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$(H(\cdot,\cdot),\eta)$-accretive operators with an application for solving set-valued variational inclusions in Banach spaces

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ABSTRACT

In this paper, we introduce a new class of accretive operators—$(H(\cdot,\cdot),\eta)$-accretive operators, which generalize many existing monotone or accretive operators. The resolvent operator associated with an $(H(\cdot,\cdot),\eta)$-accretive operator is defined and its Lipschitz continuity is presented. By using the new resolvent operator technique, we also introduce and study a new class of set-valued variational inclusions involving $(H(\cdot,\cdot),\eta)$-accretive operators and construct a new algorithm for solving this class of set-valued variational inclusions. These results are new, and improve and generalize many known corresponding results.

1. Introduction

The resolvent operator method is an important and useful tool for studying the approximation solvability of nonlinear variational inequalities and variational inclusions, which are providing mathematical models to some problems arising in optimization and control, economics and engineering science. In order to study various variational inequalities and variational inclusions, Ding [1], Huang and Fang [2], Fang and Huang [3], Fang et al. [4], Verma [5,6], Zhang [7], Sun et al. [8], Fang and Huang [9], Huang and Fang [10], Kazmi and Khan [11], Lan et al. [12] and Zou and Huang [13] have introduced the concepts of η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H,η)-monotone operators, A-monotone operators, (A,η)-monotone operators, G-η-monotone operators, M-monotone operators in Hilbert spaces, H-accretive operators, generalized m-accretive mappings, P-η-accretive operators, (A,η)-accretive mappings and $H(\cdot,\cdot)$-accretive operators in Banach spaces and their resolvent operators, respectively. Further, by using the resolvent operator technique, a number of nonlinear variational inclusions and many systems of variational inequalities and variational inclusions have been studied by some authors in recent years; see, for example, [14–21].

Motivated and inspired by the above works, we introduce a new class of accretive operators: $(H(\cdot,\cdot),\eta)$-accretive operators, which provide a unifying framework for maximal monotone operators [22], η-subdifferential operators [1], maximal η-monotone operators [2], H-monotone operators [3], (H,η)-monotone operators [4], A-monotone mappings [5], (A,η)-monotone operators [6], G-η-monotone operators [7], M-monotone operators [8], H-accretive operators [9], generalized m-accretive mappings [10], P-η-accretive operators [11], (A,η)-accretive mappings [12] and $H(\cdot,\cdot)$-accretive operators [13]. The resolvent operator associated with an $(H(\cdot,\cdot),\eta)$-accretive operator is defined and its Lipschitz continuity is presented. By using the new resolvent operator technique, we also introduce and study a new class of set-valued variational inclusions involving $(H(\cdot,\cdot),\eta)$-accretive operators and construct a new algorithm for solving this class of set-valued variational inclusions. These results are new, and improve and generalize many known corresponding results.

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2. Preliminaries

Let E be a real Banach space with dual space E^*, and the norm and the dual pair between E and E^* are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. $CB(E)$ denote the family of all the nonempty closed and bounded subsets of E and 2^E is the power set of E. $\tilde{H}(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$\tilde{H}(A,B)=\max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\},\quad \forall A,B\in CB(E).$$

The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x)=\{f^*\in E^* : \langle x,f^*\rangle=\|f^*\|\|x\|, \|f^*\|=\|x\|^{q-1}\},\quad \forall x\in E,$$

where $q>1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x)=\|x\|^{q-2}J(x)$, for all $x\neq 0$, and J_q is single-valued if E^* is strictly convex.

The modulus of smoothness of E is the function $\rho_E : [0,\infty)\to[0,\infty)$ defined by

$$\rho_E(t)=\sup\left\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1 : \|x\|\le 1, \|y\|\le t\right\}.$$

A Banach space E is called uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t}=0.$$

E is called q-uniformly smooth if there exists a constant $c>0$, such that

$$\rho_E(t)\le ct^q,\quad q>1.$$

Note that J_q is single-valued if E is uniformly smooth.

Lemma 2.1 ([23]). *Let E be a real uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c_q>0$, such that for all $x,y\in E$,*

$$\|x+y\|^q\le\|x\|^q+q\langle y,J_q(x)\rangle+c_q\|y\|^q.$$

Throughout the rest of the paper, unless otherwise stated, we assume that E is q-uniformly smooth.

Definition 2.1. Let $A,B,T : E\to E$ and $H,\eta : E\times E\to E$ be five single-valued mappings.

(1) A is said to be η-accretive if $\langle A(x)-A(y), J_q(\eta(x,y))\rangle\ge 0$;

(2) A is said to be strictly η-accretive if A is η-accretive and

$$\langle A(x)-A(y), J_q(\eta(x,y))\rangle=0$$

if and only if $x=y$;

(3) $H(A,\cdot)$ is said to be α-strongly η-accretive with respect to A if there exists a constant $\alpha>0$ such that

$$\langle H(Ax,u)-H(Ay,u), J_q(\eta(x,y))\rangle\ge\alpha\|x-y\|^q,\quad\forall x,y,u\in E;$$

(4) $H(\cdot,B)$ is said to be β-relaxed η-accretive with respect to B if there exists a constant $\beta>0$ such that

$$\langle H(u,Bx)-H(u,By), J_q(\eta(x,y))\rangle\ge-\beta\|x-y\|^q,\quad\forall x,y,u\in E;$$

(5) $H(\cdot,\cdot)$ is said to be λ-Lipschitz continuous with respect to A if there exists a constant $\lambda>0$ such that

$$\|H(Ax,u)-H(Ay,u)\|\le\lambda\|x-y\|,\quad\forall x,y,u\in E;$$

(6) A is said to be ϵ-Lipschitz continuous if there exists a constant $\epsilon>0$ such that

$$\|A(x)-A(y)\|\le\epsilon\|x-y\|,\quad\forall x,y\in E;$$

(7) η is said to be τ-Lipschitz continuous if there exists a constant $\tau>0$ such that

$$\|\eta(x,y)\|\le\tau\|x-y\|,\quad\forall x,y\in E.$$

Remark 2.1. If $\eta(x,y)=x-y, \forall x,y\in E$, then (1)–(4) of Definition 2.1 reduce to (1)–(4) of Definition 2.1 in [13], respectively.

Definition 2.2 ([12]). Let $M : E\to 2^E$ be a multi-valued mapping, $A,H : E\to E$ and $\eta : E\times E\to E$ be single-valued mappings. M is said to be

(1) accretive if $\langle x-y, j_q(u-v)\rangle\ge 0, \forall u,v\in E, x\in M(u), y\in M(v)$;

(2) η-accretive if $\langle x-y, j_q(\eta(u,v))\rangle\ge 0, \forall u,v\in E, x\in M(u), y\in M(v)$;

(3) strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;

(4) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle x - y, J_q(\eta(u, v))\rangle \geq r\|u - v\|^q, \quad \forall u, v \in E,\ x \in M(u), y \in M(v);$$

(5) m-relaxed η-accretive if there exists a constant $m > 0$ such that

$$\langle x - y, \eta(u, v)\rangle \geq -m\|u - v\|^q, \quad \forall u, v \in E, x \in M(u), y \in M(v);$$

(6) m-accretive if M is accretive and $(I + \rho M)(E) = E$ for all $\rho > 0$, where I denotes the identity operator on E;

(7) generalized m-accretive if M is η-accretive and $(I + \rho M)(E) = E$ for all $\rho > 0$, where I denotes the identity operator on E;

(8) H-accretive if M is accretive and $(H + \rho M)E = E$ for all $\rho > 0$;

(9) (H, η)-accretive if M is η-accretive and $(H + \rho M)E = E$ for all $\rho > 0$;

(10) (A, η)-accretive if M is m-η-relaxed accretive and $(A + \rho M)E = E$ for all $\rho > 0$.

Definition 2.3. Let $T, S : E \to 2^E$ be set-valued mapping, $A, B : E \to E$ and $H, F : E \times E \to E$ be single-valued mappings.

(1) F is said to be $\bar{\alpha}$-strongly accretive with respect to S and $H(A, B)$ in the first argument if there exists a constant $\bar{\alpha} > 0$ such that

$$\langle F(w_1, \cdot) - F(w_2, \cdot), J_q(H(Au, Bu) - H(Av, Bv))\rangle \geq \bar{\alpha}\|H(Au, Bu) - H(Av, Bv)\|^q$$
$$\forall u, v \in E, w_1 \in S(u), w_2 \in S(v);$$

(2) F is said to be $\bar{\beta}$-strongly accretive with respect to T and $H(A, B)$ in the second argument if there exists a constant $\bar{\beta} > 0$ such that

$$\langle F(\cdot, w_1) - F(\cdot, w_2), J_q(H(Au, Bu) - H(Av, Bv))\rangle \geq \bar{\beta}\|H(Au, Bu) - H(Av, Bv)\|^q$$
$$\forall u, v \in E, w_1 \in T(u), w_2 \in T(v);$$

(3) F is said to be ξ_1-Lipschitz in the first argument if there exists a constant $\xi_1 > 0$ such that

$$\|F(u_1, v') - F(u_2, v')\| \leq \xi_1\|u_1 - u_2\|, \quad \forall u_1, u_2, v' \in E.$$

Remark 2.2. If $H(Ax, Bx) = Ax, \forall\, x \in E$, then (1) of Definition 2.3 reduces to (3) of Definition 2.4 in [24].

3. $(H(\cdot, \cdot), \eta)$-accretive operator

In this section, we shall introduce a new class of set-valued accretive operators—$(H(\cdot, \cdot), \eta)$-accretive operators and discuss some properties of this class of operators

Definition 3.1. Let $H, \eta : E \times E \to E, A, B : E \to E$ be four single-valued mappings. Then the set-valued mapping $M : E \to 2^E$ is said to be $(H(\cdot, \cdot), \eta)$-accretive with respect to mappings A and B (or simply $(H(\cdot, \cdot), \eta)$-accretive in the sequel), if M is m-relaxed η-accretive and $(H(A, B) + \rho M)(E) = E$ for all $\rho > 0$.

Remark 3.1. (1) When $m = 0$ and $\eta(x, y) = x - y,\ \forall\, x, y \in E$, Definition 3.1 reduces to the definition of $H(\cdot, \cdot)$-accretive operators [13].

(2) If $H(Au, Bu) = Au, \forall\ u \in E$, then Definition 3.1 reduces to the definition of (A, η)-accretive operators [12,25]. Hence, the class of $(H(\cdot, \cdot), \eta)$-accretive operators in Banach spaces provides a unifying framework for the classes of maximal monotone operators, η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H, η)-monotone operators, A-monotone mappings, (A, η)-monotone operators, G-η-monotone operators, H-accretive operators, generalized m-accretive mappings, P-η-accretive operators. For details, see [1–7,9–12,22,25,15].

(3) If $E = \mathcal{H}$ is a Hilbert space, $m = 0$ and $\eta(x, y) = x - y, \forall x, y \in \mathcal{H}$, then Definition 3.1 reduces to the definition of M-monotone operators [8].

(4) A maximal monotone operator need not be $H(\cdot, \cdot)$-accretive operators, see Example 3.1 given in [13].

Theorem 3.1. *Let $\eta : E \times E \to E$ be a single-valued mapping and $H(A, B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B and $\alpha > \beta$. Let $M : E \to 2^E$ be an $(H(\cdot, \cdot), \eta)$-accretive operator with respect to A and B. If $\langle x - y, j_q(\eta(u, v))\rangle \geq 0$ holds for all $(v, y) \in$ Graph(M), where Graph(M) $= \{(a, b) \in E \times E : b \in M(a)\}$, then $(u, x) \in$ Graph(M).*

Proof. Since M is $(H(\cdot, \cdot), \eta)$-accretive with respect to A and B, we know that $(H(A, B) + \rho M)(E) = E$ holds for all $\rho > 0$ and so there exists $(u_1, x_1) \in$ Graph(M) such that

$$H(Au, Bu) + \rho x = H(Au_1, Bu_1) + \rho x_1. \tag{3.1}$$

Since $H(A, B)$ is α-strongly η-accretive with respect to A, β-relaxed accretive with respect to B and $\alpha > \beta$, we have

$$\begin{aligned}
0 &\le \rho\langle x - x_1, j_q(\eta(u, u_1))\rangle \\
&= -\langle H(Au, Bu) - H(Au_1, Bu_1), j_q(\eta(u, u_1))\rangle \\
&= -\langle H(Au, Bu) - H(Au_1, Bu), j_q(\eta(u, u_1))\rangle - \langle H(Au_1, Bu) - H(Au_1, Bu_1), j_q(\eta(u, u_1))\rangle \\
&\le -(\alpha - \beta)\|u - u_1\|^q \le 0.
\end{aligned}$$

This implies that $u = u_1$. From (3.1), we know that $x = x_1$. Thus $(u, x) = (u_1, x_1) \in \mathrm{Graph(M)}$. This completes the proof. □

Remark 3.2. Theorem 3.1 generalizes and improves Theorem 3.1 of [12,13], Proposition 2.1 of [8] (1) of Theorem 2.1 of [2], Proposition 2.1 of [3], Theorem 2.1 of [7,9] and (a) of Theorem 3.1 of [11].

Theorem 3.2. *Let $\eta : E \times E \to E$ be a single-valued mapping and $H(A, B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B and $\alpha > \beta$. Let $M : E \to 2^E$ be an $(H(\cdot, \cdot), \eta)$-accretive operator with respect to A and B. Then the operator $(H(A, B) + \rho M)^{-1}$ is single-valued for $0 < \rho < \frac{r}{m}$, where $r = \alpha - \beta$.*

Proof. For any given $u^* \in E$, let $\forall u, v \in (H(A, B) + \rho M)^{-1}(u^*)$. It follows that

$$-H(A(u), B(u)) + u^* \in \rho M(u) \quad \text{and} \quad -H(A(v), B(v)) + u^* \in \rho M(v).$$

Since $M : E \to 2^E$ is an $(H(\cdot, \cdot), \eta)$-accretive operator with respect to A and B and $H(A, B)$ is α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B and $\alpha > \beta$, we have

$$\begin{aligned}
-m\|u - v\|^q &\le \frac{1}{\rho}\langle(-H(Au, Bu) + u^*) - (-H(Av, Bv) + u^*), J_q(\eta(u, v))\rangle \\
&= -\frac{1}{\rho}\langle H(Au, Bu) - H(Av, Bv), J_q(\eta(u, v))\rangle \\
&= -\frac{1}{\rho}\langle H(Au, Bu) - H(Av, Bu), J_q(\eta(u, v))\rangle - \frac{1}{\rho}\langle H(Av, Bu) - H(Av, Bv), J_q(\eta(u, v))\rangle \\
&\le -\frac{1}{\rho}(\alpha - \beta)\|u - v\|^q = -\frac{r}{\rho}\|u - v\|^q.
\end{aligned}$$

This show that

$$m\rho\|u - v\|^q \ge r\|u - v\|^q.$$

If $u \ne v$, then $\rho \ge \frac{r}{m}$ contradicts with $0 < \rho < \frac{r}{m}$. Thus $u = v$, that is, $(H(A, B) + \rho m)^{-1}$ is singe-valued. The proof is completed. □

Remark 3.3. Theorem 3.2 generalizes and improves Theorem 3.2 of [12], Theorem 3.3 of [13], Theorem 2.1 of [8], (2) of Theorem 2.1 of [2], Theorem 2.1 of [3], Theorem 2.2 of [7,9] and (b) of Theorem 3.1 of [11].

Base on Theorem 3.2, we can define the generalized resolvent operator $R^{M,\eta}_{H(\cdot,\cdot),\rho}$ associated with an $(H(\cdot, \cdot), \eta)$-accretive mapping M as follows.

Definition 3.2. Let $\eta : E \times E \to E$ be a single-valued mapping and $H(A, B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B and $\alpha > \beta$. Let $M : E \to 2^E$ be an $(H(\cdot, \cdot), \eta)$-accretive operator with respect to A and B. Then the general resolvent operator $R^{M,\eta}_{H(\cdot,\cdot),\rho} : E \to E$ is defined by

$$R^{M,\eta}_{H(\cdot,\cdot),\rho}(u) = (H(A, B) + \rho M)^{-1}(u), \quad \forall u \in E. \tag{3.2}$$

Remark 3.4. The general resolvent operators associated with $(H(\cdot, \cdot), \eta)$-accretive operators include as special cases the corresponding resolvent operators associated with maximal monotone operators [22], η-subdifferential operators [1], maximal η-monotone operators [2], H-monotone operators [3], (H, η)-monotone operators [4], A-monotone mappings [5], (A, η)-monotone operators [6], G-η-monotone operators [7], M-monotone operators [8], H-accretive operators [9], generalized m-accretive mappings [10], P-η-accretive operators [11], (A, η)-accretive mappings [12] and $H(\cdot, \cdot)$-accretive operators [13].

Theorem 3.3. *Let $\eta : E \times E \to E$ be τ-Lipschitz continuous and $H(A, B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B and $\alpha > \beta$. Let $M : E \to 2^E$ be an $(H(\cdot, \cdot), \eta)$-accretive operator with respect to A and B. Then the resolvent operator $R^{M,\eta}_{H(\cdot,\cdot),\rho} : E \to E$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous for $0 < \rho < \frac{r}{m}$, where $r = \alpha - \beta$, that is,*

$$\|R^{M,\eta}_{H(\cdot,\cdot),\rho}(u) - R^{M,\eta}_{H(\cdot,\cdot),\rho}(v)\| \le \frac{\tau^{q-1}}{r - \rho m}\|u - v\|, \quad \forall u, v \in E.$$

Proof. Let $u, v \in H$ be any given points, it follows from (3.2) that

$$R^{M,\eta}_{H(\cdot,\cdot),\rho}(u) = (H(A,B) + \rho M)^{-1}(u)$$

and

$$R^{M,\eta}_{H(\cdot,\cdot),\rho}(v) = (H(A,B) + \rho M)^{-1}(v).$$

This implies that

$$\frac{1}{\rho}(u - H(A(R^{M,\eta}_{H(\cdot,\cdot),\rho}(u)), B(R^{M,\eta}_{H(\cdot,\cdot),\rho}(u)))) \in M(R^{M,\eta}_{H(\cdot,\cdot),\rho}(u)),$$
$$\frac{1}{\rho}(v - H(A(R^{M,\eta}_{H(\cdot,\cdot),\rho}(v)), B(R^{M,\eta}_{H(\cdot,\cdot),\rho}(v)))) \in M(R^{M,\eta}_{H(\cdot,\cdot),\rho}(v)).$$

For the sake of brevity, let $z_1 = R^{M,\eta}_{H(\cdot,\cdot),\rho}(u)$ and $z_2 = R^{M,\eta}_{H(\cdot,\cdot),\rho}(v)$.
Since M is m-relaxed η-accretive, we get

$$\begin{aligned} -m\|z_1 - z_2\|^q &\le \frac{1}{\rho}\langle u - H(Az_1, Bz_1) - (v - H(Az_2, Bz_2)), J_q(\eta(z_1, z_2))\rangle \\ &= \frac{1}{\rho}\langle u - v - (H(Az_1, Bz_1) - H(Az_2, Bz_2)), J_q(\eta(z_1, z_2))\rangle. \end{aligned}$$

From the above inequality and the conditions in the Theorem 3.3, we have

$$\begin{aligned} \tau^{q-1}\|u - v\| \cdot \|z_1 - z_2\|^{q-1} &\ge \|u - v\| \cdot \|\eta(z_1, z_2)\|^{q-1} \ge \langle u - v, J_q(\eta(z_1, z_2))\rangle \\ &\ge \langle H(Az_1, Bz_1) - H(Az_2, Bz_2), J_q(\eta(z_1, z_2))\rangle - \rho m\|z_1 - z_2\|^q \\ &\ge \langle H(Az_1, Bz_1) - H(Az_2, Bz_1), J_q(\eta(z_1, z_2))\rangle \\ &\quad + \langle H(Az_2, Bz_1) - H(Az_2, Bz_2), J_q(\eta(z_1, z_2))\rangle - \rho m\|z_1 - z_2\|^q \\ &\ge (\alpha - \beta - \rho m)\|z_1 - z_2\|^q = (r - \rho m)\|z_1 - z_2\|^q. \end{aligned}$$

Hence

$$\|R^{M,\eta}_{H(\cdot,\cdot),\rho}(u) - R^{M,\eta}_{H(\cdot,\cdot),\rho}(v)\| \le \frac{\tau^{q-1}}{r - \rho m}\|u - v\|, \quad \forall u, v \in E.$$

This completes the proof. □

Remark 3.5. Theorem 3.2 generalizes and improves Theorem 3.3 of [12], Theorem 3.4 of [13], Theorem 2.2 of [8], Theorem 2.2 of [2,3], Theorem 2.3 of [7,9] and Theorem 3.2 of [11].

4. An application for solving set-valued variational inclusions

In this section, we shall study a new class of set-valued variational inclusions involving $(H(\cdot,\cdot), \eta)$-accretive operators in Banach spaces and construct an iterative algorithm for approximating the solution of this class of variational inclusions by using the resolvent operator technique.

Let $F : E \times E \to E$ be a single-valued mapping, $S, T : E \to CB(E)$ and $M : E \to 2^E$ be set-valued mappings. For any given $a \in E$, we consider the following set-valued variational inclusion problem: find $x \in E$, $w \in S(x)$, $v \in T(x)$ such that

$$a \in F(w, v) + M(x). \tag{4.1}$$

Special cases of the problem (4.1):

(1) If $S, T : E \to E$ be single-valued mappings and $M(x) = \lambda N(x)$, where $\lambda > 0$ is a constant, then the problem (4.1) reduces to the following problem: find $x \in E$ such that

$$a \in F(S(x), T(x)) + \lambda N(x). \tag{4.2}$$

If M is an (A, η)-accretive mapping, then the problem (4.2) was introduced and studied by Lan et al. [12].

(2) If $\lambda = 1$, $a = 0$ and $F(S(x), T(x)) = T(x)$ for all $x \in E$, where $T : E \to E$ is a single-valued mapping, then the problem (4.2) reduces to the following problem: find $x \in E$ such that

$$0 \in T(x) + N(x). \tag{4.3}$$

If N is an $H(\cdot,\cdot)$-accretive mapping, then the problem (4.3) was studied by Zou and Huang [13]; If N is a generalized m-accretive mapping was studied by Bi et al. [26].

(3) When $E = \mathcal{H}$ is a Hilbert space and N is an H-monotone operators, then the problem (4.3) was introduced and studied by Fang and Huang [3] and includes many variational inequalities (inclusions) and complementarity problems as special cases. For example, see [27,28].

Definition 4.1. A set-valued mapping $A : E \to CB(E)$ is said to be $\tilde{H}$-Lipschitz continuous if there exists a constant $\mathfrak{L} > 0$ such that

$$\tilde{H}(A(x), A(y)) \leq \mathfrak{L}\|x - y\|, \quad \forall x, y \in E.$$

From Definition 3.2, we can obtain the following conclusion.

Lemma 4.1. *Let $\eta : E \times E \to E$ be a single-valued mapping and $H(A, B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B and $\alpha > \beta$. Let $M : E \to 2^E$ be an $(H(\cdot,\cdot), \eta)$-accretive operator with respect to A and B. Then $x \in E$, $w \in S(x)$, $v \in T(x)$ is a solution of the set-valued variational inclusion (4.1) if and only if $x \in E$, $w \in S(x)$, $v \in T(x)$ satisfies*

$$x = R^{M,\eta}_{H(\cdot,\cdot),\rho}(H(Ax, Bx) - \rho F(w, v) + \rho a), \tag{4.4}$$

where $R^{M,\eta}_{H(\cdot,\cdot),\rho}(u) = (H(A, B) + \rho M)^{-1}(u)$, $\forall u \in E$ and $\rho > 0$ is a constant.

Remark 4.1. The equality (4.4) can be written as

$$z = H(Ax, Bx) - \rho F(w, v) + \rho a, \qquad x = R^{M,\eta}_{H(\cdot,\cdot),\rho}(z),$$

where $a \in E$ is any given element and $\rho > 0$ is a constant. By Nadler [29], we know that this fixed point formulation enables us to suggest the following iterative algorithm.

Algorithm 4.1. For any given $z_0 \in B$, we can choose $x_0 \in B$ such that sequences $\{x_n\}$, $\{w_n\}$ and $\{v_n\}$ satisfy

$$\begin{cases} x_n = R^{M,\eta}_{H(\cdot,\cdot),\rho}(z_n), \\ w_n \in S(x_n), \qquad \|w_n - w_{n+1}\| \leq \left(1 + \dfrac{1}{n+1}\right)\tilde{H}(S(x_n), S(x_{n+1})), \\ v_n \in T(x_n), \qquad \|v_n - v_{n+1}\| \leq \left(1 + \dfrac{1}{n+1}\right)\tilde{H}(T(x_n), T(x_{n+1})), \\ z_{n+1} = H(Ax_n, Bx_n) - \rho F(w_n, v_n) + \rho a + e_n, \\ \displaystyle\sum_{j=1}^{\infty} \|e_j - e_{j-1}\|\varpi^{-j} < \infty, \quad \forall \varpi \in (0, 1), \lim_{n\to\infty} e_n = 0, \end{cases} \tag{4.5}$$

where $\rho > 0$ is a constant, $a \in E$ is any given element and $\{e_n\} \subset E$ is an error to take into account a possible inexact computation of the resolvent operator point for all $n \geq 0$, and $\tilde{H}(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$.

Theorem 4.1. *Let $\eta : E \times E \to E$ be a τ-Lipschitz continuous and $H(A, B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B and $\alpha > \beta$. Let $M : E \to 2^E$ be an $(H(\cdot,\cdot), \eta)$-accretive operator with respect to A and B. Suppose the following conditions are satisfied:*

(1) S and T are $\tilde{H}$-Lipschitz continuous with constants σ and δ, respectively;

(2) F is $\bar{\alpha}$-strongly accretive with respect to S and $H(A, B)$ in the first argument and $\bar{\beta}$-strongly accretive with respect to T and $H(A, B)$ in the second argument;

(3) $H(A, B)$ is γ_1-Lipschitz continuous with respect to A and γ_2-Lipschitz continuous with respect to B, F is ξ_1-Lipschitz continuous in the first argument and ξ_2-Lipschitz continuous in the second argument;

In addition, there exist constants $0 < \rho < \frac{r}{m}$ $(r = \alpha - \beta)$ such that

$$\tau^{q-1}\sqrt[q]{(\gamma_1 + \gamma_2)^q + c_q\rho^q(\xi_1\sigma + \xi_2\delta)^q - \rho q(\bar{\alpha} + \bar{\beta})(\gamma_1 + \gamma_2)^q} < r - \rho m. \tag{4.6}$$

Then the iterative sequences $\{x_n\}$, $\{w_n\}$ and $\{v_n\}$ generated by Algorithm 4.1 converge strongly to $\{x\}$, $\{w\}$ and $\{v\}$, respectively, and (x, w, v) is a solution of the problem (4.1).

Proof. It follows from (4.5) and Theorem 3.3 that

$$\|x_{n+1} - x_n\| = \|R^{M,\eta}_{H(\cdot,\cdot),\rho}(z_{n+1}) - R^{M,\eta}_{H(\cdot,\cdot),\rho}(z_n)\| \leq \frac{\tau^{q-1}}{r - \rho m}\|z_{n+1} - z_n\|. \tag{4.7}$$

From (4.5), we can get

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|H(Ax_n, Bx_n) - \rho F(w_n, v_n) + \rho a + e_n \\ &\quad - (H(Ax_{n-1}, Bx_{n-1}) - \rho F(w_{n-1}, v_{n-1}) + \rho a + e_{n-1})\| \\ &\leq \|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1}) - (\rho F(w_n, v_n) - \rho F(w_{n-1}, v_{n-1}))\| + \|e_n - e_{n-1}\|. \end{aligned} \tag{4.8}$$

By Lemma 2.1, we have

$$\begin{aligned}&\|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1}) - \rho(F(w_n, v_n) - F(w_{n-1}, v_{n-1}))\|^q\\ &\quad \le \|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1})\|^q + c_q\rho^q\|F(w_n, v_n) - F(w_{n-1}, v_{n-1})\|^q\\ &\qquad - \rho q\langle F(w_n, v_n) - F(w_{n-1}, v_{n-1}), J_q(H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1}))\rangle. \end{aligned} \tag{4.9}$$

From (4.5) and the conditions (1) and (3), we have

$$\begin{aligned}\|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1})\| &\le \|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_n)\| + \|H(Ax_{n-1}, Bx_n) - H(Ax_{n-1}, Bx_{n-1})\|\\ &\le (\gamma_1 + \gamma_2)\|x_n - x_{n-1}\| \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\|F(w_n, v_n) - F(w_{n-1}, v_{n-1})\| &\le \|F(w_n, v_n) - F(w_{n-1}, v_n)\| + \|F(w_{n-1}, v_n) - F(w_{n-1}, v_{n-1})\|\\ &\le \xi_1\|w_n - w_{n-1}\| + \xi_2\|v_n - v_{n-1}\|\\ &\le \xi_1\left(1 + \frac{1}{n}\right)\tilde{H}(S(x_n), S(x_{n-1})) + \xi_2\left(1 + \frac{1}{n}\right)\tilde{H}(T(x_n), T(x_{n-1}))\\ &\le \left(\xi_1\left(1 + \frac{1}{n}\right)\sigma + \xi_2\left(1 + \frac{1}{n}\right)\delta\right)\|x_n - x_{n-1}\|. \end{aligned} \tag{4.11}$$

By the conditions (2) and (3), we have

$$\begin{aligned}&-\langle F(w_n, v_n) - F(w_{n-1}, v_{n-1}), J_q(H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1}))\rangle\\ &\quad = -[\langle F(w_n, v_n) - F(w_{n-1}, v_n), J_q(H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1}))\rangle\\ &\qquad + \langle F(w_{n-1}, v_n) - F(w_{n-1}, v_{n-1}), J_q(H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1}))\rangle]\\ &\quad \le -(\bar{\alpha} + \bar{\beta})\|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1})\|^q\\ &\quad \le -(\bar{\alpha} + \bar{\beta})(\gamma_1 + \gamma_2)^q\|x_n - x_{n-1}\|^q. \end{aligned} \tag{4.12}$$

From (4.9)–(4.12), it follows that

$$\begin{aligned}&\|H(Ax_n, Bx_n) - H(Ax_{n-1}, Bx_{n-1}) - \rho(F(w_n, v_n) - F(w_{n-1}, v_{n-1}))\|\\ &\quad \le \sqrt[q]{(\gamma_1 + \gamma_2)^q + c_q\rho^q\left(\xi_1\left(1 + \frac{1}{n}\right)\sigma + \xi_2\left(1 + \frac{1}{n}\right)\delta\right)^q - \rho q(\bar{\alpha} + \bar{\beta})(\gamma_1 + \gamma_2)^q} \cdot \|x_n - x_{n-1}\|. \end{aligned} \tag{4.13}$$

Combining (4.7), (4.8) and (4.13), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \frac{\tau^{q-1}}{r - \rho m}\|z_{n+1} - z_n\|\\ &\le \theta_n\|x_n - x_{n-1}\| + \frac{\tau^{q-1}}{r - \rho m}\|e_n - e_{n-1}\|, \end{aligned} \tag{4.14}$$

where

$$\theta_n = \frac{\tau^{q-1}}{r - \rho m}\sqrt[q]{(\gamma_1 + \gamma_2)^q + c_q\rho^q(\xi_1\left(1 + \frac{1}{n}\right)\sigma + \xi_2\left(1 + \frac{1}{n}\right)\delta)^q - \rho q(\bar{\alpha} + \bar{\beta})(\gamma_1 + \gamma_2)^q}.$$

Let

$$\theta = \frac{\tau^{q-1}}{r - \rho m}\sqrt[q]{(\gamma_1 + \gamma_2)^q + c_q\rho^q(\xi_1\sigma + \xi_2\delta)^q - \rho q(\bar{\alpha} + \bar{\beta})(\gamma_1 + \gamma_2)^q}.$$

Then we know that $\theta_n \to \theta$ as $n \to \infty$.

By (4.6), we know that $0 < \theta < 1$ and hence there exist $n_0 > 0$ and $\theta_0 \in (0, 1)$ such that $\theta_n \le \theta_0$ for all $n \ge n_0$. Therefore, by (4.14), we have

$$\|x_{n+1} - x_n\| \le \theta_0\|x_n - x_{n-1}\| + \frac{\tau^{q-1}}{r - \rho m}\|e_n - e_{n-1}\|, \quad \forall n \ge n_0. \tag{4.15}$$

(4.15) implies that

$$\|x_{n+1} - x_n\| \le \theta_0^{n-n_0}\|x_{n_0+1} - x_{n_0}\| + \frac{\tau^{q-1}}{r - \rho m}\sum_{j=1}^{n-n_0}\theta_0^{j-1}t_{n-(j-1)}, \tag{4.16}$$

where $t_n = \|e_n - e_{n-1}\|$ for all $n \geq n_0$. Hence, for any $m \geq n > n_0$, we have

$$\begin{aligned} \|x_m - x_n\| &\leq \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \\ &\leq \sum_{k=n}^{m-1} \theta_0^{k-n_0} \|x_{n_0+1} - x_{n_0}\| + \frac{\tau^{q-1}}{r - \rho m} \sum_{k=n}^{m-1} \theta_0^k \left[\sum_{j=1}^{k-n_0} \frac{t_{k-(j-1)}}{\theta_0^{k-(j-1)}} \right]. \end{aligned} \tag{4.17}$$

Since $\sum_{j=1}^{\infty} \|e_j - e_{j-1}\| \varpi^{-j} < \infty, \forall \varpi \in (0, 1)$ and $0 < \theta_0 < 1$, it follows that $\|x_m - x_n\| \to 0$ as $n \to \infty$, and so $\{x_n\}$ is a Cauchy sequence in E. Thus, there exists $x \in E$ such that $x_n \to x$ as $n \to \infty$. By Algorithm 4.1 and the $\tilde{H}$-Lipschitz continuity of S and T, we get

$$\begin{cases} \|w_n - w_{n+1}\| \leq \left(1 + \dfrac{1}{n+1}\right) \tilde{H}(S(x_n), S(x_{n+1})) \leq \left(1 + \dfrac{1}{n+1}\right) \sigma \|x_{n+1} - x_n\|, \\ \|v_n - v_{n+1}\| \leq \left(1 + \dfrac{1}{n+1}\right) \tilde{H}(T(x_n), T(x_{n+1})) \leq \left(1 + \dfrac{1}{n+1}\right) \delta \|x_{n+1} - x_n\|. \end{cases} \tag{4.18}$$

It follows that $\{w_n\}$ and $\{v_n\}$ are also Cauchy sequences. Thus, there exist w and v such that $w_n \to w$ and $v_n \to v$, as $n \to \infty$. In the sequel, we will show that $w \in S(x)$. Noting $w_n \in S(x_n)$, we have

$$\begin{aligned} d(w, S(x)) &\leq \|w - w_n\| + d(w_n, S(x)) \\ &\leq \|w - w_n\| + \tilde{H}(S(x_n), S(x)) \\ &\leq \|w - w_n\| + \sigma \|x_n - x\| \to 0, \quad (n \to \infty). \end{aligned}$$

Since $S(x)$ is closed, it implies $w \in S(x)$. Similarly, one can show that $v \in T(x)$.
By continuity, we know that x, w, v satisfy

$$x = R_{H(\cdot,\cdot),\rho}^{M,\eta} (H(Ax, Bx) - \rho F(w, v) + \rho a). \tag{4.19}$$

By Lemma 4.1, (x, w, v) is a solution of the problem (4.1). This completes the proof. □

Remark 4.2. (1) Theorem 4.1 unifies, improves and extends the results of [3,12,13] in several aspects.

(2) By Algorithm 4.1 and Theorem 4.1, it is easy to obtain the convergence results for iterative algorithms for special cases of problem (4.1) with $(H(\cdot, \cdot), \eta)$-accretive operators. We omit them here.

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