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Submaps of Maps. I. General 0–1 Laws*

EDWARD A. BENDER† AND ZHI-CHENG GAO

Department of Mathematics, University of California, San Diego, La Jolla, California 92093

AND

L. BRUCE RICHMOND‡

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

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Let $\mathcal{M}_n$ be the set of n edge maps of some class on a surface of genus g. When $g=0$ (planar maps) we show how to prove that $\lim_{n\to\infty} |\mathcal{M}_n|^{1/n}$ exists for many classes of maps. Let P be a particular map that can appear as a submap of maps in our class. There is often a strong 0–1 law for the property that P is a submap of a randomly chosen map in $\mathcal{M}_n$: If P is planar, then almost all $\mathcal{M}_n$ contain at least cn disjoint copies of P for small enough c; while if P is not planar, almost no $\mathcal{M}_n$ contain a copy of P. We show how to establish this for various classes of maps. For planar P, the existence of $\lim_{n\to\infty} |\mathcal{M}_n|^{1/n}$ suffices. For nonplanar P, we require more detailed asymptotic information. © 1992 Academic Press, Inc.

1. INTRODUCTION

A 0–1 law for an infinite collection of structures is a statement that, as we look at random structures of "size" n, the probability of their having some property is asymptotically zero or one. We will establish a 0–1 law for various classes of n edged maps on a surface when the property is "P is a submap of the given map."

Compton [8] has established 0–1 laws for a variety of structures that are closed under disjoint union when the property is expressible in the language of first order logic. Labeled n vertex graphs having given sub-

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graphs fall into this framework. Fagin [9] and Bollobás [6, II.2] have extended this to labeled n vertex graphs where the number of edges depends on n. Kolaitis, Prömel, and Rothschild [13, Theorem 3] strengthen Compton's results for certain classes of labeled graphs. Although maps do not fall into these frameworks, there may well be a general 0–1 law for submaps of maps.

Richmond, Robinson, and Wormald [14] and Richmond and Wormald [15] have shown that all but an exponentially small fraction of various classes of planar triangulations with n edges contain at least cn copies of a given map in the class, thereby establishing a 0–1 law. The classes considered are 2- and 3-connected triangulations and 2-face colorable 2- and 3-connected ones. Similar results hold for certain classes of cubic maps by taking the planar dual. We extend some of these results to general surfaces and to other classes of maps. We also show that nonplanar submaps are rare.

Our methods will work for many reasonable classes of maps provided that asymptotic results are known. In particular, we will obtain 0–1 laws for all of the following classes of maps.

(a) all rooted maps [1],
(b) smooth rooted maps [1],
(c) 2-connected rooted maps [4],
(d) triangular rooted maps [10],
(e) 2-connected rooted triangulations [11], and
(f) the maps in Theorem 1 below.

Let $\mathscr{M}_n$ be the set of n edge maps of some class on a given surface $\mathscr{S}$ of type g and let $m_n = |\mathscr{M}_n|$. *Throughout this paper*, all limits involving m_n are understood to be taken through those values of n for which $\mathscr{M}_n \neq \varnothing$ unless explicitly stated otherwise. We say that the class of maps *grows smoothly* if $\lim_{n\to\infty} m_n^{1/n}$ exists. (Rooting is immaterial for smooth growth since a map can be rooted in at most $4n$ ways.) The references in the above list show that $m_n \sim A(\mathscr{S})\, n^{5(g-1)/2} B^n$, where $A(\mathscr{S})$ and B depend on the class. Those results are much stronger than smooth growth.

In Section 2, we show how to prove that many classes of planar maps grow smoothly. In Section 3, we prove a simple result about composition of power series that is useful for showing that one subject contains many copies of another. We thank Carl FitzGerald for the proofs in that section. In Section 4, we explain how to use the power series result to establish results for planar submaps in classes that grow smoothly. We apply the idea in Section 5. In Section 6, we show that nonplanar submaps are rare; i.e., the probability that a random rooted map contains a given nonplanar map as a submap is asymptotically zero for certain classes of maps. To do this we need more than smooth growth. We briefly consider threshold functions in Section 7. Applications of these techniques to cyclically k-connected planar cubic maps are discussed in [2].

2. Many Classes of Planar Maps Grow Smoothly

Many classes of planar maps can be proved to grow smoothly by means of a three step argument. We will illustrate the argument by proving the following theorem. The reader is invited to adapt the method to his or her favorite class, if possible.

Theorem 1. *Let $F \subseteq \{1, 2, ...\}$ contain 3 and some f which is not a multiple of 3. Let k be 1, 2, or 3. The class of k-connected planar maps, rooted or not, all of whose face degrees lie in F, grows smoothly.*

Proof. We prove the theorem for rooted maps. As observed in the Introduction, it then follows immediately for nonrooted maps. Let m_n be the number of n edge maps in the class and let t_n be the number of those with triangular root face. Since the number of planar maps is bounded by 12^n [17], it follows that $M(x) = \sum m_n x^n$ has a radius of convergence r satisfying $1/12 \leqslant r \leqslant 1$. Let $C_i > 0$ and $1 - r > \delta > 0$ be arbitrary. We will show:

Step 1. For some n, $m_n > C_1(r+\delta)^{-n}$.

Step 2. For some m, $t_m > C_2(r+\delta)^{-m}$ and $t_{m+1} > C_2(r+\delta)^{-(m+1)}$.

Step 3. For some N and all $n > N$, $m_n > C(r+\delta)^{-n}$.

The theorem follows easily from Step 3 and the fact that for a power series

$$\limsup_{n \to \infty} m_n^{1/n} = 1/r.$$

Step 1 follows immediately from the lim sup result. We now give constructions to prove Steps 2 and 3.

The maps shown in Fig. 1 have root ρ. The other labels are used only for ease of discussion and should not regarded as part of the maps. The last map, T_m, exists only for $m > 2$. The central face of T_m is an m-gon whose vertices are connected to the outer triangle by $m+3$ edges as shown so that the remaining faces are triangles and the map is 3-connected.

Recursively construct a set $\mathscr{S}$ of maps containing T and T_f by choosing two maps $S, S' \in \mathscr{S}$ and embedding S in the face B of S' by identifying the root face boundary of S with the boundary of B. The new map inherits the root face of S' and the B face of S. The labels b and B are then removed

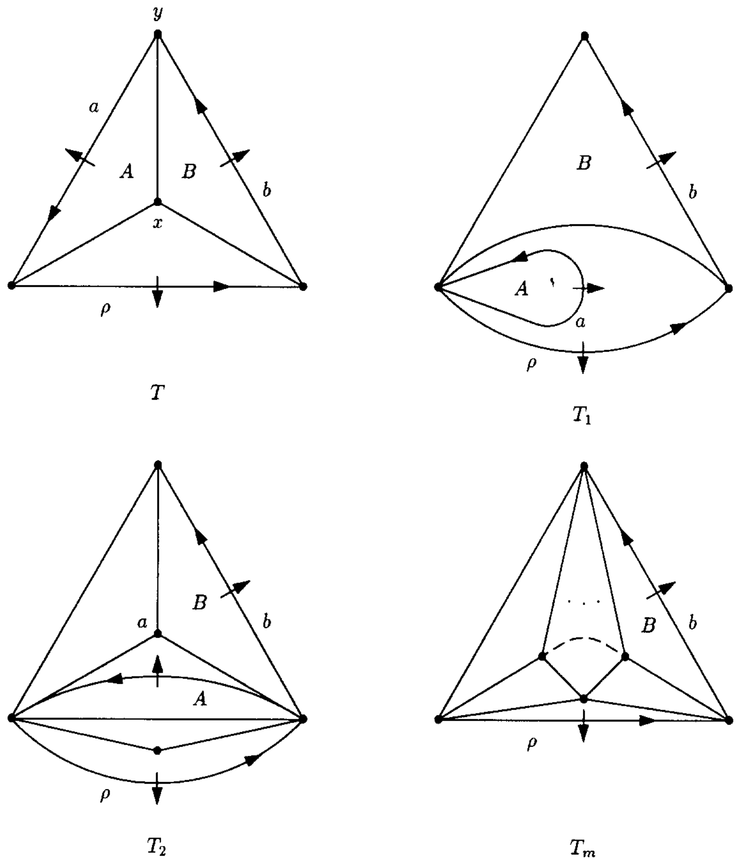

FIGURE 1

from what was S'. The new map has $e(S)+e(S')-3$ edges, where $e(M)$ denotes the number of edges of the map M. All the labels a and A are removed from the maps in $\mathscr{S}$.

Since $e(T_f)$ is not a multiple of 3, it follows that for all sufficiently large e, $\mathscr{S}$ contains a map with e edges. Thus, for all sufficiently large e, we can construct maps with e edges by embedding a map $S \in \mathscr{S}$ in the face B of T, T_1, or T_2. Call the results $T(e)$ and $T_i(e)$.

Suppose that $M \in \mathscr{M}_n$ and let d be the degree of the root face of M. If $d=1$ or 2, embed M in the face A of $T_i(e)$. If $d>2$, replace the edge $\{x, y\}$ of $T(e)$ by a path of length $d-2$ and embed M in the face A of $T(e)$. In all cases, the embedding is to be done so that the root edge of M is identified with the edge a. The labels A, B, a, and b do not appear on the final map. This process is reversible: Given such a map we can determine M.

The map constructed in this way has $e(M)+e-\min(d, 3)$. Thus, for some E and all $e \geqslant E$, we have injections from $\mathcal{M}_n$ to the maps in $\mathcal{M}_{n+e}$ with triangular root faces. Let $C_1 = C_2(r+\delta)^{-E-1}$. Step 2 follows from Step 1 with $m = n+E$.

For Step 3, we use the map T in Fig. 1 and we let C_2 be the maximum of C_3 and 1. Given two maps M_1 and M_2 with triangular root faces, define a new map by embedding M_1 in the face A of T and M_2 in the face B of T so that the root edges are identified with a and b. Call this operation $M_1 + M_2$. This process is reversible: Given a map M, there is at most one ordered pair (M_1, M_2) such that $M = M_1 + M_2$. Note that $e(M_1 + M_2) = e(M_1) + e(M_2)$. By iterating this construction, it follows from Step 2 and the fact that $C_2^k \geqslant C_2$ that $m_n > C_2(r+\delta)^{-n}$ whenever $n > 0$ can be written as a linear combination of m and $m+1$ with non-negative integer coefficients. Step 3 follows with $N = m(m+1)$. This completes the proof. ∎

3. The Composition of Power Series

For an analytic function with power series $f(z) = \sum a_n z^n$, let $r(f)$ be the radius of convergence of f. For the sake of completeness, we give a proof of a well known result (see Titchmarsh [16]) that we shall need to prove a theorem on composition of power series.

Lemma 1. *If $f(z)$ is analytic, has a power series with non-negative coefficients, and satisfies $0 < r(f) < \infty$, then $z = r(f)$ is a singularity of $f(z)$.*

Proof. Suppose that $z = r = r(f)$ is not a singularity. Then the Taylor series expansion of f about $r/2$ must have a radius of convergence $R > r/2$. If $|z_0| = r/2$, it follows from the non-negativity of the coefficients in $f(z_0) = \sum a_n z^n$ that $|f^{(k)}(z_0)| \leqslant f^{(k)}(r/2)$. Therefore the radius of convergence of the Taylor series expansion of f about z_0 is at least R. Consequently $f(z)$ has no singularities in the circle $|z| < R + r/2$. Since $R > r/2$, this is a contradiction. ∎

The hypotheses on $F(z)$ in the next lemma can be weakened, but we have no need for that.

Lemma 2. *If*

(a) *$F(z) \neq 0$ is a polynomial with non-negative coefficients and $F(0) = 0$,*

(b) *$H(w)$ has a power series expansion with non-negative coefficients and $0 < r(H) < \infty$, and*

(c) $G(z) = H(F(z))$,

then $r(H) = F(r(G))$ or, equivalently, $r(G) = F^{-1}(r(H))$, where F^{-1} denotes the unique inverse whose domain and range is the positive reals.

Proof. The uniqueness of F^{-1} follows from the fact that F is a strictly increasing map from the positive reals onto the positive reals. Thus $F^{-1}(w)$ is analytic for all positive real w.

Since G is the composition of functions analytic at the origin and $F(0) = 0$, it follows that G is analytic at the origin. Furthermore, since $G(z)$ is the composition of power series with non-negative coefficients, it follows that $G(z)$ has a power series expansion with non-negative coefficients.

In view of Lemma 1, the radii of convergence of G and H are the smallest positive reals for which they are not analytic. Since it is the composition of analytic functions, $H(x) = G(F^{-1}(x))$ is analytic for all positive real x with $F^{-1}(x) < r(G)$. By Lemma 1, $H(x)$ has a singularity at $x = r(H)$ and so, by the previous sentence, $F^{-1}(r(H)) \geqslant r(G)$ and so $r(H) \geqslant F(r(G))$. Furthermore, if $r(H) > F(r(G))$, we would have that $G(x) = H(F(x))$ is analytic at $x = r(G)$, which contradicts Lemma 1. ∎

LEMMA 3. *If*

(a) *$F(z) \neq 0$ is a polynomial with non-negative coefficients and $F(0) = 0$,*

(b) *$H(w)$ has a power series expansion with non-negative coefficients and $0 < r(H) < \infty$,*

(c) *for some positive integer k, the linear operator $\mathscr{L}$ is given by $\mathscr{L}(w^n) = z^n(F(z)/z)^{[n/k]}$, and*

(d) $G(z) = \mathscr{L}(H(w))$,

then $r(H)^k = r(G)^{k-1} F(r(G))$.

Proof. There are uniquely determined power series H_i such that

$$H(w) = \sum_{i=0}^{k-1} w^i H_i(w^k).$$

Since the coefficients of H are non-negative, the same is true for the H_i and at least one of them has radius of convergence $r(H)^{1/k}$. Since

$$G(z) = \mathscr{L}(H(w)) = \sum_{i=0}^{k-1} z^i H_i(z^{k-1}F(z))$$

and non-negativity of coefficients of the H_i and F prevents cancellation of positive real singularities, it follows from Lemma 2 that $G(z)$ has its smallest positive real singularity at that value of z for which $z^{k-1}F(z) = r(H)^k$. By Lemma 1, this is the radius of convergence of G. ∎

4. A Method for Planar Submaps

Let $\mathscr{S}$ be a surface without boundary. We recall that a map on $\mathscr{S}$ is an embedding of an unlabeled graph in $\mathscr{S}$ such that $\mathscr{S}$ minus the embedding is a set of discs. The discs are the faces of the map. The map is rooted if an edge, a direction along the edge, and a side of the edge are distinguished. The face on the distinguished side of the edge is the root face of the map. Two maps are considered the same if one embedding is mapped to the other by a homeomorphism of the surfaces. If the map is rooted, the rooting must also be preserved. If M is a map, then $e(M)$ is the number of edges of M.

Let M be a map on a surface $\mathscr{S}$ of type g and let C be a cycle formed by a subset of the edges of M. (We use type rather than genus because our results also apply to nonorientable surfaces.) Imagine that the edges of M have a non-zero width so that we can cut the surface by running a cut along C through the middle of the edges. (In effect, we are duplicating the edges of C so that they appear on both sides of the cut.) Suppose that this process separates $\mathscr{S}$ into two pieces. Each piece will have a hole which we fill in with a disc. This gives two new surfaces $\mathscr{S}_1$ and $\mathscr{S}_2$ of types g_1 and $g_2 = g - g_1$ containing maps M_1 and M_2. We call M_1 and M_2 *submaps* of M with respect to C. It follows readily from the generalized Euler relation $(v - e + f = 2 - 2g)$ that $g_1 + g_2 = g$. We say that M contains a *copy* of P if P is a submap of M. If P is rooted, the disc that was added must correspond to the root face of P. We say that two submaps M' and M'' of M are *disjoint* if they have no faces in common. (The holes that are filled are never considered to be common.)

In this section we prove the following theorem and corollary. In the next section we establish the assumption needed by the theorem for various classes of maps. We have decided to describe the assumption after the theorem as it is rather technical and deserves some motivation.

Theorem 2. *Let $\mathscr{M}$ be some class of maps on a surface of type g and let P be a planar map that can be found as a submap of maps in $\mathscr{M}$. Let $M(x)$ be the generating function by number of edges for $\mathscr{M}$. Let $H(x)$ be the generating function by number of edges for those maps M in $\mathscr{M}$ that contain less than $ce(M)$ pairwise disjoint copies of P. Suppose that the assumption* *given later in this section is true. If $c>0$ is sufficiently small, then $r(M)<r(H)$. The maps may be rooted or not.*

COROLLARY 1. *Under the above condtions, let m_n and h_n be the coefficients of x^n in $M(x)$ and $H(x)$, respectively. If $\mathscr{M}$ grows smoothly, then $h_n/m_n=o(d^{-n})$ for some $d>1$. In other words, an exponentially small fraction of the maps contain less than cn copies of P for some sufficiently small $c>0$.*

Before discussing how to prove the theorem, we note that the corollary follows immediately from the theorem because

$$\limsup_{n\to\infty}(h_n)^{1/n}=1/r(H)<1/r(M)=\lim_{n\to\infty}(m_n)^{1/n},$$

the last equality following from smooth growth.

The idea of the theorem's proof is as follows. In some manner associate with each edge of a map K counted by $H(x)$ a way of introducing a copy of P so that the resulting map lies in $\mathscr{M}$. One way this might be done is by identifying an edge of P with the edge of K, in which case the generating function for the result is given by $G(x)=H(x+x^{e(P)-1})$. By Lemma 2, $r(H)=r(G)+r(G)^{e(P)-1}>r(G)$. Since the maps produced in this fashion form a subset of $\mathscr{M}$, this appears to complete the proof. We cannot always do this with every edge, but only with some positive fraction of them, so we will use Lemma 3 instead.

Unfortunately, there is a problem: Maps in $\mathscr{M}$ can arise in many ways by this construction. If we attempt to count how many ways a map $M\in\mathscr{M}$ can arise, we run into difficulties caused by copies sharing faces. To prevent such overlap, we assume that the following is possible:

Assumption. We can embed P in a larger rooted map Q and attach copies of Q to each map K counted by $H(x)$ in such a way that (i) for some fixed positive integer k, at least $[e(K)/k]$ possible non-conflicting places of attachment exist; (ii) only maps in $\mathscr{M}$ are produced; (iii) for any map so produced, we can identify the copies of Q that may have been added and they are all pairwise disjoint; and (iv) given the copies that have been added, the original map and the associated places of attachment are uniquely determined.

The means of attachment is purposely left vague because different situations require different means. What is important is that the attachment keep the map in the class and be reversible as described by (iv).

The method of attachment leads to the generating function $G(z)=\mathscr{L}(H(w))$ in the notation of Lemma 3, where $F(z)=z+z^q$ and q is the number of edges that are added when a copy of Q is attached. Let g_n be the coefficients of $G(z)$.

Suppose $M \in \mathscr{M}$ contains m copies of Q. By (iii), $m \leqslant n$. If M was produced from some K by our process, we can find all possible K by removing at least $m - cn$ copies of Q from M. The number of ways this can be done is bounded above by

$$\sum_{j \geqslant m-cn} \binom{m}{j} = \sum_{k<cn} \binom{m}{k} < \sum_{k<cn} \binom{n}{k}$$
$$\leqslant n \binom{n}{cn} \leqslant nn^{cn}/(cn/e)^{cn} = n((e/c)^c)^n.$$

Call this last number t_n.

If $M(x) = \sum m_n x^n$, it follows that $m_n \geqslant g_n/t_n$. Thus

$$1/r(M) \geqslant \limsup_{n \to \infty} (g_n/t_n)^{1/n} = \lim_{n \to \infty} (t_n)^{-1/n} \limsup_{n \to \infty} (g_n) \geqslant (c/e)^c/r(G).$$

By Lemma 3, $r(H)^k = r(G)^k\,(1 + r(G)^{q-1})$ and so

$$r(H)/r(M) \geqslant (1 + r(G)^{q-1})^{1/k}\,(c/e)^c.$$

Since $\lim_{c \to 0^+} (c/e)^c = 1$ and $r(G)^k\,(1 + r(G)^{q-1}) = r(H)^k \geqslant 1/12^k$ by [1], it follows that $r(H)/r(M) > 1$ for sufficiently small c. This completes the proof of the theorem.

5. Planar Submaps

For simplicity, we will assume that our maps are allowed to have faces of degree 3. If this is not the case for some class of maps one is interested in, the constructions in this section can probably be altered to fit the class. The basic idea behind our constructions is to construct a Q that will be attached to a given map rather loosely and arrange it so that paths within Q between points of attachment are too long to be part of the root face of another Q.

There are two somewhat different definitions of k-connectedness for a connected graph G. The Tutte definition states that a graph is *not* k-connected if for some $0 < m < k$ one can partition the edges of G into two sets H and K such that $|H| \geqslant m$, $|K| \geqslant m$, and $H \cap K$ consists of exactly m vertices. The other common definition contains the condition that each of H and K must contain vertices not found in the other. We adopt Tutte's definition. See [12] for further discussion.

We begin with maps which are not required to be 3-connected. In this case, we will embed our submap P in 2-cycle whose vertices are not equal, shown shaded in Fig. 2. This can be done in a variety of ways and it is

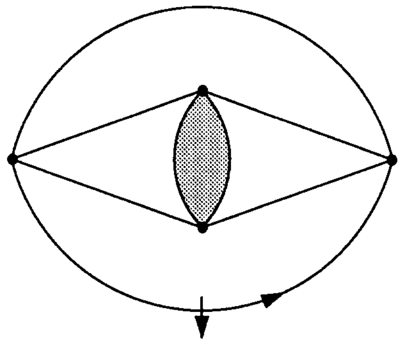

FIGURE 2

irrelevant how it is done. Next, the 2-cycle is embedded in another 2-cycle to produce Q as shown. Given an edge e of a map K, we arbitrarily select a side of e, place Q in the face on that side of e, and identify e with the root edge of Q. This does not change the degree of the face in which Q was placed. Let v_1 and v_2 be the, not necessarily distinct, ends of e. If there is some overlap between this copy Q' of Q and another copy Q'', then there must be a path in Q' that is part of the root face of Q''. Hence there must be a path in Q' from v_i to v_j (possibly $i=j$) of length at most 2. This can happen only if $v_1=v_2$. In this case, Q'' will contain one of the boundary edges of Q' attached to one of its root vertices as a loop. Such a map cannot be a copy of Q, so we are done.

This establishes the assumption needed for Theorem 2 for a wide class of maps. In particuar, Corollary 1 holds for all the classes in the list in Section 1 except for 3-connected planar maps.

We now provide a construction for 3-connected maps that are allowed to have faces of degrees 3 and $k>3$. By the definition of 3-connectivity, everry face has degree at least 3. We will construct Q to have root face degree 4 and will attach it to two adjacent edges of a face by its root edge and the following edge. Thus, the places of attachment can be thought of as corners: vertices where two edges of a face meet. To avoid overlap, we must not have adjacent corners on a face as planes of attachment. Consequently we can choose $[d/2]$ corners on a face of degree d. Since $d \geqslant 3$ and there are twice as many corners as edges, there are at least $2n/3 > n/2$ places of attachment on an n edge map. When specifying a place of attachment, we must also specify which edge of the corner is to be identified with the root edge of Q.

The construction of Q is shown in Fig. 3. First P is embedded in a quadrilateral in some fashion, shown shaded in the figure. Next this quadrilateral is embedded in another quadrilateral as shown. The dashed lines in the figure represent paths of length $k-3$. Edges from the vertices

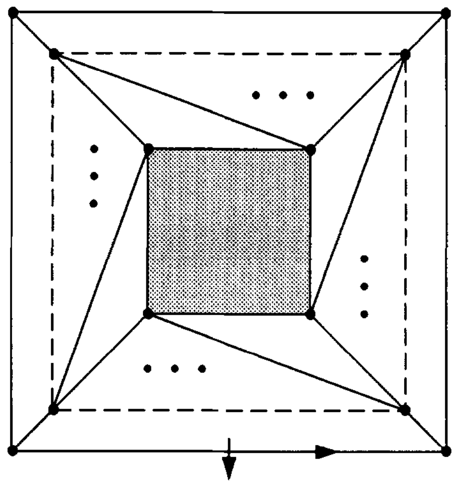

FIGURE 3

of the shaded quadrilateral triangulate the four inner k-gons. Using arguments like that for Fig. 2, it is a straightforward matter to complete the proof of the assumption needed in Theorem 2.

6. NONPLANAR SUBMAPS

We will prove the following result, which applies to all the classes of maps in the list in Section 1.

THEOREM 3. *Let $\mathcal{M}$ be some class of rooted maps and let $P \in \mathcal{M}$ be nonplanar. Suppose that the number of n edged rooted maps in the class that lie on a particular surface $\mathcal{S}$ of type g is given asymptotically by $A(\mathcal{S})\, n^{5(g-1)/2} B^n$ for some $A(\mathcal{S}) > 0$ and $B > 1$. One may require that n be in some congruence class (as must be done for triangulations). Suppose that if $M \in \mathcal{M}$ and C is a cycle such that one submap with respect to C is a copy of P, then, the other submap is in $\mathcal{M}$. (If the root of M lies in the copy of P, the other submap may be rooted at any edge of C.) Under these conditions, almost no maps in $\mathcal{M}$ on a given surface contain copies of P. If the type of P exceeds $\frac{1}{2}$, the maps in $\mathcal{M}$ need not be rooted.*

Proof. Because of the assumptions, we may produce any map in $\mathcal{M}$ containing a copy of P by choosing a map $N \in \mathcal{M}$, choosing a face of N with the same degree as the root face of P, and, using its boundary as C, attach P to N. The root of the resulting map will be either the original root or an edge of P. The number of n edged maps on $\mathcal{S}$ that are produced in this manner has the form $O(n \times n^{5(g-g_1-1)/2}B^n)$, where g_1 is the type of P. The factor of n arises as a bound on the number of faces of N. Since P is nonplanar, $g_1 \geqslant \frac{1}{2}$ and so the number of maps produced is

$$O(n^{1-5g_1/2+5(g-1)/2}B^n) = o(n^{5(g-1)/2}B^n).$$

If the type of P exceeds $\frac{1}{2}$, the requirement that the maps be rooted can easily be eliminated even though we lack an asymptotic formula for the number of unrooted maps. In this case $1-5g_1/2>1$ and the ratio of unrooted to rooted maps is at least $1/4n$. Thus the number of rooted maps containing copies of P divided by the number of unrooted maps approaches 0. ▌

For convenience, we state the consequences for our list of maps as a theorem. Clearly it implies a 0–1 law for those classes.

THEOREM 4. *Let $\mathcal{M}$ be any of the classes of maps in the list in Section* 1, *let $\mathcal{M}(\mathcal{S}, n)$ be the subset that lies on the surface $\mathcal{S}$ and has n edges, let P be a map which can occur as a submap of maps in the class, let $M \in \mathcal{M}(\mathcal{S}, n)$ be chosen uniformly at random, and let $S_P(M)$ be the largest set of pairwise disjoint disjoint copies of P that can be found in M. There are numbers $c>0$ and $d>1$ such that*

(i) *if P is planar,* $\text{Prob}\{S_P(M)<cn\}<d^{-n}$ *for all sufficiently large n;*

(ii) *if P is nonplanar* $\text{Prob}\{S_P(M)\neq 0\}\to 0$ *as $n\to\infty$.*

If P is not a map on the projective plane, we need not require that the maps in $\mathcal{M}$ be rooted.

7. THRESHOLD FUNCTIONS

If the number of vertices, v, is forced to depend on the number of edges (e.g., by requiring that $v<f(n)$ or that $v>f(n)$) we may expect the appearance of a threshold function as happens for subgraphs of random graphs [6, IV.2]. Of course, if the submap P is nonplanar, this is unlikely by the previous section. Therefore, we restrict our attention to planar P.

There is an important difference between a submap and a subgraph: we cannot remove edges or vertices from inside the submap. In this way a submap is more like a subgraph induced by a set of vertices. As a result, we get transitions from probability 1 to 0 as well as from probability 0 to 1 as the number of edges increases. To illustrate this, consider 3-connected maps on the plane and let P consist of a quadrilateral and a pentagon with a common edge. When there are few edges, almost all faces are hexagons and we expect the rest to be pentagons. Hence the expected number of copies of P should approach 0. When there are many edges, almost all faces are triangles and we expect the rest to be quadrilaterals. Hence the expected number of copies of P should again approach 0. In between, we expect to see many copies of P in almost all of the maps.

It is likely that the threshold functions have their transitions when v is near its maximum or minimum possible value as a function of n. To see this, suppose we impose a constraint of either of the forms $v < rn$ or $v > rn$, where r is strictly between $\liminf(v/n)$ and $\limsup(v/n)$, the limits being taken over all maps in the class without regard to number of edges. Usually, one can embed P in a map Q satisfying the assumption and having sufficiently small or large $|V|/|E|$, respectively, so that attaching copies to a map that satisfies the v/n constraint leads to another map satisfying the constraint.

To apply Corollary 1 and obtain a 0–1 law, we need asymptotic information about the number of maps with a given number of vertices and edges in some class. At present, this is available only for 2- and 3-connected rooted maps in the plane ([7] and [5], respectively). The case of all rooted maps on the plane can also be done. Tutte [17] obtained an implicit generating function for $f_{i,j}$, the number of rooted maps with $i+1$ vertices and $j+1$ faces,

$$xyf(x, y) = \sigma\tau(1 - 2\sigma - 2\tau),$$

where

$$x = \sigma(1 - \sigma - 2\tau) \qquad \text{and} \qquad y = \tau(1 - 2\sigma - \tau).$$

Lagrange inversion can be used in various ways to write $f_{i,j}$ as a linear combination of several sums of positive terms. It is likely that asymptotics for $f_{i,j}$ can be worked out in this case. An analytic approach can also be taken as in [3]. In this case a central limit theorem suffices because we are interested in a sum of the $f_{i,n-i}$ over all i with $i < rn$ or $i > rn$. Having only a central limit theorem will give the sum to within a factor that is exponential in the square root of the variance; i.e., a factor of the form $\exp(o(n))$. This is enough since our estimates for Corollary 1 allow us a factor of $\exp(\varepsilon n)$ for sufficiently small ε.

Note added in proof. The list of classes of maps given in the Introduction has grown [18]. Asymptotic enumeration of all maps on a surface by vertices and edges is done in [19].

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