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Criteria for Asymptotic Constancy of Solutions of Functional Differential Equations

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1. INTRODUCTION

The purpose of this paper is to develop conditions which ensure that all solutions of certain retarded functional differential equations are asymptotically constant as $t \to \infty$. The results developed here are tailored for systems (and perturbations of systems) for which each constant function is a solution. Such systems have appeared recently in the literature as models of various phenomena and have often been examined as delay differential equations of the form

$$x'(t) = g(t, x(t) - x(t-r)), \qquad g(t, 0) \equiv 0, \tag{1.1}$$

or

$$x'(t) = g(t, x(t)) - g(t, x(t-r)) \tag{1.2}$$

or as perturbations of these equations (see, for example, [1, 3, 5, 6]).) It should be mentioned, however, that previous results have not always been restricted to equations with a single delay. Likewise, the techniques developed in this paper do not impose such a restriction.

To partially set the stage, we refer to a recent result for linear systems

$$x'(t) = P(t)[x(t) - x(t-r)] + Q(t)\, x(t) + R(t)\, x(t-r), \tag{1.3}$$

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where P, Q, R are continuous $n \times n$ matrix functions. In [3] it was shown that each solution of (1.3) is asymptotically constant as $t \to \infty$ provided $|P| \in L^2[0, \infty)$ and $|Q|, |R| \in L^1[0, \infty)$, where $|\cdot|$ represents an appropriate norm. By focusing on the "unperturbed" part $P(t)[x(t) - x(t-r)]$ and observing that it satisfies an inequality

$$|P(t)[x(t) - x(t-r)]| \leqslant |P(t)| \int_{t-r}^{t} |x'(s)| \, ds$$

whenever x' is continuous on $[t-r, t]$, we will be able to readily apply our main results to significantly improve and generalize the above result (cf. Example 3.1).

Since we wish to obtain results that have a wider range of applicability than merely to Eqs. (1.1). (1.2), or (1.3), we employ the standard setting for functional differential equations. Let $r \geqslant 0$ be given and let $C = C([-r, 0], R^n)$ denote the space of continuous functions that map the interval $[-r, 0]$ into R^n. For $\phi \in C$, the norm of ϕ is defined by $\|\phi\| = \max_{-r \leqslant s \leqslant 0} |\phi(s)|$, where $|\cdot|$ denotes any convenient norm in R^n. Also, for $k \in R^n$, let $\phi_k \in C$ denote the constant function defined by $\phi_k(s) = k$, $-r \leqslant s \leqslant 0$. If $x: [t_0 - r, t_0 + A) \to R^n$ is continuous ($t_0 \geqslant 0$, $0 < A \leqslant \infty$), then for each $t \in [t_0, t_0 + A)$, $x_t \in C$ is defined by $x_t(s) = x(t+s)$, $-r \leqslant s \leqslant 0$.

We consider a system of functional differential equations

$$x' = N(t, x_t) + M(t, x_t), \tag{1.4}$$

where N, M: $[0, \infty) \times C \to R^n$ are continuous and map closed and bounded sets into bounded sets. The *prime* notation ($'$) denotes the right-hand derivative with respect to t. These conditions guarantee that each initial value problem (with initial condition $x_{t_0} = \phi$; $t_0 \geqslant 0$, $\phi \in C$) has at least one solution defined on an interval $[t_0, t_0 + A)$, $0 < A \leqslant \infty$ (cf. [4]). Furthermore, if a solution is bounded on $[t_0, t_0 + A)$ with $A < \infty$, then it can be extended as a solution "past" $t_0 + A$. We denote a solution of (1.4) that satisfies $x_{t_0} = \phi$ by $x(\cdot; t_0, \phi)$. Whenever there is no possibility of confusion, we write $x(\cdot)$ or simply x instead of $x(\cdot; t_0, \phi)$.

In Section 2 we establish the main theorem for asymptotic constancy of solutions of (1.4). Previous results along these lines are scarce; some of them will be discussed in Sections 3 and 4. Also, in Section 3 we provide special criteria regarding the main result of Section 2. Finally, in Section 4 we see that conditions used to establish asymptotic constancy of solutions of (1.4) are frequently sufficient to prove that each solution is uniformly stable.

2. A Condition for Asymptotic Constancy

In this section we consider system (1.4) and we assume there exist nonnegative continuous functions $p(\cdot)$, $v(\cdot)$ on $[0, \infty)$ such that

$$|N(t, y_t)| \leqslant p(t) \int_{t-r}^{t} |y'(s)| \, ds \tag{2.1}$$

for $y(\cdot)$ continuously differentiable on $[t-r, t]$ and

$$|M(t, \phi)| \leqslant v(t) \|\phi\| \qquad \text{for all} \quad t \geqslant 0, \quad \phi \in C. \tag{2.2}$$

We further assume

$$\text{each solution } x(\cdot\,; t_0, \phi) \text{ is defined on the interval } [t_0 - r, \infty). \tag{2.3}$$

Condition (2.3) holds, for example, if there exists a continuous nonnegative function $w(\cdot)$ on $[0, \infty)$ such that

$$|N(t, \phi) + M(t, \phi)| \leqslant w(t) \|\phi\| \qquad \text{for all} \quad t \geqslant 0, \quad \phi \in C. \tag{2.4}$$

It is often the case that (2.3) holds even if (2.4) is not valid, so we do not impose the latter condition.

At first glance it might appear to the reader that (2.1) is an unusual condition since it imposes the property that $N(t, y) \equiv 0$ for constant functions y: $[-r, 0] \to R^n$. This condition does, however, often arise in a natural way. For example, if there exists a nonnegative function $p(\,\cdot\,)$ such that

$$|g(t, u)| \leqslant p(t) |u| \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad u \in R^n$$

in (1.1) or

$$|g(t, u) - g(t, v)| \leqslant p(t) |u - v| \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad u, v \in R^n$$

in (1.2), then (2.1) is satisfied. (Here, $M \equiv 0$.) Likewise, (2.1)–(2.4) are clearly satisfied for (1.3) with $p(t) = |P(t)|$, $v(t) = |Q(t)| + |R(t)|$, and $w(t) = 2|P(t)| + |Q(t)| + |R(t)|$.

The following lemma is fundamental to the work presented here; we include the proof for the sake of completeness:

Lemma 2.1. *Suppose f and g are continuous on $[0, \infty)$ with $f(t) \geqslant 0$, $g(t) \geqslant 0$ for all $t \geqslant 0$. If, for some $t_0 \geqslant 0$,*

$$f(t) \leqslant g(t) \int_{t-r}^{t} f(s) \, ds \qquad \text{for all} \quad t \geqslant t_0 + r, \tag{2.5}$$

then

$$\int_{t^*}^{T} f(t)\,dt \leqslant \int_{t^*-r}^{T} f(s)\,ds \int_{s}^{s+r} g(t)\,dt \qquad \textit{for all} \quad t^*, T \quad \textit{with} \quad t^* \leqslant T. \tag{2.6}$$

Proof. Let t_0 be such that (2.5) holds for all $t \geqslant t_0 + r$ and choose t^*, T such that $t_0 + r \leqslant t^* \leqslant T$. By integrating (2.5) from t^* to T, we obtain

$$\int_{t^*}^{T} f(t)\,dt \leqslant \int_{t^*}^{T} g(t)\,dt \int_{t-r}^{t} f(s)\,ds.$$

(i) If $T - r \geqslant t^*$, then

$$\begin{aligned}
&\int_{t^*}^{T} g(t)\,dt \int_{t-r}^{t} f(s)\,ds \\
&\quad = \int_{t^*-r}^{t^*} f(s)\,ds \int_{t^*}^{s+r} g(t)\,dt \\
&\qquad + \int_{t^*}^{T-r} f(s)\,ds \int_{s}^{s+r} g(t)\,dt + \int_{T-r}^{T} f(s)\,ds \int_{s}^{T} g(t)\,dt \\
&\quad \leqslant \int_{t_0}^{t^*} f(s)\,ds \int_{s}^{s+r} g(t)\,dt + \int_{t^*}^{T-r} f(s)\,ds \int_{s}^{s+r} g(t)\,dt \\
&\qquad + \int_{T-r}^{T} f(s)\,ds \int_{s}^{s+r} g(t)\,dt.
\end{aligned}$$

(ii) If $T - r < t^*$, then

$$\begin{aligned}
&\int_{t^*}^{T} g(t)\,dt \int_{t-r}^{t} f(s)\,ds \\
&\quad = \int_{t^*-r}^{T-r} f(s)\,ds \int_{t^*}^{s+r} g(t)\,dt \\
&\qquad + \int_{T-r}^{t^*} f(s)\,ds \int_{t^*}^{T} g(t)\,dt + \int_{t^*}^{T} f(s)\,ds \int_{s}^{T} g(t)\,dt \\
&\quad \leqslant \int_{t_0}^{T-r} f(s)\,ds \int_{s}^{s+r} g(t)\,dt + \int_{T-r}^{t^*} f(s)\,ds \int_{s}^{s+r} g(t)\,dt \\
&\qquad + \int_{t^*}^{T} f(s)\,ds \int_{s}^{s+r} g(t)\,dt.
\end{aligned}$$

In either case, (2.6) holds and the proof is complete.

For a function $h: [t_0, \infty) \to R$ (or R^n), if $\int_{t_0}^{\infty} |h(t)|^p \, dt < \infty$, then $h \in L^p[t_0, \infty)$, $p > 0$.

THEOREM 2.1. *In addition to* (2.1)–(2.3), *suppose there exists a positive nondecreasing function* $q(\cdot)$ *such that, for sufficiently large* t, $q(t) \geqslant 1$,

$$\int_t^{t+r} p(s) \, ds \leqslant [q(t) - 1]/q(t+r) \tag{2.7}$$

and

$$qv \in L^1[0, \infty). \tag{2.8}$$

Then solutions $x = x(\cdot; 0, \phi)$ *of* (1.4) *satisfy*

$$x' \in L^1[0, \infty); \tag{2.9}$$

in particular, every solution of (1.4) *tends to a finite limit as* $t \to \infty$.

Proof. For a given solution $x = x(\cdot; 0, \phi)$, we have from (2.1)–(2.3) that, for $t \geqslant r$,

$$|x'(t)| \leqslant p(t) \int_{t-r}^{t} |x'(s)| \, ds + v(t) \, \|x_t\|. \tag{2.10}$$

For some $t_0 \geqslant r$ (to be determined later) and $t \geqslant t_0$, we multiply inequality (2.10) by $q(t)$ and integrate over $[t_0, t]$. This gives easily (from Lemma 2.1)

$$\begin{aligned} &\int_{t_0}^{t} q(s) \, |x'(s)| \, ds \\ &\qquad \leqslant \int_{t_0-r}^{t} |x'(s)| \, ds \int_s^{s+r} q(u) \, p(u) \, du + \int_{t_0}^{t} q(s) \, v(s) \, \|x_s\| \, ds. \end{aligned} \tag{2.11}$$

Here, $\|x_s\| \leqslant \|x_{t_0}\| + \int_{t_0}^{s} |x'(u)| \, du$. Hence, if

$$K = \int_{t_0-r}^{t_0} |x'(s)| \, ds \int_s^{s+r} q(u) \, p(u) \, du + \|x_{t_0}\| \int_{t_0}^{\infty} q(s) \, v(s) \, ds, \tag{2.12}$$

we have from (2.11) that

$$\begin{aligned} \int_{t_0}^{t} q(s) \, |x'(s)| \, ds \leqslant K &+ \int_{t_0}^{t} |x'(s)| \, ds \int_s^{s+r} q(u) \, p(u) \, du \\ &+ \int_{t_0}^{t} q(s) \, v(s) \, ds \int_{t_0}^{s} |x'(u)| \, du. \end{aligned} \tag{2.13}$$

Here, the last double integral does not exceed

$$\int_{t_0}^{t} |x'(s)|\, ds \int_{s}^{\infty} q(u)\, v(u)\, du,$$

and so from (2.13) we deduce that

$$\int_{t_0}^{t} |x'(s)|\, ds \left\{ q(s) - \int_{s}^{s+r} q(u)\, p(u)\, du - \int_{s}^{\infty} q(u)\, v(u)\, du \right\} \leqslant K. \qquad (2.14)$$

We suppose now that t_0 is such that (2.7) holds for $t \geqslant t_0$, and is so large that

$$\int_{t_0}^{\infty} q(t)\, v(t)\, dt < \tfrac{1}{2}.$$

The expression in braces in (2.14) is then not less than

$$q(s) - q(s+r) \int_{s}^{s+r} p(u)\, du - \tfrac{1}{2} > \tfrac{1}{2},$$

where we have used the fact that q is nondecreasing. Result (2.9) now follows from (2.14).

The above proof is easily extended to include solutions $x = x(\cdot\,; t_1, \phi)$ for any $t_1 \geqslant 0$.

Remark 2.1. We are able to deduce from (2.12) and (2.14) more specifically that

$$\int_{t_0}^{\infty} |x'(s)|\, ds \leqslant 2q(t_0) \int_{t_0 - r}^{t_0} |x'(s)|\, ds + \|x_{t_0}\|, \qquad (2.15)$$

where t_0 is as above. Also, the requirement that $q(t)$ be nondecreasing may be omitted if (2.7) is replaced by

$$\int_{t}^{t+r} q(s)\, p(s)\, ds \leqslant q(t) - 1, \qquad (2.16)$$

for large t. As we note later, this can serve as a source of special criteria.

Remark 2.2. If M is not present in (1.4), then a straightforward modification of the proof of Theorem 2.1 will yield the same result if (2.7) is replaced by the weaker condition

$$\int_{t}^{t+r} p(s)\, ds \leqslant [q(t) - 1]/q(t+r) + h(t), \qquad (2.17)$$

for some h with $q(t+r)\,h(t) \leqslant \beta < 1$. This fact will be employed in Theorem 3.3.

3. Special Criteria and Examples

This section is devoted to obtaining special criteria for asymptotic constancy of solutions of (1.4). We obtain these by particular choices of the function q in Theorem 2.1. For example, assuming (2.1)–(2.3), we have

THEOREM 3.1. *Let* $v \in L^1[0, \infty)$ *and*

$$\limsup_{t\to\infty} \int_t^{t+r} p(s)\,ds < 1. \tag{3.1}$$

Then solutions of (1.4) *satisfy* (2.9).

Proof. For this case we take $q(t) \equiv k > 0$, a constant. Then (2.7) becomes

$$\int_t^{t+r} p(s)\,ds \leqslant 1 - 1/k,$$

and since k may be chosen arbitrarily large, we have the result.

EXAMPLE 3.1. Theorem 3.1 provides a significant improvement of a recent result of Haddock and Sacker relating to linear nonautonomous system (1.3)

$$x'(t) = P(t)[x(t) - x(t-r)] + Q(t)\,x(t) + R(t)\,x(t-r). \tag{3.2}$$

As was mentioned in Section 1, it was shown in [3] that each solution of (1.3) is asymptotically constant as $t \to \infty$ whenever $|P| \in L^2[0, \infty)$ and $|Q|$, $|R| \in L^1[0, \infty)$. Theorem 3.1 indicates that the condition $|P| \in L^2[0, \infty)$ can be weakened to

$$\limsup_{t\to\infty} \int_t^{t+r} |P(s)|\,ds < 1. \tag{3.3}$$

This would also include the case $|P| \in L^p[0, \infty)$ if $p \geqslant 1$ (and not just $p = 2$). This follows since

$$\lim_{t\to\infty} \int_t^{t+r} |P(s)|\,ds = 0 \qquad \text{for} \quad |P| \in L^p(0, \infty), \qquad p \geqslant 1.$$

(Of course, (3.3) can be affected by the choice of matrix norm.)

To see that the number *one* in (3.3) is the best possible as far as a general result is concerned, consider the scalar equation

$$x'(t) = p(t)[x(t) - x(t-r)]. \tag{3.4}$$

If $p(t) \equiv b > 0$, a constant, and $\lim_{t\to\infty} \sup \int_t^{t+r} p(s)\, ds = br = 1$, then $x(t) = t$ is a solution of (3.4). Some results for (3.4) of a more general character, for which the pointwise behavior of $p(t)$ is restricted in various ways, have been given by Slater [6].

As a final comment regarding Example 3.1, we should mention that there is certainly no need to avoid a variable delay in (3.2). For instance, if r is replaced by $r(t)$ in (3.2) with $0 \leqslant r(t) \leqslant r$, then the same results hold. Likewise, if both $|P(\cdot)|$ and $r(\cdot)$ are bounded on $[0, \infty)$, if $|Q|$, $|R| \in L^1[0, \infty)$, and if either $|P(t)| \to 0$ or $r(t) \to 0$ as $t \to \infty$, then each solution of (3.2) tends to a constant as $t \to \infty$.

As the next theorem indicates, condition (3.1) can in some cases be weakened to allow

$$\lim_{t\to\infty} \sup \int_t^{t+r} p(s)\, ds = 1.$$

THEOREM 3.2. *Suppose* (2.1)–(2.3) *hold, where*

$$\int_0^\infty tv(t)\, dt < \infty. \tag{3.5}$$

Further, suppose there exists $K > r$ *such that for sufficiently large* t

$$\int_t^{t+r} p(s)\, ds \leqslant 1 - K/t. \tag{3.6}$$

Then solutions of (1.4) *satisfy* (2.9).

Proof. Here, we let $q(t) = \alpha t$. Then

$$[q(t) - 1]/q(t+r) = [t - 1/\alpha]/[t + r] \geqslant 1 - [r + 1/\alpha]/t,$$

and since α may be chosen arbitrarily large, the result follows.

Condition (3.6) can be further weakened if M is absent from (1.4), so that $v = 0$ and (2.8) may be eliminated from the considerations.

THEOREM 3.3. *Suppose* (2.1) *and* (2.3) *hold and, for sufficiently large* t,

$$\int_t^{t+r} p(s)\,ds \leqslant 1 - r/t - K/t \ln t, \tag{3.7}$$

for some $K > r$. *Then solutions of* (1.4) *satisfy* (2.9).

Proof. For this case, we let $q(t) = \alpha t \ln t$.

$$\begin{aligned}[q(t)-1]/q(t+r) &= 1 - r/[t+r] - [r+1/\alpha]/[t+r]\ln(t+r)\\ &\quad + r + t\ln(t/[t+r])/[t+r]\ln(t+r)\\ &= 1 - r/t - [r+1/\alpha]/t\ln t\\ &\quad + [r + t\ln(t/t+r)]/[t+r]\ln(t+r).\end{aligned} \tag{3.8}$$

It is straightforward to show that for

$$h(t) = [r + t\ln(t/t+r)]/[t+r]\ln(t+r), \tag{3.9}$$

$q(t+r)\,h(t) \to 0$ as $t \to \infty$. Since α may be chosen arbitrarily large, the remainder of the proof follows from (3.8), (3.9), and Remark 2.2.

Another criterion for (1.4) is obtained by using (2.16) in place of (2.7), with $q(t) = \alpha t$. In partiular, (2.16) holds if $p(t)$ satisfies the pointwise bound

$$p(t) \leqslant 1/r - K/t, \qquad K > \tfrac{1}{2}, \tag{3.10}$$

for large t. That is

$$\int_t^{t+r} \alpha s p(s)\,ds \leqslant \alpha t + \alpha r(\tfrac{1}{2} - K) \leqslant q(t) - 1$$

for $K > \frac{1}{2}$ and α sufficiently large. It would seem that the value "$\frac{1}{2}$" here is the best possible.

4. STABILITY OF SOLUTIONS

For the sake of convenience, we let $M = 0$ in this section and consider the system

$$x' = N(t, x_t). \tag{4.1}$$

In studying stability of solutions of (4.1), we need a condition of the nature of a Lipschitz hypothesis.

THEOREM 4.1. *Let* (2.1) *hold, where* $p(t)$ *satisfies* (3.1). *Further, let there exist a nonnegative locally integrable function* $u(\cdot)$ *such that*

$$|N(t, \phi_1) - N(t, \phi_2)| \leqslant u(t) \|\phi_1 - \phi_2\| \qquad \text{for all} \quad t \geqslant 0, \quad \phi_1, \phi_2 \in C \tag{4.2}$$

and

$$\int_t^{t+r} u(s)\, ds \leqslant c_1 < \infty, \qquad t \geqslant 0 \quad \text{for some} \quad c_1\,(\text{const}) > 0. \tag{4.3}$$

Then solutions of (4.1) *are uniformly stable.*

Proof. In view of (3.1), we may take it that

$$\int_t^{t+r} p(s)\, ds \leqslant c < 1, \qquad t \geqslant t^* \quad \text{for some} \quad t^* \geqslant 0 \text{ and some constant } c. \tag{4.4}$$

Let $x(t)$ be a solution defined for $t \geqslant t_0 \geqslant 0$. We denote a second solution by $y(t)$, $t \geqslant t_1 \geqslant t_0$. We need to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that the inequality

$$|y(t) - x(t)| < \delta, \qquad t_1 - r \leqslant t \leqslant t_1 \tag{4.5}$$

implies that

$$|y(t) - x(t)| < \varepsilon \qquad \text{for all} \quad t \geqslant t_1. \tag{4.6}$$

Here, δ should not depend on t_1 nor on y.

We will arrange that, for some $T \geqslant t_0$,

$$|y(t) - x(t)| \leqslant \tfrac{1}{2}(1 - c)\, \varepsilon, \qquad t_1 \leqslant t \leqslant T, \tag{4.7}$$

$$\int_T^{\infty} |x'(t)|\, dt \leqslant \tfrac{1}{2}(1 - c)\, \varepsilon, \tag{4.8}$$

$$\int_T^{\infty} |y'(t)|\, dt \leqslant c\varepsilon. \tag{4.9}$$

These will together ensure that (4.6) holds.

Since (2.9) holds, by Theorem 3.1 there will be a $t_2 \geqslant t_0$ such that (4.8) holds if $T \geqslant t_2$. We write

$$T_0 = \max(t_2, t^*) \tag{4.10}$$

and choose

$$T = \max(t_1, T_0) + r. \tag{4.11}$$

Thus (4.8) is assured. We now determine δ by

$$\delta \exp \left\{ c_1 + \int_{t_0}^{T_0 - r} u(s)\, ds \right\} = \tfrac{1}{2}(1 - c)\, \varepsilon, \tag{4.12}$$

and claim that (4.7) and (4.9) hold. We have, for $t \geqslant t_1$,

$$\begin{aligned} |y'(t) - x'(t)| &= |N(t, y_t) - N(t, x_t)| \leqslant u(t) \,\|y_t - x_t\| \\ &< u(t) \left\{ \delta + \int_{t_1}^{t} |y'(s) - x'(s)|\, ds \right\}. \end{aligned}$$

Hence, by a Gronwall-type argument,

$$|y'(t) - x'(t)| < \delta u(t) \exp \left\{ \int_{t_1}^{t} u(s)\, ds \right\}, \qquad t \geqslant t_1. \tag{4.13}$$

Integration of this yields

$$\int_{t_1}^{t} |y'(s) - x'(s)|\, ds < \delta \left\{ \exp \int_{t_1}^{t} u(s)\, ds - 1 \right\}, \qquad t \geqslant t_1. \tag{4.14}$$

Using (4.5), we then have

$$|y(t) - x(t)| < \delta \exp \left\{ \int_{t_1}^{t} u(s)\, ds \right\}, \qquad t \geqslant t_1. \tag{4.15}$$

Suppose first that $t_1 \geqslant T_0$, so that, by (4.11), $T = t_1 + r$. We then have from (4.15) that

$$|y(t) - x(t)| < \delta \exp \int_{t_1}^{t_1 + r} u(s)\, ds, \qquad t_1 \leqslant t \leqslant T,$$

which, in view of (4.2) and (4.12), proves (4.7). Suppose next that $t_0 \leqslant t_1 < T_0$, so that $T = T_0 + r$. We then have from (4.15) that

$$|y(t) - x(t)| < \delta \exp \int_{t_1}^{T_0 + r} u(s)\, ds, \qquad t_1 \leqslant t \leqslant T,$$

and this again proves (4.7) in view of (4.12).

It remains to establish (4.9). Arguing as in (2.11) and (2.12), we have

$$\begin{aligned} \int_{T}^{t} |y'(s)|\, ds &\leqslant \int_{T}^{t} p(s)\, ds \int_{s-r}^{s} |y'(u)|\, du \\ &\leqslant \int_{T-r}^{t} |y'(s)|\, ds \int_{s}^{s+r} p(u)\, du \leqslant c \int_{T-r}^{t} |y'(s)|\, ds. \end{aligned}$$

Hence,

$$\int_T^t |y'(s)|\,ds \leqslant c(1-c)^{-1}\int_{T-r}^T |y'(s)|\,ds.$$

Here we may take $t \to \infty$ and deduce that

$$\int_T^\infty |y'(s)|\,ds \leqslant c(1-c)^{-1}\int_{T-r}^T |y'(s)-x'(s)|\,ds$$
$$+\,c(1-c)^{-1}\int_{T-r}^T |x'(s)|\,ds. \tag{4.16}$$

We have now

$$\int_{T-r}^T |y'(s)-x'(s)|\,ds \leqslant \int_{t_1}^T |y'(s)-x'(s)|\,ds$$

by (4.11), and so, by (4.14),

$$\int_{T-r}^T |y'(s)-x'(s)|\,ds \leqslant \delta \exp\int_{t_1}^T u(s)\,ds \leqslant \tfrac{1}{2}(1-c)\,\varepsilon,$$

as in the proof of (4.7), whether $t_1 \geqslant T_0$ or $t_1 < T_0$. A similar bound for the last integral in (4.16) has already been arranged; we may in (4.8) replace T by $T-r$, since $T \geqslant t_2 + r$. Combining these results, we obtain (4.9), thereby completing the proof of Theorem 4.1.

COROLLARY. *Let* (4.2) *and* (4.3) *hold with* $c_1 < 1$. *If, in addition,*

$$N(t,\phi_k) = 0 \qquad \textit{for all} \quad t \geqslant 0 \quad \textit{and all constant functions} \quad \phi_k \in C, \tag{4.17}$$

then solutions of (4.1) *are uniformly stable.*

Proof. In view of Theorem 4.1, it suffices to show that (2.1) holds with $p(t) = u(t)$.

Let $y(\cdot)$ be continuously differentiable on $[t-r,t]$, let $\theta \in [t-r,t]$, and let $k = y(\theta)$. For some $\omega \in [-r,0]$, we have

$$\|y_t - \phi_k\| = |y_t(\omega) - \phi_k(\omega)| = |y(t+\omega) - y(\theta)|.$$

Thus,

$$\begin{aligned} |N(t,y_t)| &= |N(t,y_t) - N(t,\phi_k)| \leqslant p(t)\,\|y_t - \phi_k\| \\ &= p(t)\,|y(t+\omega) - y(\theta)| \\ &= p(t)\left|\int_\theta^{t+\omega} y'(s)\,ds\right| \leqslant p(t)\int_{t-r}^t |y'(s)|\,ds, \end{aligned}$$

from which the conclusion follows.

Remark 4.1. There is no difficulty in extending Theorem 4.1 to include Eq. (1.4) whenever (2.2) holds with $v \in L^1[0, \infty)$. In particular, if (4.2) holds, then we can also include in Theorem 3.1 that solutions of (1.4) are uniformly stable. We use this fact in the next example.

EXAMPLE 4.1. In [1], Cooke and Yorke obtained rather interesting results for the autonomous scalar equations

$$x'(t) = g(x(t)) - g(x(t-r)) \tag{4.18}$$

and

$$x'(t) = g(x(t)) - \int_{-r}^{0} P_1(s)\, g(x(t+s))\, ds, \tag{4.19}$$

where $r > 0$ and $\int_{-r}^{0} P_1(s)\, ds = 1$. These equations have been proposed as models for the growth of certain populations and for the spread of epidemics. As a simple illustration of (4.18), suppose a single species has birthrate $g(x(t))$, where $x(t)$ denotes the number of the species at time t. If each member has a life span $r > 0$, then (4.18) becomes a reasonable growth model. For the case that g is locally Lipschitzian, Cooke and Yorke have shown in [1] that each solution $x: [t_0 - r, \omega) \to R$, $t_0 < \omega \leqslant \infty$, of (4.18) (or (4.19)) satisfies

$$\begin{array}{lll} x(t) \to \infty & \text{as} & t \to \omega; \quad \text{or} \\ x(t) \to \text{const} & \text{as} & t \to \omega; \quad \text{or} \\ x(t) \to -\infty & \text{as} & t \to \omega. \end{array}$$

These results have recently been extended to include a more general (but still scalar and autonomous) setting (cf. [5]).

Our previous results can be employed to supplement the results of [1] and [5]. For example, if g satisfies a global Lipschitz condition

$$|g(x) - g(y)| \leqslant L\, |x - y| \quad \text{for} \quad x, y \in R \ (\text{or } R^n) \quad \text{and some} \quad L > 0, \tag{4.20}$$

and if $Lr < 1$, then from Theorems 3.1 and 4.1, each solution of (4.18) is uniformly stable and tends to a constant limit as $t \to \infty$. (The function g is not restricted to the scalar case.)

If the birthrate in (4.18) were altered to allow for *seasonal* fluctuations (variable coefficients), then the growth model could take the form

$$\begin{aligned} x'(t) &= p(t)\, g(x(t)) - p(t-r)\, g(x(t-r)) \\ &= p(t)[\, g(x(t)) - g(x(t-r))] + v(t)\, g(x(t-r)), \end{aligned} \tag{4.21}$$

where $v(t) \stackrel{\text{def}}{=} p(t) - p(t-r)$. In view of Remark 4.1, if g satisfies (4.20), then each solution of (4.21) is uniformly stable and asymptotically constant as $t \to \infty$, provided $|v| \in L^1[r, \infty)$ and $\lim_{t\to\infty} \sup \int_{t-r}^{t} p(s)\, ds < 1/L$.

Even if g does not satisfy a global Lipschitz condition in (4.18), local results can still be salvaged. For example, the results of the previous sections can be readily modified and made applicable to an equation such as

$$x'(t) = p(t)[x^{\gamma}(t) - x^{\gamma}(t-r)]; \qquad r > 0, \tag{4.22}$$

where $\gamma = a/b > 1$ is the quotient of positive odd integers. In particular, if $|p(\cdot)|$ is bounded on $[0, \infty)$, then the zero solution of (4.22) is uniformly stable. Furthermore, each solution with sufficiently small initial condition tends to a constant as $t \to \infty$. Also, for each fixed $\gamma > 1$ and each constant $K > 0$, there exists $r_0 > 0$ such that, for each $r \leqslant r_0$, the solutions that satisfy $\lim_{t\to\infty} \sup |x(t)| < K$ are uniformly stable and asymptotically constant as $t \to \infty$.

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