{3.22}$$

i.e., (B_1) is satisfied.

(B_2): For $v \in Y_k$, $\|v\| = 1$ and $0 < t < \rho_k < 1$, we have

$$\begin{aligned} I(tv) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|t\nabla v|^{p(x)}\,\mathrm{d}x\right) - \int_\Omega F(x,tv)\,\mathrm{d}x \\ &\leqslant c\left(\int_\Omega |t\nabla v|^{p(x)}\,\mathrm{d}x\right)^{\frac{1}{1-\mu}} - c\int_\Omega |tv|^{\gamma(x)}\,\mathrm{d}x \\ &\leqslant ct^{\frac{p^-}{1-\mu}}\left(\int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x\right)^{\frac{1}{1-\mu}} - ct^{\gamma^+}\int_\Omega |v|^{\gamma(x)}\,\mathrm{d}x. \end{aligned}$$

Condition $\gamma^+ < \frac{p^-}{1-\mu}$ implies that there exists a $r_k \in (0, \rho_k)$ such that $I(tv) < 0$ when $t = r_k$. Hence we get

$$b_k := \max_{u \in Y_k,\, \|u\| = r_k} I(u) < 0,$$

so (B_2) is satisfied.

(B_3): Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k := \inf_{u \in Z_k,\, \|u\| \leqslant \rho_k} I(u) \leqslant b_k := \max_{u \in Y_k,\, \|u\| = r_k} I(u) < 0.$$

From (3.21), for $v \in Z_k$, $\|v\| = 1$, $0 \leqslant t \leqslant \rho_k$ and $u = tv$, we have

$$\begin{aligned} I(u) = I(tv) &\geqslant \frac{m_0}{2p^+} t^{p^+} - \begin{cases} c\beta_k^{\alpha^-} t^{\alpha^-} & \text{if } |u|_{\alpha(x)} \leqslant 1, \\ c\beta_k^{\alpha^+} t^{\alpha^-} & \text{if } |u|_{\alpha(x)} > 1 \end{cases} \\ &\geqslant - \begin{cases} c\beta_k^{\alpha^-} t^{\alpha^-} & \text{if } |u|_{\alpha(x)} \leqslant 1, \\ c\beta_k^{\alpha^+} t^{\alpha^-} & \text{if } |u|_{\alpha(x)} > 1, \end{cases} \end{aligned}$$

hence $d_k \to 0$, i.e., (B_3) is satisfied.

Conclusion of Theorem 3.4 is reached by the Dual fountain theorem. □

4. Corollaries for a special problem

In this section we will give some typical consequences of Theorems 3.1–3.4. We consider the following special problem

$$\begin{cases} -\left(a + b\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where a, b are two positive constants.

Let $M(t) = a + bt$ with $t = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx$. It is clear that

$$M(t) \geqslant a > 0. \tag{4.2}$$

Taking $\mu = \frac{1}{2}$, we have

$$\widehat{M}(t) = \int_0^t M(s)\, ds = at + \frac{1}{2} bt^2 \geqslant \frac{1}{2}(a + bt)t = (1 - \mu) M(t) t. \tag{4.3}$$

So the conditions (M_0), (M_1) are satisfied. Therefore, corresponding to Theorems 3.1–3.4, we have the following corollaries.

Corollary 4.1. *If M satisfies (M_0) and*

$$\left| f(x,t) \right| \leqslant C_1 + C_2 |t|^{\beta - 1}, \tag{4.4}$$

where $1 \leqslant \beta < p^-$, then (4.1) has a weak solution.

Corollary 4.2. *If M satisfies (M_0), (M_1), and f satisfies (f_0), (f_1) and (f_2), where $\alpha^- > p^+$, then (4.1) has a nontrivial solution.*

Corollary 4.3. *Assume that the conditions (M_0), (M_1), (f_0), (f_1) and (f_3) hold. Then Eq. (4.1) has a sequence of solutions $\{\pm u_k\}_{k=1}^\infty$ such that $I(\pm u_k) \to +\infty$ as $k \to +\infty$.*

Corollary 4.4. *Assume that the conditions (M_0), (M_1), (f_0), (f_2), (f_3) and (f_4) hold. Then Eq. (4.1) has a sequence of solutions $\{\pm u_k\}_{k=1}^\infty$ such that $I(\pm u_k) \to +\infty$ as $k \to +\infty$.*

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