Journal of Algebra 347 (2011) 214–223

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Centralizers in semisimple algebras, and descent spectrum in Banach algebras ☆

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ARTICLE INFO

Article history:
Received 21 June 2011
Available online 13 July 2011
Communicated by Efim Zelmanov

MSC:
16B99
46H99
47A10
47A55

Keywords:
Centralizer
Descent spectrum
Semisimple Banach algebra

ABSTRACT

We prove that semisimple algebras containing some algebraic element whose centralizer is semiperfect are artinian. As a consequence, semisimple complex Banach algebras containing some element whose centralizer is algebraic are finite-dimensional. This answers affirmatively a question raised in Burgos et al. (2006) [4], and is applied to show that an element a in a semisimple complex Banach algebra A does not perturb the descent spectrum of every element commuting with a if and only if some of power of a lies in the socle of A. This becomes a Banach algebra version of a theorem in Burgos et al. (2006) [4], Kaashoek and Lay (1972) [9] for bounded linear operators on complex Banach spaces.

1. Introduction

Let T be a linear operator on a vector space X over a field K. The descent $d(T)$ of T is defined by the equality

$$d(T) := \min\{n \in \mathbb{N} \cup \{0\}:\ R(T^n) = R(T^{n+1})\},$$

☆ The authors are partially supported by the projects I+D MCYT MTM2004-03882, MTM-2006-15546-C02-02, MTM2007-65959, with FEDER founds, AECI PCI A/4044/05, A/5037/06, and the Junta de Andalucia grants FQM-194, FQM-199, FQM-1215.
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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.06.021

with the convention that $\min \emptyset = \infty$. Here $R(\cdot)$ denotes the range. The descent spectrum of T is the set

$$\sigma_{\text{desc}}(T) := \{\lambda \in K\colon d(T-\lambda) = \infty\}.$$

We note that $\sigma_{\text{desc}}(T)$ is a subset of the usual spectrum

$$\sigma(T) := \{\lambda \in K\colon T - \lambda \text{ is not bijective}\}.$$

As a consequence of a theorem of M.A. Kaashoek and D.C. Lay in perturbation theory [9, Theorem 2.2], we are provided with the following

Theorem 1.1. *Let T be a linear operator on a vector space X over a field K. If, T^m has finite rank for some $m \in \mathbb{N}$, then we have*

$$\sigma_{\text{desc}}(T+S) = \sigma_{\text{desc}}(S)$$

for every linear operator S on X commuting with T.

By putting together Theorem 1.1 and [4, Theorem 3.1], we are provided also with the following.

Theorem 1.2. *Let X be a complex Banach space, and let T be a bounded linear operator on X. Then some power of T has finite rank if and only if the equality $\sigma_{\text{desc}}(T+S) = \sigma_{\text{desc}}(S)$ holds for every bounded linear operator S on X commuting with T.*

As main result, we prove in the present paper a variant of Theorem 1.2, where an arbitrary semisimple complex Banach algebra A replaces the algebra $\mathrm{BL}(X)$ of all bounded linear operators on the complex Banach space X, the socle of A replaces the ideal of all finite-rank bounded linear operators on X, and the descent spectrum of an element $a \in A$ (denoted by $\sigma_{\text{desc}}(a, A)$) is defined as the descent spectrum of the operator of left multiplication by a on A (see Theorem 3.6). Essentially, the "only if" part of Theorem 3.6 just reviewed is of a purely algebraic nature, since socles of complex Banach algebras are algebraic, and we prove that, if some power of an element a in an arbitrary algebra A lies in the socle of A, and if the socle of A is algebraic, then a does not perturb the descent spectrum of any element of A commuting with a (Corollary 3.5). This last result becomes in fact a wide generalization of Theorem 1.1, since it is easily realized that, for a linear operator T on an arbitrary vector space X, we have $\sigma_{\text{desc}}(T) = \sigma_{\text{desc}}(T, \mathcal{L}(X))$, where $\mathcal{L}(X)$ stand for the algebra of all linear operators on X (Proposition 3.7).

The key tool in the proof of (the "if part" of) Theorem 3.6 is the affirmative answer to [4, Question 2] provided by Theorem 2.3. Indeed, we prove in that theorem that, if a semisimple complex Banach algebra A contains an element whose centralizer is algebraic, then A is finite-dimensional. Again, most ingredients in the proof of Theorem 2.3 are of a purely algebraic nature. Indeed, Theorem 2.3 follows quickly from the fact that, if A is a semisimple algebra over a field of characteristic zero, and if A contains an algebraic element whose centralizer is semiperfect, then A is artinian (Theorem 2.2). This result follows the line of some classical papers (see [5,8]), where their authors study the structure of centralizers and the information that they can provide on the whole algebra.

2. Centralizers in semisimple algebras

For algebraic background, one may consult standard books on ring theory [2,6] or [12].

Throughout this paper, all algebras will be assumed to be associative and to have a unit element. Let A be an algebra over a field K. We denote by $J(A)$ its Jacobson radical, and we say that A is semisimple if $J(A) = 0$. We say that A is local if the set of all noninvertible elements of A is an ideal of A or, equivalently, if for every $x \in A$, either x or $1 - x$ is invertible [12, Theorem 19.1]. A is called semiperfect if $A/J(A)$ is artinian and the idempotents of $A/J(A)$ can by lifted to A or, equivalently, if A contains a finite set of pairwise orthogonal idempotents $e_1, \dots, e_n$ such that $e_1 + \cdots + e_n = 1$, and $e_i A e_i$ is a local algebra for every $i = 1, \dots, n$ [12, Theorem 23.6]. An element a of A is called algebraic if there exists a nonzero polynomial $f \in K[\mathbf{X}]$, such that $f(a) = 0$. The monic polynomial of smallest degree with this property is called the minimum polynomial of a, and shall be denoted by ϕ_a. A subset of A is said to be algebraic if all its elements are algebraic. The centralizer of an element $a \in A$ is the subalgebra $\mathrm{Cent}_A(a) = \{x \in A\colon\ xa = ax\}$. When no confusion is possible, we write $\mathrm{Cent}(a)$ instead of $\mathrm{Cent}_A(a)$.

Some authors (see [5,8]) have studied the structure of centralizers and the information that they can provide on the whole algebra. In [5, Theorem 2.4.], it is shown that, if A is semiprime and has an algebraic element a such that $\mathrm{Cent}(a)$ is semisimple artinian, then A is semisimple artinian. Theorem 2.2 below becomes a result in this direction. For the proof, we need the following result on nilpotent endomorphisms.

Lemma 2.1. *Let M be a semisimple module over a ring R, let T be a nilpotent endomorphism of index k of M, and put $N_i = \ker T^i$. Then there exists a family $V_1, V_2, \dots, V_k$ of nonzero submodules of M such that:*

(1) *$V_1 = N_1$, and V_i is contained in N_i for $i = 2, \dots, k$.*
(2) *For $i = 2, \dots, k$, $T(V_i) \subset V_{i-1}$, and the restriction of T to V_i is injective.*
(3) *$M = V_1 \oplus V_2 \oplus \cdots \oplus V_k$.*

If in addition $\ker T$ *is of finite length (i.e., has a finite number of simple components), then so is M.*

Proof. The proof is an adaptation of the Jordan reduction of a nilpotent endomorphism in a vector space. First take V_k a submodule of M such that $V_k \oplus N_{k-1} = M$ (the existence is ensured by the semisimplicity of M). The restriction of T to V_k is injective and $T(V_k) \cap N_{k-2} = \{0\}$, so we can find a submodule V_{k-1} of N_{k-1} containing $T(V_k)$ and such that $V_{k-1} \oplus N_{k-2} = N_{k-1}$. The process is repeated analogously for $k-2, k-3, \dots, 1$, until we get the desired family $V_k, V_{k-1}, \dots, V_1$.

Suppose now that $\ker T = V_1$ has finite length. Since the restrictions $T_{|V_i} : V_i \to V_{i-1}$ are injective, we deduce that each V_i has finite length, and consequently M has finite length. $\square$

We come now to a theorem on centralizers.

Theorem 2.2. *Let A be a semisimple algebra over a field K of characteristic zero, and let a be an algebraic element of A. If the centralizer of a is semiperfect, then A is artinian.*

Proof. In a first step, we assume that the minimum polynomial ϕ_a of a splits over K and that A is not artinian, and we prove that in such a case $\mathrm{Cent}(a)$ is not a local algebra.

Put $\phi_a(\mathbf{X}) = \prod_{i=1}^{m} (\mathbf{X} - \lambda_i)^{k_i}$. If $m > 1$, then there exists two coprime non-constant polynomials f and g such that $\phi_a = fg$, and therefore we have

$$K[a] \cong \frac{K[\mathbf{X}]}{(\phi_a)} \cong \frac{K[\mathbf{X}]}{(f)} \times \frac{K[\mathbf{X}]}{(g)}.$$

Since the last algebra in the above chain of isomorphisms contains a nontrivial idempotent, $K[a]$ also contains a nontrivial idempotent. It follows from the inclusion $K[a] \subset \mathrm{Cent}(a)$ that $\mathrm{Cent}(a)$ contains a nontrivial idempotent, which implies that it is not a local algebra.

Now assume that $m = 1$, that is $\phi_a(\mathbf{X}) = (\mathbf{X} - \lambda)^k$. Since $\mathrm{Cent}(a) = \mathrm{Cent}(a - \lambda)$, we may suppose that a is nilpotent of index k. For every $x \in A$, put $T_a(x) = \sum_{i=0}^{k-1} a^i x a^{k-i-1}$. It is easy to see that $T_a(x) \in \mathrm{Cent}(a)$.

Let $\mathcal{P}$ denote the set of those primitive ideals P of A such that $a^{k-1} \notin P$. Since A is semisimple, and $a^{k-1} \neq 0$, the set $\mathcal{P}$ is nonempty.

We shall distinguish two cases:

Case one. There exists some $P \in \mathcal{P}$ such that A/P is not artinian. Take a simple A-module S with $P = \mathrm{Ann}(S)$, and put $D = \mathrm{End}_A(S)$. If S were finite-dimensional over D, then, by the Jacobson density theorem (see for example [2, 14.49]), A/P would be isomorphic to a matrix algebra over the division ring D (a nice example of an artinian ring), arriving thus in a contradiction. Therefore S is infinite-dimensional over D. Consider the map $L_a : S \to S$ defined by $L_a(x) = ax$. Then $L_a \in \mathrm{End}_D(S)$ and, since $a^k = 0$ and $a^{k-1}S \neq 0$, L_a is nilpotent of index k. Since S is a semisimple D-module (because D is a division ring), we can use the results of Lemma 2.1. Take $V_k, V_{k-1}, \ldots, V_1$ as in that lemma and $u \in V_k$ nonzero. We have $a^i u = L_a^i(u) \in V_{k-i}$. Consequently $u, au \ldots, a^{k-1}u$ are linearly independent over D. On the other hand, since S has not finite length over D, we can choose an element v of V_1 D-linearly independent of $a^{k-1}u$. We obtain finally a family $u, au, \ldots, a^{k-1}u, v$ of D-linearly independent vectors such that $av = 0$. Now, by the Jacobson density theorem, there exists $x \in A$ such that $xa^{k-1}u = u$, $xa^i u = 0$ for every $i = 0, 1, \ldots, k-2$, and $xv = 0$. Therefore we have that $T_a(x)u = u$ and $T_a(x)v = 0$. Putting $b := T_a(x)$, we realize that $b \in \mathrm{Cent}(a)$ and that neither b nor $1-b$ is invertible. This implies that $\mathrm{Cent}(a)$ is not a local algebra.

Case two. For every $P \in \mathcal{P}$, A/P is artinian (which implies that all elements of $\mathcal{P}$ are maximal ideals of A). Let us fix $P \in \mathcal{P}$. Assume that for all primitive ideals Q of A we have $Q + P \neq A$. Then for such a primitive ideal Q we have $Q + P = P$ (by the maximality of P) or, equivalently, $Q \subset P$, which implies that Q belongs to $\mathcal{P}$ (and hence is a maximal ideal of A), and then that $P = Q$. Thus P is the unique primitive ideal of A which equals its Jacobson radical $J(A)$. Since A is semisimple, $J(A) = 0$ and A is artinian, which is a contradiction. In this way we have proved that there exists some primitive ideal Q of A satisfying $Q + P = A$. Take S a simple A-module such that $\mathrm{Ann}(S) = P$, and $u \in S$ such that $a^{k-1}u, \ldots, au, u$ are linearly independent. Again by the Jacobson density theorem, there exists $x \in A$ such that $xa^{k-1}u = u$ and $xa^i u = 0$ for all $i = 0, \ldots, k-2$. Then $T_a(x)u = u$. Now $x = x_1 + x_2$ where $x_1 \in Q$ and $x_2 \in P$, hence

$$T_a(x_1 + x_2)u = \big(T_a(x_1) + T_a(x_2)\big)u = T_a(x_1)u = u,$$

since $T_a(x_2) \in P$ and $PS = 0$. If we put $b = T_a(x_1)$, then $bu = u$ and $b \in Q$. Thus $1-b$ and b are not invertible.

Now that the first step in the proof is concluded, assume that $B := \mathrm{Cent}(a)$ is semiperfect and that the minimum polynomial ϕ_a of a splits over K. Let $e_1, \ldots, e_n$ be a finite set of pairwise orthogonal idempotents of B such that $e_1 + \cdots + e_n = 1$, and $e_i A e_i$ is a local algebra for every $i = 1, \ldots, n$. Then one can see that the centralizer of $e_i a e_i = a e_i$ in $e_i A e_i$ is equal to $e_i B e_i$ which is local. Moreover, the minimum polynomial of ae_i relative to $e_i A e_i$ splits over K because it is a divisor of ϕ_a. It follows from the first step in the proof that $e_i A e_i$ is semisimple artinian for all $i = 1, \ldots, n$. Consequently, A is artinian.

Finally, remove the assumption above that ϕ_a splits over K. Take a splitting field L of this polynomial over K. We consider the tensor product $R = A \otimes_K L$. It is easy to see that $\mathrm{Cent}_R(a \otimes 1) = \mathrm{Cent}_A(a) \otimes L$. Since $\mathrm{Cent}_A(a)$ is semiperfect, so is $\mathrm{Cent}_A(a) \otimes L$ [10]. Hence R is artinian. A is then artinian. □

We are now ready to answer Question 2 of [4].

Theorem 2.3. *Let A be a semisimple complex Banach algebra containing an element a such that* $\mathrm{Cent}(a)$ *is algebraic. Then A is finite-dimensional.*

Proof. Since $\mathrm{Cent}(a)$ is an algebraic Banach algebra, $J(\mathrm{Cent}(a))$ is nil [12, Corollary 4.19] and $\mathrm{Cent}(a)/J(\mathrm{Cent}(a))$ is finite-dimensional [3, Theorem 5.4.2]. Therefore, by [12, Theorem 21.28], $\mathrm{Cent}(a)$ is semiperfect. By Theorem 2.2, A is artinian. But artinian semisimple complex Banach algebras are finite-dimensional (see for example [1, Corollary 4]). □

Let X be a complex Banach space, and let $\mathrm{BL}(X)$ stand for the complex Banach algebra of all bounded linear operators on X. Since $\mathrm{BL}(X)$ is semisimple, Theorem 2.3 applies, giving that X is finite-dimensional whenever there exists $T \in \mathrm{BL}(X)$ whose centralizer in $\mathrm{BL}(X)$ is algebraic. In this way, we rediscover [4, Proposition 3.3]. As another consequence of Theorem 2.3, we have the following.

Corollary 2.4. *The centralizer of every element of an infinite-dimensional C^*-algebra with identity contains a non-algebraic element.*

Proof. It is well known that every C^*-algebra is semisimple. □

One may ask if a complex normed algebra A is algebraic whenever $\mathrm{Cent}(a)$ is algebraic for some $a \in A$, and A is complete (possibly non-semisimple) or semisimple (possibly non-complete). The answer is negative in the two cases, as the following examples show.

Example 2.5. Let R be any non-algebraic complex Banach algebra. Consider the algebra $A = \begin{pmatrix} R & R \\ 0 & \mathbb{C} \end{pmatrix}$ which can be endowed with a complete algebra norm. If we take $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $a \in J(A)$ and $\mathrm{Cent}(a) = \mathbb{C} + \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$ which is algebraic, but A is not.

Example 2.6. Let X be any infinite-dimensional complex normed space. Denote by R the algebra of all bounded linear operator on X, and by F the ideal of all bounded linear operators of finite rank. Let $S = \mathbb{C} + F$. Define $A = \begin{pmatrix} R & R \\ F & S \end{pmatrix}$ and $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$\mathrm{Cent}(a) = \left\{ \begin{pmatrix} \lambda + u & v \\ 0 & \lambda + u \end{pmatrix} \in A\colon \lambda \in \mathbb{C},\ u \in F,\ v \in R \right\}.$$

Let $b = \begin{pmatrix} \lambda+u & v \\ 0 & \lambda+u \end{pmatrix} \in \mathrm{Cent}(a)$. Since u is of finite rank, u is algebraic. Thus $\lambda + u$ is algebraic. If ϕ stands for the minimum polynomial of $\lambda + u$, then $\phi(b)$ is in the form $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, and hence $(\phi(b))^2 = 0$. This means that b is algebraic. Hence $\mathrm{Cent}(a)$ is algebraic. On the other hand, A is primitive because it is isomorphic to a subalgebra of $\mathrm{BL}(X \oplus X)$ containing all finite rank operators.

Despite the above examples, we are going to prove in Theorem 2.8 below a purely algebraic result in the spirit of Theorem 2.3.

Lemma 2.7. *Let X be a vector space over a field K, and let T be an algebraic linear operator on X. If T is injective or surjective, then T is actually bijective.*

Proof. Let ϕ denote the minimum polynomial of T, and write $\phi(\mathbf{X}) = \lambda + \mathbf{X}\psi(\mathbf{X})$ with $\lambda \in K$ and $\psi \in K[\mathbf{X}]$. If $\lambda = 0$, then we have $T\psi(T) = 0$ with $\psi(T) \neq 0$ (which implies that T is not injective) and $\psi(T)T = 0$ with $\psi(T) \neq 0$ (which implies that T is not surjective). Therefore, if T is injective or surjective, then we have $\lambda \neq 0$, so that, by putting $F = -\lambda^{-1}\psi(T)$, we obtain $TF = FT = 1$, and T becomes bijective. □

Theorem 2.8. *Let X be a vector space over a field K, and let $\mathcal{L}(X)$ stand for the algebra of all linear operators on X. Then the following assertions are equivalent*:

(1) *$\mathcal{L}(X)$ contains an element whose centralizer is algebraic.*
(2) *X is finite-dimensional.*

Proof. We need only to show that (1) $\Rightarrow$ (2). Let T be in $\mathcal{L}(X)$ such that $\mathrm{Cent}(T)$ is algebraic, and put $R := K[T]$. Since T is algebraic, we have $R \cong K[\mathbf{X}]/(\phi)$, where ϕ stands for the minimum polynomial of T. On the other hand, X can be naturally regarded as an R-module in such a way that

$\mathrm{End}_R(X) = \mathrm{Cent}(T)$. Since $\mathrm{Cent}(T)$ is algebraic, it follows from Lemma 2.7 that X is hopfian and co-hopfian as an R-module (i.e., every injective or surjective endomorphism is an isomorphism). Since R is a commutative artinian principal ideal ring, [11, Theorem 9] applies, giving us that the R-module X is noetherian, and hence finitely generated over R. Since R is finitely generated over K, we finally deduce that X is finite-dimensional. □

3. Application to perturbations

In this section, we shall prove a variant of Theorem 1.2 for semisimple complex Banach algebras. We recall the notion of descent spectrum of an element of an algebra, as introduced in [4]. Let A be an algebra over a field K, and let a be in A. The descent $d(a, A)$ of a is defined by the equality

$$d(a, A) := \min\{n \in \mathbb{N} \cup \{0\}:\ a^n \in a^{n+1}A\},$$

with the convention that $\min \emptyset = \infty$. The descent spectrum of a is the set

$$\sigma_{\mathrm{desc}}(a, A) := \{\lambda \in K:\ d(a - \lambda, A) = \infty\}.$$

It is easily seen that $d(a, A) = d(L_a)$, and consequently that $\sigma_{\mathrm{desc}}(a, A) = \sigma_{\mathrm{desc}}(L_a)$, where L_a stands for the operator of left multiplication by a on A (see [4, Remark 2.1.(i)]). The next result follows from the observation just done and Theorem 1.1. Nevertheless, we include a proof for the sake of self-containment.

Lemma 3.1. *Let A be an algebra over a field K, let a be a nilpotent element of A, and let b be in* $\mathrm{Cent}(a)$. *Then we have*

$$\sigma_{\mathrm{desc}}(a + b, A) = \sigma_{\mathrm{desc}}(b, A).$$

Proof. It is enough to show that, if $d(b, A)$ is finite, then so is $d(a + b, A)$. Let k be in $\mathbb{N} \cup \{0\}$ such that $a^k = 0$. Then, for all $x \in \mathrm{Cent}(a)$ and $n \in \mathbb{N} \cup \{0\}$ we have

$$(a + x)^{n+k} = \sum_{i=0}^{n+k} \binom{n+k}{i} x^i a^{n+k-i} = \sum_{i=n+1}^{n+k} \binom{n+k}{i} x^i a^{n+k-i} \in x^n A.$$

Now assume that $m := d(b, A) < \infty$. By choosing successively in the above fact $(x, n) = (b, m)$ and $(x, n) = (-(a + b), m + k + 1)$, it follows that

$$(a + b)^{m+k} \in b^m A = b^{m+2k+1} A = [a - (a + b)]^{m+2k+1} A \subseteq (a + b)^{m+k+1} A,$$

and hence that $d(a + b) \leqslant m + k < \infty$. □

Lemma 3.2. *Let A be an algebra over a field K, and let a be an algebraic element of A. Then we have*:

(1) *There exists an idempotent $e \in K[a]$ such that $a(1 - e)$ is nilpotent, and ae is invertible in $K[a]e$.*
(2) $\sigma_{\mathrm{desc}}(a, A) = \emptyset$.

Proof. Let ϕ_a stand for the minimum polynomial of a, and write $\phi_a(\mathbf{X}) = \mathbf{X}^{\mathbf{k}} \psi(\mathbf{X})$ where $k \in \mathbb{N} \cup \{0\}$ and $\psi \in K[\mathbf{X}]$ with $\psi(0) \neq 0$. Then we have a natural isomorphism

$$K[a] \cong \frac{K[\mathbf{X}]}{(\mathbf{X}^k)} \times \frac{K[\mathbf{X}]}{(\psi)}$$

which provides us with an idempotent $e \in K[a]$ such that $\psi(ae) = 0$ in $K[a]e$, and $(a(1-e))^k = 0$. Therefore, to conclude the proof of (1), it is enough to show that ae is invertible in $K[a]e$. But, since $\psi(ae) = 0$ and $\psi(0) \neq 0$, we can argue as in the conclusion of the proof of Lemma 2.7 to obtain that ae is invertible in $K[a]e$.

Let b denote the inverse of ae in $K[a]$. Then, since

$$a^k(1-e) = \big(a(1-e)\big)^k = 0,$$

we have that $a^k = a^{k+1}b$, and hence that $d(a, A)$ is finite. By replacing a with $a - \lambda$, with λ arbitrarily in K, we deduce that $\sigma_{\text{desc}}(a, A) = \emptyset$. □

We shall need also the following result.

Proposition 3.3. *Let A be an algebra over a field K, let a be in A, and let e be an idempotent of A commuting with a. Then we have*

$$\sigma_{\text{desc}}(ae, eAe) = \sigma_{\text{desc}}(ae, A). \tag{3.1}$$

As a consequence, the equality

$$\sigma_{\text{desc}}(a, A) = \sigma_{\text{desc}}(ae, A) \cup \sigma_{\text{desc}}\big(a(1-e), A\big) \tag{3.2}$$

holds.

Proof. Let λ be in $K \setminus \sigma_{\text{desc}}(ae, A)$. Then we have

$$(ae - \lambda)^{n+1}c = (ae - \lambda)^n$$

for some $c \in A$ and $n \in \mathbb{N} \cup \{0\}$. Therefore we have

$$(ae - \lambda e)^{n+1}ece = (ae - \lambda e)^n,$$

which implies that λ is not in $\sigma_{\text{desc}}(ae, eAe)$. Conversely, let λ be in $K \setminus \sigma_{\text{desc}}(ae, eAe)$. Then we have $(ae - \lambda e)^{n+1}c = (ae - \lambda e)^n$ for some $c \in eAe$ and $n \in \mathbb{N} \cup \{0\}$. Take $\mu \in K$ such that $\lambda^{n+1}\mu e' = \lambda^n e'$, where $e' = 1 - e$. Then we may write

$$\begin{aligned}(ae - \lambda)^n &= \big(ae - \lambda e + \lambda e'\big)^n = (ae - \lambda e)^n + \lambda^n e' \\ &= (ae - \lambda e)^{n+1}c + \lambda^{n+1}\mu e' = (ae - \lambda)^{n+1}\big(c + \mu e'\big),\end{aligned}$$

which implies that λ is not in $\sigma_{\text{desc}}(ae, A)$. Now, equality (3.1) has been proved.

If λ is in $K \setminus \sigma_{\text{desc}}(a, A)$, then we have $(a - \lambda)^{n+1}c = (a - \lambda)^n$ for some $c \in A$ and $n \in \mathbb{N} \cup \{0\}$, so $(ae - \lambda e)^{n+1}ece = (ae - \lambda e)^n$, and so λ is not in $\sigma_{\text{desc}}(ae, eAe) = \sigma_{\text{desc}}(ae, A)$. Therefore we have $\sigma_{\text{desc}}(ae, A) \subseteq \sigma_{\text{desc}}(a, A)$, and analogously $\sigma_{\text{desc}}(ae', A) \subseteq \sigma_{\text{desc}}(a, A)$. Now let λ be in $K \setminus [\sigma_{\text{desc}}(ae, A) \cup \sigma_{\text{desc}}(ae', A)]$. Since $\sigma_{\text{desc}}(ae, A) = \sigma_{\text{desc}}(ae, eAe)$ and $\sigma_{\text{desc}}(ae', A) = \sigma_{\text{desc}}(ae', e'Ae')$, there exist $b \in eAe$, $c \in e'Ae'$, and $n \in \mathbb{N} \cup \{0\}$ such that

$$(ae - \lambda e)^{n+1}b = (ae - \lambda e)^n \quad \text{and} \quad \big(ae' - \lambda e'\big)^{n+1}c = \big(ae' - \lambda e'\big)^n.$$

Therefore we have $(a - \lambda)^{n+1}(b + c) = (a - \lambda)^n$, which implies that λ is not in $\sigma_{\text{desc}}(a, A)$. □

Theorem 3.4. *Let A be an algebra over a field K, and let a be in A such that there exists $n \in \mathbb{N}$ in such a way that Aa^n is an algebraic subset of A. Then we have that $\sigma_{\text{desc}}(a+b, A) = \sigma_{\text{desc}}(b, A)$ for every $b \in \text{Cent}(a)$.*

Proof. Put $I := Aa^n$, which is an algebraic left ideal of A. Since $a^n \in I$, and I is algebraic, a is algebraic. Let e be the idempotent in $K[a]$ given by Lemma 3.2(1), so that $a(1-e)$ is nilpotent. Since ae is invertible in $K[a]e$, we deduce that $a^n e = (ae)^n$ is invertible in $K[a]e$, and hence that, for some $c \in K[a]e$, we have $e = ca^n \in cI \subseteq I$. Now, let b be in $\text{Cent}(a)$. Then, by (3.2), we have

$$\begin{aligned}\sigma_{\text{desc}}(a+b, A) &= \sigma_{\text{desc}}\big((a+b)e, A\big) \cup \sigma_{\text{desc}}\big((a+b)(1-e), A\big)\\ &= \sigma_{\text{desc}}\big((a+b)e, A\big) \cup \sigma_{\text{desc}}\big(b(1-e), A\big),\end{aligned}$$

the last equality being true because $a(1-e)$ is nilpotent, and Lemma 3.1 applies. On the other hand, $\sigma_{\text{desc}}((a+b)e, A) = \emptyset$ because $(a+b)e \in I$, I is algebraic, and Lemma 3.2(2) applies. It follows that

$$\sigma_{\text{desc}}(a+b, A) = \sigma_{\text{desc}}\big(b(1-e), A\big).$$

But, again by (3.2) and Lemma 3.2(2), we have $\sigma_{\text{desc}}(b(1-e), A) = \sigma_{\text{desc}}(b, A)$. □

Theorem 3.4 remains true if the requirement that Aa^n is algebraic is relaxed to the one that $\text{Cent}(A)a^n$ is algebraic.

We recall that the socle of an algebra A (denoted by $\text{Soc}(A)$) is defined as the sum of all minimal left ideals of A. If A is semiprime, then $\text{Soc}(A)$ coincides with the sum of all minimal right ideals of A, and is indeed an ideal of A. The next corollary follows straightforwardly from Theorem 3.4.

Corollary 3.5. *Let A be an algebra over a field K such that $\text{Soc}(A)$ is algebraic, and let a be in A such that $a^n \in \text{Soc}(A)$ for some $n \in \mathbb{N}$. Then we have that $\sigma_{\text{desc}}(a+b, A) = \sigma_{\text{desc}}(b, A)$ for every $b \in \text{Cent}(a)$.*

Now we are ready to state and prove the desired variant of Theorem 1.2.

Theorem 3.6. *Let A be a semisimple complex Banach algebra, and let a be in A. Then the following assertions are equivalent:*

(1) *The equality $\sigma_{\text{desc}}(a+b, A) = \sigma_{\text{desc}}(b, A)$ holds for every $b \in \text{Cent}(a)$.*
(2) *There exists an idempotent $e \in \mathbb{C}[a] \cap \text{Soc}(A)$ such that $a(1-e)$ is nilpotent.*
(3) *There exists an idempotent $e \in \mathbb{C}[a]$ such that ae belongs to $\text{Soc}(A)$ and $a(1-e)$ is nilpotent.*
(4) *There exists $c \in A$ such that ac belongs to $\text{Soc}(A)$ and $a(1-c)$ is nilpotent.*
(5) *There exist $c \in A$ and $n \in \mathbb{N}$ such that both ac and $(a(1-c))^n$ belong to $\text{Soc}(A)$.*
(6) *There exists $n \in \mathbb{N}$ such that $a^n \in \text{Soc}(A)$.*

Proof. (1) $\Rightarrow$ (2). Taking $b = 0$ in the assumption (1), we obtain that $\sigma_{\text{desc}}(a, A) = \emptyset$, and hence (as a consequence of [4, Theorem 1.5]) that a is algebraic. Let $\phi_a(\mathbf{X}) = \prod_{i=1}^m (\mathbf{X} - \lambda_i)^{k_i}$ be the minimum polynomial of a. Then we have a natural isomorphism

$$\mathbb{C}[a] \cong \frac{\mathbb{C}[\mathbf{X}]}{((\mathbf{X}-\lambda_1)^{k_1})} \times \cdots \times \frac{\mathbb{C}[\mathbf{X}]}{((\mathbf{X}-\lambda_m)^{k_m})}$$

which provides us with a set $\{e_1, e_2, \ldots, e_m\}$ of pairwise orthogonal idempotents of $\mathbb{C}[a]$ such that $1 = \sum_{i=1}^m e_i$ and $(a - \lambda_i)e_i$ is nilpotent for every $i = 1, \ldots, m$. Let e denote the sum of those e_i such that $\lambda_i \neq 0$. Then, clearly, e is an idempotent in $\mathbb{C}[a]$ such that $a(1-e)$ is nilpotent. Therefore, to conclude the proof of the present implication, it is enough to show that e lies in $\text{Soc}(A)$. To this end, we argue by contradiction, and hence we assume that there exists $p := e_i$ such that $\lambda := \lambda_i \neq 0$ and $p \notin \text{Soc}(A)$

(equivalently, pAp is infinite-dimensional [13]). Then, by Theorem 2.3, $\mathrm{Cent}_{pAp}(ap)$ contains a non-algebraic element b. Again by [4, Theorem 1.5], that means that $\sigma_{\mathrm{desc}}(b, pAp)$ is nonempty. Since $b \in \mathrm{Cent}_{pAp}(ap)$, we have also that $b \in \mathrm{Cent}_A(a)$, and then, by applying again the assumption (1), that

$$\sigma_{\mathrm{desc}}(b, A) = \sigma_{\mathrm{desc}}(a + b, A) = \sigma_{\mathrm{desc}}(ap + b, A) \cup \sigma_{\mathrm{desc}}\big(a(1 - p), A\big),$$

the last equality being true by Proposition 3.3. But

$$\sigma_{\mathrm{desc}}(ap + b, A) = \sigma_{\mathrm{desc}}\big(\lambda p + (a - \lambda)p + b, A\big) = \sigma_{\mathrm{desc}}(\lambda p + b, A)$$

because $(a - \lambda)p$ is nilpotent, and Lemma 3.1 applies. By applying again Proposition 3.3, it follows that

$$\begin{aligned}\lambda + \sigma_{\mathrm{desc}}(b, pAp) &= \sigma_{\mathrm{desc}}(\lambda p + b, pAp)\\ &= \sigma_{\mathrm{desc}}(\lambda p + b, A) \subseteq \sigma_{\mathrm{desc}}(b, A) = \sigma_{\mathrm{desc}}(b, pAp),\end{aligned}$$

which is a contradiction because $\lambda \neq 0$ and $\sigma_{\mathrm{desc}}(b, pAp)$ is bounded and nonempty.

The implications (2) ⇒ (3), (3) ⇒ (4), and (4) ⇒ (5) are clear.

(5) ⇒ (6). Let c and n be the elements of A and $\mathbb{N}$, respectively, whose existence is assumed in (5). Let $\pi : A \to A/\mathrm{Soc}(A)$ stand for the natural homomorphism. Since both ac and $(a(1 - c))^n$ lie in $\mathrm{Soc}(A)$, we have

$$0 = \pi\big(\big(a(1 - c)\big)^n\big) = \big(\pi\big(a(1 - c)\big)\big)^n = \big(\pi(a)\big)^n = \pi\big(a^n\big).$$

(6) ⇒ (1). By Corollary 3.5 (since the socle of any semisimple complex Banach algebra is algebraic [13, Theorem 3.2]). □

Given a linear operator T on a vector space X, we can consider the descent spectrum of T as an operator on X, $\sigma_{\mathrm{desc}}(T)$, as well as its descent spectrum $\sigma_{\mathrm{desc}}(T, \mathcal{L}(X))$ as an element of the algebra $\mathcal{L}(X)$ (of all linear operators on X). Actually, we have the following.

Proposition 3.7. *Let T be a linear operator on vector space X over a field K. Then we have $\sigma_{\mathrm{desc}}(T) = \sigma_{\mathrm{desc}}(T, \mathcal{L}(X))$.*

Proof. The inclusion $\sigma_{\mathrm{desc}}(T) \subseteq \sigma_{\mathrm{desc}}(T, \mathcal{L}(X))$ is clear. To show the converse inclusion, we recall that, given $F, G \in \mathcal{L}(X)$ with $R(F) \subseteq R(G)$, there exists $S \in \mathcal{L}(X)$ such that $F = GS$. Indeed, taking a subspace N of X such that $X = \ker G \oplus N$, and denoting by p the projection from X onto N corresponding to the decomposition $X = \ker G \oplus N$, for $x \in X$, the set $\widehat{S}(x) := \{y \in X\colon\ F(x) = G(y)\}$ becomes an element of $X/\ker G$, and the set $p(\widehat{S}(x))$ reduces to a singleton (say $S(x) \in \widehat{S}(x)$), so that the mapping $S : x \to S(x)$ becomes a linear operator on X satisfying $F = GS$. Now, let λ be in $K \setminus \sigma_{\mathrm{desc}}(T)$. Then we have $R((T - \lambda)^{n+1}) = R((T - \lambda)^n)$ for some $n \in \mathbb{N} \cup \{0\}$. By the above, there exists $S \in \mathcal{L}(X)$ such that

$$(T - \lambda)^{n+1} S = (T - \lambda)^n,$$

which implies that λ does not belong to $\sigma_{\mathrm{desc}}(T, \mathcal{L}(X))$. □

Let X be a vector space over a field K. Then $\mathrm{Soc}(\mathcal{L}(X))$ is precisely the ideal of all finite-rank linear operators on X, and hence is algebraic. It follows from Corollary 3.5 and Proposition 3.7 that, if T is a linear operator on X some power of which has a finite rank, then we have

$$\sigma_{\text{desc}}(T+S)=\sigma_{\text{desc}}(S)$$

for every linear operator S on X commuting with T. Thus our results contain Theorem 1.1, and hence also the "only if" part of Theorem 1.2.

Now, let X be a complex Banach space, and let $\mathrm{BL}(X)$ stand for the Banach algebra of all bounded linear operators on X. Then $\mathrm{Soc}(\mathrm{BL}(X))$ is precisely the ideal of all finite-rank bounded linear operators on X. Therefore, since $\mathrm{BL}(X)$ is semisimple (it is in fact primitive), we can apply Theorem 3.6 to obtain that, given $T \in \mathrm{BL}(X)$, some power of T has finite rank if and only if the equality $\sigma_{\text{desc}}(T+S, \mathrm{BL}(X)) = \sigma_{\text{desc}}(S, \mathrm{BL}(X))$ holds for every bounded linear operator S on X commuting with T. Although this result mimics Theorem 1.2, both results would read identically if and only if the equality $\sigma_{\text{desc}}(T) = \sigma_{\text{desc}}(T, \mathrm{BL}(X))$ were true for every $T \in \mathrm{BL}(X)$. As we show in [7], this happens for many Banach spaces X (including ℓ_1 and all Hilbert spaces), but, for suitable choices of X (including the one $X = \ell_p$, for $1 < p \leqslant \infty$ with $p \neq 2$), the above equality fails for some T. Thus, Theorems 3.6 and 1.2 seem to be independent results. Anyway, as a byproduct of our discussion, we have the following.

Corollary 3.8. *Let X be a complex Banach space, and let T and F be in $\mathrm{BL}(X)$ such that F commutes with T, and F^n has a finite rank for some $n \in \mathbb{N}$. Then $\sigma_{\text{desc}}(T) = \sigma_{\text{desc}}(T, \mathrm{BL}(X))$ if and only if $\sigma_{\text{desc}}(F+T) = \sigma_{\text{desc}}(F+T, \mathrm{BL}(X))$.*

Proof. By Theorem 1.1, we have $\sigma_{\text{desc}}(T) = \sigma_{\text{desc}}(F+T)$, whereas the equality $\sigma_{\text{desc}}(T, \mathrm{BL}(X)) = \sigma_{\text{desc}}(F+T, \mathrm{BL}(X))$ follows from Theorem 3.6. By putting together these equalities, the result follows. □

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