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Note

The No-Four-on-Circle Problem

TORSTEN THIELE*

Freie Universität Berlin, Institut für Mathematik II, Arnimallee 3, D-14195 Berlin, Germany

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Erdős and Purdy ask how many points can be chosen from the $n \times n$-grid with no four of them on a circle. They proved a lower bound of $n^{2/3-\varepsilon}$. In this note we improve the lower bound to $(\frac{1}{4}-\varepsilon) \cdot n$. © 1995 Academic Press, Inc.

1. INTRODUCTION

Erdős and Purdy ask the following question [Rkg]: consider the set of n^2 grid points (x, y) with $0 \leqslant x,\ y < n$. How many points can you choose, s.t. there are no four of them on a common circle; in particular, there are no four points on a line. Let $C(n)$ denote the maximum number of such points. They mentioned that it is easy to show that $n^{2/3-\varepsilon}$ is a lower bound but they conjectured that more is possible. We will show the following.

THEOREM 1.1. *Let $\varepsilon > 0$ be a real constant. Then, for n sufficiently large*

$$C(n) \geqslant (\tfrac{1}{4}-\varepsilon) \cdot n.$$

This bound is optimal up to the constant since $3n$ is an obvious upper bound. For the proof we construct a subset of the $n \times n$-grid with the desired property. In fact, our point set will have a further property: there are no three points on a line.

Such point sets are useful for perturbating degenerated point sets. The aim of a perturbation algorithm is to perturb the input of a geometric algorithm (such as computation of convex hulls or Voronoi diagrams) designed under the hypothesis of non-degeneracy of the input so that it can execute on arbitrary instances (see, for example, [EC]).

Related to our problem is the so-called no-three-in-line problem where one seeks for large subsets of the grid with no three points on a common line. The best known lower bound is $(1.5-\varepsilon) \cdot n$ (see [HJSW]). Up to now

* E-mail: thiele@math.fu-berlin.de.

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it is an unsolved problem whether it is always possible to select $2n$ points with this property (see [F1]).

2. Construction and Proof

Let p be a prime number. Define the point set $P(p)$ as

$$P(p) := \{(t, t^2 \bmod p) : 0 \leqslant t < p/4\}. \tag{1}$$

Clearly $P(p)$ is a subset of the $p \times p$-grid.

Lemma 2.1. *Let p be a prime number. Then there are no four points in $P(p)$ on a common circle and no three points on a common line.*

Proof. Let p be a given prime and $P(p)$ defined by (1). Assume first that there are three points $(t_i, t_i^2 \bmod p)$, $i = 1, 2, 3$, in $P(p)$ on a line. Then the determinant

$$\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix} = \prod_{i<j} (t_j - t_i)$$

would be zero modulo p. But this is impossible since p is a prime

Now assume there are four points $(t_i, t_i^2 \bmod p)$, $i = 1, 2, 3, 4$, in $P(p)$ on a circle. Four points in the plane lie on a common circle if and only if their projections onto the unit paraboloid $\{(x, y, x^2 + y^2) : x, y \in \mathbf{R}\}$ lie on a common hyperplane. Thus the determinant

$$\begin{vmatrix} 1 & t_1 & t_1^2 & t_1^2 + t_1^4 \\ 1 & t_2 & t_2^2 & t_2^2 + t_2^4 \\ 1 & t_3 & t_3^2 & t_3^2 + t_3^4 \\ 1 & t_4 & t_4^2 & t_4^2 + t_4^4 \end{vmatrix} = (t_1 + t_2 + t_3 + t_4) \prod_{i<j} (t_j - t_i)$$

would be zero modulo p. Again this is impossible since p is a prime and the t_i's are less than $p/4$. ▮

Theorem 1.1 is an immediate consequence of Lemma 2.1 by taking p to be the largest prime less than or equal to n.

Acknowledgments

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Note added in proof. The construction can be generalized to dimension $d \geqslant 2$ to determine a set of $\Omega(n^{1/(d-1)})$ points in the n^d-grid with no $d+2$ points on a sphere and no $d+1$ points on a hyperplane.

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