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Generalized competition indices of symmetric primitive digraphs

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ABSTRACT

For a primitive digraph D of order n and a positive integer m such that $m \leq n$, the m-competition index of D is defined as the smallest positive integer k such that for every pair of vertices x and y, there exist m distinct vertices $v_1, v_2, \ldots, v_m$ such that there are directed walks of length k from x to v_i and from y to v_i for $1 \leq i \leq m$ in D. In this study, we investigate m-competition indices of symmetric primitive digraphs and provide the upper and lower bounds. We also characterize the set of m-competition indices of symmetric primitive digraphs.

1. Preliminaries and notations

In this paper, we follow the terminology and notation used in [1,3,8]. Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$, and order n. Loops are permitted but multiple arcs are not. A *walk* from x to y in a digraph D is a sequence of vertices $x, v_1, \ldots, v_t, y \in V(D)$ and a sequence of arcs $(x, v_1), (v_1, v_2), \ldots, (v_t, y) \in E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is a walk from x to y where $x = y$. A *cycle* is a closed walk from x to y with distinct vertices except for $x = y$. The *length of a walk* W is the number of arcs in W. The notation $x \xrightarrow{k} y$ is used to indicate that there exists a walk from x to y of length k. The *distance* from vertex x to vertex y in D is the length of the shortest walk from x to y, and it is denoted by $d_D(x, y)$. For a vertex x and a set $Y \subset V(D)$, let $d_D(x, Y) = \min\{d_D(x, y) : y \in Y\}$. For $x \in Y$, we define $d_D(x, Y) = 0$.

A digraph D is called *strongly connected* if for each pair of vertices x and y in $V(D)$, there exists a walk from x to y. For a strongly connected digraph D, the *index of imprimitivity* of D is the greatest common divisor of the lengths of the cycles in D, and it is denoted by $l(D)$. If D is a trivial digraph of order 1, $l(D)$ is undefined. A strongly connected digraph D is *primitive* if $l(D) = 1$.

If D is a primitive digraph of order n, there exists some positive integer k such that there exists a walk of length exactly k from each vertex x to each vertex y. The smallest such k is called the *exponent* of D, and it is denoted by $\exp(D)$. For a positive integer m where $1 \leq m \leq n$, we define the *m-competition index* of a primitive digraph D, it is denoted by $k_m(D)$, as the smallest positive integer k such that for every pair of vertices x and y, there exist m distinct vertices $v_1, v_2, \ldots, v_m$ such that $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ for $1 \leq i \leq m$ in D.

Cho et al. [6] introduced the concept of the m-step competition graph of a digraph. Kim [9] introduced the m-competition index as a generalization of the competition index presented in [8]. Akelbek and Kirkland [1,2] introduced the scrambling

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index of a primitive digraph D, denoted by $k(D)$. In the case of primitive digraphs, the definitions of the scrambling index and competition index are identical. Furthermore, we have $k(D) = k_1(D)$. Huang and Lie [7] studied the scrambling index of primitive digraphs. Lie and Huang [10] introduced the concept of the generalized scrambling index of a primitive digraph D, denoted by $k(D, \lambda, \mu)$. This concept is a generalization of m-competition index of a primitive digraph D since $k(D, 2, m) = k_m(D)$.

On the basis of definitions of the m-competition index and the exponent of D of order n, we can write $k_m(D) \leq \exp(D)$, where m is a positive integer such that $1 \leq m \leq n$. Furthermore, we have $k_n(D) = \exp(D)$ and

$$k(D) = k_1(D) \leq k_2(D) \leq \cdots \leq k_n(D) = \exp(D). \tag{1}$$

This is a generalization of the scrambling index and exponent. Several studies such as [11,15] have investigated exponents and their generalization. Some studies such as [9,14] have also investigated generalized competition indices.

Definition 1. Let k be a positive integer and D be a primitive digraph of order n. Let m be a positive integer such that $1 \leq m \leq n$. For a pair of vertices x and y in $V(D)$, we define the following notation:

$$N^+(D^k : x) = \left\{ v \in V(D) : x \xrightarrow{k} v \text{ in } D \right\},$$
$$N^+(D^k : x, y) = N^+(D^k : x) \cap N^+(D^k : y),$$
$$k_m(D : x, y) = \min \left\{ t : |N^+(D^a : x, y)| \geq m \text{ for each } a \text{ such that } a \geq t \right\},$$
$$k_m(D : x) = \max \left\{ k_m(D : x, y) : y \in V(D) \right\}.$$

Then, from the definitions of $k_m(D)$, $k_m(D : x)$, and $k_m(D : x, y)$, we have

$$k_m(D : x, y) \leq k_m(D : x) \leq k_m(D),$$

and

$$\begin{aligned} k_m(D) &= \max\{k_m(D : x) : x \in V(D)\} \\ &= \max\{k_m(D : x, y) : x, y \in V(D)\}. \end{aligned}$$

A primitive digraph D is called *symmetric* if and only if the adjacency matrix of D is symmetric. If D is symmetric and $(x, y) \in E(D)$, then we have $(y, x) \in E(D)$, which is represented by $x \leftrightarrow y$. When D is symmetric, the notation $x \xleftrightarrow{k} y$ is used to indicate that there exists a walk of length k from x to y.

Let A_n denote the set of all symmetric primitive digraphs of order n. If $D \in A_n$, the s-cycle denotes the induced subdigraph by distinct vertices $v_1, v_2, \ldots, v_s$ such that $v_1 \leftrightarrow v_2 \leftrightarrow \cdots \leftrightarrow v_s \leftrightarrow v_1$. If $D \in A_n$, there exists an s-cycle in D where s is odd since D is primitive. Let $s(D)$ denote the smallest odd number s such that there exists an s-cycle in D. In this study, we investigate $k_m(D)$ where $D \in A_n$.

Definition 2. For positive integers n and m such that $1 \leq m \leq n$, we define the following notation:

$$A_n^s = \{D \in A_n : s(D) = s\},$$
$$I_{n,m} = \{k_m(D) : D \in A_n\},$$
$$I_{n,m}^s = \{k_m(D) : D \in A_n^s\}.$$

Definition 3. For positive integers n and m such that $1 \leq m \leq n$, we define $P_{n,s} = (V, E)$ where

$$V = \{v_1, v_2, \ldots, v_n\},$$
$$E = \{v_i \leftrightarrow v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n \leftrightarrow v_{n-s+1}\}.$$

Then, we have $s(P_{n,s}) = s$. The notation $[a, b]^o$ indicates the set of integers in $[a, b]$.

Proposition 4 (*Shao [13]*)**.** *If $D \in A_n$ where $n \geq 4$, then we have*

$$I_{n,n} = [1, 2n-2]^o \setminus S,$$

where $S = \{k : k$ is an odd integer and $n \leq k \leq 2n-3\}$. Further, $\exp(D) = 2n-2$ if and only if D is isomorphic to $P_{n,1}$.

Proposition 5 (*Liu et al. [12]*)**.** *Let n be a positive integer where $n \geq 4$. Then, we have*

$$\cup_{s \geq 3} I_{n,n}^s = [2, 2n-4]^o \setminus S',$$

where $S' = \{k : k$ is an odd integer and $n-2 \leq k \leq 2n-5\}$.

Proposition 6 (*Chen and Liu [4]*). *Let n be a positive integer where $n \geq 2$. Then, we have*

$$I^s_{n,1} = \left[\delta_s, n - \frac{s+1}{2}\right]^o,$$

where $\delta_s = \begin{cases} 1, & \text{for } s = 1, \\ \frac{s-1}{2}, & \text{for } s \equiv 1 \pmod 2 \text{ and } s \geq 3. \end{cases}$

These results give us the upper and lower bounds on $I_{n,n}$ or $I^s_{n,1}$. In this paper, we extend these bounds to $I_{n,m}$.

2. Main results

Proposition 7 (*Chen and Liu [4], Cho and Kim [5]*). *If $D \in A_n$, then we have*

$$k_1(D) = \left\lceil \frac{\exp(D)}{2} \right\rceil = \left\lceil \frac{k_n(D)}{2} \right\rceil.$$

Theorem 8. *If $D \in A_n$ where $n \geq 4$, then we have*

$$I_{n,1} = [1, n-1]^o.$$

Furthermore, $k_1(D) = n - 1$ if and only if D is isomorphic to $P_{n,1}$.

Proof. By Propositions 4 and 7, we have $I_{n,1} = [1, n-1]^o$. By Proposition 4, there is no symmetric primitive digraph whose exponent is $2n - 3$. Further, by Proposition 7, we have $k_1(P_{n,1}) = n - 1$. Therefore, we have $k_1(D) = n - 1$ if and only if D is isomorphic to $P_{n,1}$. This establishes the result. □

Example 9. Let D_1 and D_2 be digraphs in A_n such that $k_n(D_1) = k_n(D_2)$. By Proposition 7, we have $k_1(D_1) = k_1(D_2)$. However, it is possible that $k_m(D_1) \neq k_m(D_2)$ when $1 < m < n$. For example, consider two digraphs D_1 and D_2 whose adjacency matrices are respectively given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, we have $D_1 \in A_n$ and $D_2 \in A_n$. Further, $k_5(D_1) = k_5(D_2) = 6$ and $k_1(D_1) = k_1(D_2) = 3$. However,

$$3 = k_2(D_1) \neq k_2(D_2) = 4,$$
$$4 = k_3(D_1) \neq k_3(D_2) = 5,$$
$$5 = k_4(D_1) \neq k_4(D_2) = 6.$$

Lemma 10. *Suppose $D \in A^s_n$ where $s \geq 3$. Let C be an s-cycle and m be a positive integer such that $m \leq s$. For each pair of vertices x and y in $V(C)$, we have*

$$k_m(D : x, y) \leq \frac{s-1}{2} + \left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. Let $t = \frac{s-1}{2} + \left\lfloor \frac{m}{2} \right\rfloor$ and $t' = \left\lfloor \frac{t}{2} \right\rfloor$. We have that C^2 is an s-cycle where each vertex in $V(C^2)$ has a loop. Consider the primitive digraph C.

Case 1. t is even.

Then, $t = 2t'$. We also have $|N^+\left(\left(C^2\right)^{t'} : x\right)| \geq 2t' + 1$ since C^2 is an s-cycle in which each vertex has a loop. Then, we have

$$\begin{aligned} |N^+(C^t : x, y)| &\geq |N^+(C^t : x)| + |N^+(C^t : y)| - |V(C)| \\ &= |N^+(C^{2t'} : x)| + |N^+(C^{2t'} : y)| - |V(C)| \\ &\geq (2t' + 1) + (2t' + 1) - s \\ &= 2t + 2 - s \geq m. \end{aligned}$$

Case 2. t is odd.

Then, $t = 2t' + 1$. We also have $|N^+(C^1 : x)| = 2$ for each vertex $x \in V(C)$. Suppose $N^+(C^1 : x) = \{u_x, v_x\}$. Then,

$$N^+(C^{2t'+1} : x) = N^+\left(\left(C^2\right)^{t'} : u_x\right) \cup N^+\left(\left(C^2\right)^{t'} : v_x\right).$$

As a result, we have $|N^+(C^{2t'+1} : x)| \geq 2t' + 2$ since C^2 is an s-cycle in which each vertex has a loop. Therefore,

$$\begin{aligned} |N^+(C^t : x, y)| &\geq |N^+(C^t : x)| + |N^+(C^t : y)| - |V(C)| \\ &= |N^+(C^{2t'+1} : x)| + |N^+(C^{2t'+1} : y)| - |V(C)| \\ &\geq (2t' + 2) + (2t' + 2) - s \\ &= 2t + 2 - s \geq m. \end{aligned}$$

In all cases, we have

$$|N^+(C^t : x, y)| \geq m. \tag{2}$$

Then, $|N^+(D^t : x, y)| \geq |N^+(C^t : x, y)| \geq m$. Therefore,

$$k_m(D : x, y) \leq t = \frac{s-1}{2} + \left\lfloor \frac{m}{2} \right\rfloor.$$

This establishes the result. □

Lemma 11. *Suppose $D \in A_n^s$ where $s \geq 3$. Let C be an s-cycle and m be a positive integer such that $s \leq m \leq n$. For each pair of vertices x and y in $V(C)$, we have*

$$k_m(D : x, y) \leq m - 1.$$

Proof. By (2), we have $k_s(C : x, y) \leq s - 1$ and

$$V(C) \subset N^+(C^{s-1} : x, y) \subset N^+(D^{s-1} : x, y).$$

Further, for each positive integer i such that $i \geq s$, $V(C) \subset N^+(D^i : x, y)$ and $|N^+(D^i : x, y)| \geq s + \{i - (s - 1)\} = i + 1$. Therefore,

$$k_m(D : x, y) \leq m - 1.$$

This establishes the result. □

Lemma 12. *Let $D \in A_n^s$ where $s \geq 3$. For a positive integer m such that $m \leq n$, we have*

$$k_m(D) \geq s - 1 - \left\lfloor \frac{n-m}{2} \right\rfloor.$$

Proof. Suppose $k = k_m(D) < s - 1 - \left\lfloor \frac{n-m}{2} \right\rfloor$. By Proposition 6 and (1), we have $k = k_m(D) \geq k_1(D) \geq \frac{s-1}{2}$. Let C be an s-cycle in D. Without loss of generality, we can assume that

$$\begin{aligned} V(C) &= \{v_0, v_1, v_2, \ldots, v_{s-1}\}, \\ E(C) &= \{v_i \leftrightarrow v_{i+1} : 1 \leq i \leq s-2\} \cup \{v_{s-1} \leftrightarrow v_0\}. \end{aligned}$$

Let $t = s - 1 - k$. Then, we have $0 < t < \frac{s-1}{2}$ since $t = s - 1 - k > \left\lfloor \frac{n-m}{2} \right\rfloor \geq 0$. Further,

$$N^+(C^k : v_0, v_{s-1}) = \left\{ v_t, \ldots, v_{\frac{s-3}{2}}, v_{\frac{s-1}{2}}, v_{\frac{s+1}{2}}, \ldots, v_{s-1-t} \right\}.$$

For a nonnegative integer a such that $0 \leq a < t$, we claim $v_a \notin N^+(D^k : v_0, v_{s-1})$. Otherwise, there are two closed walks in D, expressed as

$$v_0 \overset{k}{\longleftrightarrow} v_a \overset{a}{\longleftrightarrow} v_0 \quad \text{and} \quad v_{s-1} \overset{k}{\longleftrightarrow} v_a \overset{a}{\longleftrightarrow} v_0 \overset{1}{\longleftrightarrow} v_{s-1},$$

whose lengths are $k + a$ and $k + a + 1$, respectively. Since $k + a + 1 = s - 1 - t + a + 1 < s$, we have a closed walk of odd length less than s. This is a contradiction to $s(D) = s$. Therefore, $v_a \notin N^+(D^k : v_0, v_{s-1})$ for each a such that $a < t$.

Similarly, for a nonnegative integer a such that $s - 1 - t < a \leq s - 1$, we claim $v_a \notin N^+(D^k : v_0, v_{s-1})$. Otherwise, there are two closed walks in D, expressed as

$$v_{s-1} \overset{k}{\longleftrightarrow} v_a \overset{s-a-1}{\longleftrightarrow} v_{s-1} \quad \text{and} \quad v_0 \overset{k}{\longleftrightarrow} v_a \overset{s-a-1}{\longleftrightarrow} v_{s-1} \overset{1}{\longleftrightarrow} v_0,$$

whose lengths are $k + s - a - 1$ and $k + s - a$, respectively. Since $k + s - a = k + (s - a - 1) + 1 < k + t + 1 = s$, we have a closed walk of odd length less than s. This is a contradiction to $s(D) = s$. Therefore, $v_a \notin N^+(D^k : v_0, v_{s-1})$ for each a such that $s - 1 - t < a \leq s - 1$.

Then, we have

$$\{v_0, \ldots, v_{t-1}\} \cup \{v_{s-t}, \ldots, v_{s-1}\} \subset V(D) - N^+(D^k : v_0, v_{s-1}).$$

Since $t = s - 1 - k < \frac{s-1}{2}$, we have $\{v_0, \ldots, v_{t-1}\} \cap \{v_{s-t}, \ldots, v_{s-1}\} = \phi$. Further,

$$|\{v_0, \ldots, v_{t-1}\} \cup \{v_{s-t}, \ldots, v_{s-1}\}| = 2t.$$

Then,

$$2t \le |V(D)| - |N^+(D^k : v_0, v_{s-1})|.$$

Since $|N^+(D^k : v_0, v_{s-1})| \ge m$,

$$2t \le |V(D)| - |N^+(D^k : v_0, v_{s-1})| \le n - m,$$
$$t \le \left\lfloor \frac{n-m}{2} \right\rfloor.$$

Since $t = s - 1 - k$, we have

$$k \ge s - 1 - \left\lfloor \frac{n-m}{2} \right\rfloor.$$

This is a contradiction. Therefore, the result is established. □

Lemmas 10 and 11 give us the upper bounds on $I^s_{n,m}$ and $I_{n,m}$. Further, (1) and Lemma 12 give us the lower bounds on $I^s_{n,m}$ and $I_{n,m}$.

Definition 13. Let n, s, and m be positive integers such that $m, s \le n$ and s is odd. We denote

$$K(n, s, m) = \begin{cases} n - \dfrac{s+1}{2} + \left\lfloor \dfrac{m}{2} \right\rfloor, & \text{when } m < s, \\ n + m - s - 1, & \text{when } m \ge s, \end{cases}$$

and

$$k(n, s, m) = \begin{cases} \dfrac{s-1}{2}, & \text{when } m < n - s, \\ s - 1 - \left\lfloor \dfrac{n-m}{2} \right\rfloor, & \text{when } m \ge n - s. \end{cases}$$

Theorem 14. *Let $D \in A^s_n$. For a positive integer m such that $m \le n$, we have*

$$k(n, s, m) \le k_m(D) \le K(n, s, m).$$

Further, $k_m(D) = K(n, s, m)$ only if D is isomorphic to $P_{n,s}$, and $k_m(P_{n,1}) = K(n, 1, m)$.

Proof. Let C be an s-cycle.
Case 1. $s = 1$.

There exists a vertex z having a loop. Then, $|N^+(D^t : z)| \ge t + 1$ for each t such that $1 \le t < n$. For each pair of vertices x and y, we have two directed walks expressed as $x \xrightarrow{n-1} z$ and $y \xrightarrow{n-1} z$. Therefore, we have

$$k_m(D : x, y) \le n - 1 + m - 1 = K(n, s, m),$$

and $k_m(D : x, y) \ge k(n, s, m)$ since $k(n, 1, m) = 0$. Therefore, we have

$$k(n, s, m) \le k_m(D) \le K(n, s, m).$$

Suppose $k_m(D) = n + m - 2 = K(n, 1, m)$. Let $V(C) = \{z\}$, where z has a loop. Consider a pair of vertices x and y such that $k_m(D : x, y) = k_m(D)$. If $d_D(x, z) < n - 1$ and $d_D(y, z) < n - 1$, then $z \in N^+(D^{n-2} : x, y)$. Further, if there exists another vertex z' having a loop, then $z \in N^+(D^{n-2} : x, y)$ or $z' \in N^+(D^{n-2} : x, y)$. Furthermore, if $z \in N^+(D^{n-2} : x, y)$ or $z' \in N^+(D^{n-2} : x, y)$, then we have $k_m(D : x, y) \le (n - 2) + (m - 1) < n + m - 2$, which is a contradiction. Therefore, there exists a vertex x such that $d_D(x, z) = n - 1$ and z is the only vertex having a loop in D. Then, D is isomorphic to $P_{n,1}$.

Conversely, if D is isomorphic to $P_{n,1}$, then we have

$$k_m(D) = n + m - 2 = K(n, 1, m).$$

Case 2. $s \ge 3$.

For each pair of vertices x and y, there exist directed walks such that

$$x \xrightarrow{n-s} x' \in V(C) \quad \text{and} \quad y \xrightarrow{n-s} y' \in V(C).$$

If $m < s$, then we have $k_m(D : x', y') \leq \frac{s-1}{2} + \lfloor \frac{m}{2} \rfloor$ by Lemma 10. Therefore, we have

$$\begin{aligned} k_m(D : x, y) &\leq n - s + k_m(D : x', y') \\ &\leq n - s + \frac{s-1}{2} + \left\lfloor \frac{m}{2} \right\rfloor = K(n, s, m). \end{aligned}$$

If $m \geq s$, then we have $k_m(D : x', y') \leq m - 1$ by Lemma 11. Therefore, we have

$$\begin{aligned} k_m(D : x, y) &\leq n - s + k_m(D : x', y') \\ &\leq n - s + m - 1 = K(n, s, m). \end{aligned}$$

By Proposition 6 and (1), we have

$$k_m(D) \geq k_1(D) \geq \frac{s-1}{2}.$$

By Lemma 12, we have

$$k_m(D) \geq s - 1 - \left\lfloor \frac{n-m}{2} \right\rfloor.$$

Since $m \geq n - s$ if and only if $\frac{s-1}{2} \leq s - 1 - \lfloor \frac{n-m}{2} \rfloor$, we have $k_m(D) \geq k(n, s, m)$. Therefore,

$$k(n, s, m) \leq k_m(D) \leq K(n, s, m).$$

Suppose $k_m(D) = K(n, s, m)$. Consider a pair of vertices x and y such that $k_m(D : x, y) = k_m(D)$. If $d_D(x, V(C)) < n - s$ and $d_D(y, V(C)) < n - s$, then there exist vertices x' and y' in $V(C)$ such that $x \xrightarrow{n-s-1} x'$ and $y \xrightarrow{n-s-1} y'$, respectively. By Lemmas 10 and 11, we have

$$k_m(D : x', y') \leq \begin{cases} \frac{s-1}{2} + \left\lfloor \frac{m}{2} \right\rfloor, & \text{when } m < s, \\ m - 1, & \text{when } m \geq s. \end{cases}$$

Then, $k_m(D : x, y) \leq K(n, s, m) - 1$, which is a contradiction. Therefore, we have that $P_{n,s}$ is a subdigraph of D. We can also have $k_m(D) < K(n, s, m)$ if there is another edge that is not in $E(P_{n,s})$, Therefore, D is isomorphic to $P_{n,s}$. This establishes the result. □

Corollary 15. *If $D \in A_n^s$, then we have*

$$s - 1 \leq k_n(D) = \exp(D) \leq 2n - s - 1.$$

Corollary 16. *Let $D \in A_n$ and m be a positive integer such that $m \leq n$. Then, we have*

$$k_m(D) \leq n + m - 2.$$

The equality holds if and only if D is isomorphic to $P_{n,1}$.

Theorem 17. *Let n and m be positive integers such that $n \geq 4$ and $m < n$. Then, we have*

$$I_{n,m} = [1, m + n - 2]^o.$$

Proof. By Theorem 8, we have the result for $m = 1$. Suppose $1 < m < n$. For a positive integer k such that $1 \leq k \leq n+m-2$, we claim that $k \in I_{n,m}$.

We have $n + m - 2 \in I_{n,m}$ since $k_m(P_{n,1}) = n + m - 2$ by Corollary 16.

We have $1 \in I_{n,m}$ since $k_m(\bar{J}_n) = 1$, where $\bar{J}_n$ is the digraph whose adjacency matrix is $n \times n$ all-ones matrix.

Consider the digraph D' given by

$$\begin{aligned} &V(D') = \{v_1, v_2, \ldots, v_{n-1}, v_n\}, \\ &E(D') = \{v_i \leftrightarrow v_j : 1 \leq i, j \leq n-1, i \neq j\} \cup \{v_{n-1} \leftrightarrow v_n\}. \end{aligned}$$

Then, we have $k_m(D') = 2 \in I_{n,m}$.

We claim that $k \in I_{n,m}$ for each k such that $3 \leq k < n + m - 2$.

Case 1. $2(m-1) < k < n + m - 2$.

Consider the symmetric primitive digraph D_1 given by

$$\begin{aligned} &V(D_1) = \{v_1, v_2, \ldots, v_n\}, \\ &E(D_1) = \{v_i \leftrightarrow v_{i+1} : 1 \leq i \leq k - m + 1\} \cup \{v_{k-m+2} \leftrightarrow v_{k-m+2}\} \cup \{v_i \leftrightarrow v_2 : k - m + 3 \leq i \leq n\}. \end{aligned}$$

For a pair of positive integers i and j such that $1 \le i, j \le n$, we have $k_m(D_1 : v_i, v_j) \le k_m(D_1 : v_1, v_2)$. For each positive integer l such that $l \ge k$,

$$N^+(D_1^{k-1} : v_1, v_2) = \{v_{k-m+2}, v_{k-m+1}, \ldots, v_{k-2m+4}\},$$
$$N^+(D_1^{l} : v_1, v_2) \supset \{v_{k-m+2}, v_{k-m+1}, \ldots, v_{k-2m+3}\}.$$

Then, we have $k_m(D_1) = k_m(D_1 : v_1, v_2) = k$. Further, $k \in I_{n,m}$.
Case 2. $k \le 2(m-1)$ and k is even.

Consider the symmetric primitive digraph D_2 given by

$$V(D_2) = \{v_1, v_2, \ldots, v_n\},$$
$$E(D_2) = \left\{v_i \leftrightarrow v_{i+1} : 1 \le i \le \left\lfloor \frac{k}{2} \right\rfloor - 1\right\} \cup \left\{v_i \leftrightarrow v_i : \left\lfloor \frac{k}{2} \right\rfloor + 1 \le i \le m\right\} \cup \left\{v_{\left\lfloor \frac{k}{2} \right\rfloor} \leftrightarrow v_i : \left\lfloor \frac{k}{2} \right\rfloor + 1 \le i \le m\right\} \cup \{v_i \leftrightarrow v_2 : m+1 \le i \le n\}.$$

For a pair of positive integers i and j such that $1 \le i, j \le n$, we have $k_m(D_2 : v_i, v_j) \le k_m(D_2 : v_1, v_2)$. For each positive integer l such that $l \ge k$,

$$N^+(D_2^{k-1} : v_1, v_2) = \{v_2, v_3, \ldots, v_m\},$$
$$N^+(D_2^{l} : v_1, v_2) \supset \{v_1, v_2, \ldots, v_m, \ldots, v_n\}.$$

Then, we have $k_m(D_2) = k_m(D_2 : v_1, v_2) = k$. Further, $k \in I_{n,m}$.
Case 3. $k \le 2(m-1)$ and k is odd.

Since $m+1 \le n$, we can consider the symmetric primitive digraph D_3 given by

$$V(D_3) = \{v_1, v_2, \ldots, v_n\},$$
$$E(D_3) = \left\{v_i \leftrightarrow v_{i+1} : 1 \le i \le \left\lfloor \frac{k}{2} \right\rfloor\right\} \cup \left\{v_i \leftrightarrow v_i : \left\lfloor \frac{k}{2} \right\rfloor + 2 \le i \le m+1\right\} \cup \left\{v_{\left\lfloor \frac{k}{2} \right\rfloor + 1} \leftrightarrow v_i : \left\lfloor \frac{k}{2} \right\rfloor + 2 \le i \le m+1\right\} \cup \{v_i \leftrightarrow v_2 : m+2 \le i \le n\}.$$

For a pair of positive integers i and j such that $1 \le i, j \le n$, we have $k_m(D_3 : v_i, v_j) \le k_m(D_3 : v_1, v_2)$. For each positive integer l such that $l \ge k$,

$$N^+(D_3^{k-1} : v_1, v_2) = \{v_3, v_4, \ldots, v_{m+1}\},$$
$$N^+(D_3^{l} : v_1, v_2) \supset \{v_2, v_3, \ldots, v_{m+1}\}.$$

Then, we have $k_m(D_3) = k_m(D_3 : v_1, v_2) = k$. Further, $k \in I_{n,m}$.

We have $k \in I_{n,m}$ for each k such that $1 \le k \le n+m-2$. This establishes the result. □

If $1 \le m < n$, then there is no gap in $I_{n,m}$. However, there is a gap in $I_{n,n}$ by Proposition 4. It should be noted that the condition of $m < n$ is essential for constructing the digraph D_3 in Theorem 17.

3. Closing remark

Akelbek and Kirkland [1] introduced the concept of scrambling index of a primitive digraph. Kim [9] introduced the generalized competition index $k_m(D)$ as another generalization of exponent $\exp(D)$ and scrambling index $k(D)$ for a primitive digraph D. In this study, we investigated $k_m(D)$ for a symmetric primitive digraph D as an extension of the results in [4,13].

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