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w-Divisorial domains

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Abstract

We study the class of domains in which each w-ideal is divisorial, extending several properties of divisorial and totally divisorial domains to a much wider class of domains. In particular we consider *PvMD*s and Mori domains.

Keywords: Divisorial domains; Prüfer v-multiplication domains; Mori domains

Introduction

The class of domains in which each nonzero ideal is divisorial has been studied, independently and with different methods, by H. Bass [2], E. Matlis [25] and W. Heinzer [17] in the sixties. Following S. Bazzoni and L. Salce [3,4], these domains are now called *divisorial domains*. Among other results, Heinzer proved that an integrally closed domain is divisorial if and only if it is a Prüfer domain with certain finiteness properties [17, Theorem 5.1].

Twenty years later E. Houston and M. Zafrullah introduced in [20] the class of domains in which each t-ideal is divisorial, which they called *TV-domains*, and characterized *PvMD*s with this property [20, Theorem 3.1]. However they observed that an integrally

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closed *TV*-domain need not be a *PvMD* [20, Remark 3.2]; thus in some sense the class of *TV*-domains is not the right setting for extending to *PvMD*s the properties of divisorial Prüfer domains.

The purpose of this paper is to investigate *w-divisorial domains*, that is domains in which each w-ideal is divisorial. This class of domains proves to be the most suitable t-analogue of divisorial domains. In fact, by using this concept we are able to improve and generalize several results proved for Noetherian and Prüfer divisorial domains in [3,17,28,31].

The main result of Section 1 is Theorem 1.5. It states that R is a w-divisorial domain if and only if R is a weakly Matlis domain (that is a domain with t-finite character such that each t-prime ideal is contained in a unique t-maximal ideal) and R_M is a divisorial domain, for each t-maximal ideal M. In this way we recover the characterization of divisorial domains given in [3, Proposition 5.4].

In Section 2, we study the transfer of the properties of w-divisoriality and divisoriality to certain (generalized) rings of fractions, such as localizations at (t-)prime ideals, (t-)flat overrings and (t-)subintersections.

In Section 3 we consider w-divisorial *PvMD*s. We prove that R is an integrally closed w-divisorial domain if and only if R is a weakly Matlis *PvMD* and each t-maximal ideal is t-invertible (Theorem 3.3). This is the t-analogue of [17, Theorem 5.1]. We also prove that when R is integrally closed, each t-linked overring of R is w-divisorial if and only if R is a generalized Krull domain and each t-prime ideal is contained in a unique t-maximal ideal (Theorem 3.5). Since in the Prüfer case generalized Krull domains coincide with generalized Dedekind domains [7], we obtain that an integrally closed domain is totally divisorial if and only if it is a divisorial generalized Dedekind domain [28, Section 4].

The last section is devoted to Mori w-divisorial domains. A Mori w-divisorial domain is necessarily of t-dimension one and each of its localizations at a height-one prime is Noetherian (Corollary 4.3). Noetherian divisorial and totally divisorial domains were intensely studied in [2,3,25,31]. It turns out that several of the results proved there can be extended to the Mori case by using different technical tools. In Theorem 4.2 we characterize w-divisorial Mori domains and in Theorems 4.5 and 4.11 we study w-divisoriality of their overrings. In particular, we show that generalized rings of fractions of w-divisorial Mori domains are w-divisorial and we prove that a domain whose t-linked overings are all w-divisorial is Mori if and only if it has t-dimension one.

Throughout this paper R will denote an integral domain with quotient field K and we will assume that $R \neq K$.

We shall use the language of star operations. A *star operation* is a map $I \to I^*$ from the set $F(R)$ of nonzero fractional ideals of R to itself such that:

(1) $R^* = R$ and $(aI)^* = aI^*$, for all $a \in K \setminus \{0\}$;
(2) $I \subseteq I^*$ and $I \subseteq J \Rightarrow I^* \subseteq J^*$;
(3) $I^{**} = I^*$.

General references for systems of ideals and star operations are [13,15,16,21].

A star operation $*$ is of *finite type* if $I^* = \bigcup\{J^*;\ J \subseteq I$ and J is finitely generated$\}$, for each $I \in F(R)$. To any star operation $*$, we can associate a star operation $*_f$ of finite type by defining $I^{*_f} = \bigcup J^*$, with the union taken over all finitely generated ideals J contained in I. Clearly $I^{*_f} \subseteq I^*$. A nonzero ideal I is $*$-*finite* if $I^* = J^*$ for some finitely generated ideal J.

The identity is a star operation, called the d-operation. The v- and the t-operations are the best known nontrivial star operations and are defined in the following way. For a pair of nonzero ideals I and J of a domain R we let $(J : I)$ denote the set $\{x \in K;\ xI \subseteq J\}$. We set $I_v = (R : (R : I))$ and $I_t = \bigcup J_v$ with the union taken over all finitely generated ideals J contained in I. Thus the t-operation is the finite type star operation associated to the v-operation.

A nonzero fractional ideal I is called a $*$-*ideal* if $I = I^*$. If $I = I_v$ we say that I is *divisorial*. For each star operation $*$, we have $I^* \subseteq I_v$, thus each divisorial ideal is a $*$-ideal.

The set $F_*(R)$ of $*$-ideals of R is a semigroup with respect to the $*$-*multiplication*, defined by $(I, J) \to (IJ)^*$, with unity R. We say that an ideal $I \in F(R)$ is $*$-*invertible* if I^* is a unit in the semigroup $F_*(R)$. In this case the $*$-*inverse* of I is $(R : I)$. Thus I is $*$-invertible if and only if $(I(R : I))^* = R$. Invertible ideals are ($*$-invertible) $*$-ideals.

A prime $*$-ideal is also called a $*$-*prime*. A $*$-*maximal* ideal is an ideal that is maximal in the set of the proper $*$-ideals. A $*$-maximal ideal (if it exists) is a prime ideal. If $*$ is a star operation of finite type, an easy application of Zorn's Lemma shows that the set $*$-Max(R) of the $*$-maximal ideals of R is not empty. Moreover, for each $I \in F(R)$, $I^* = \bigcap_{M \in *\text{-Max}(R)} I^* R_M$; in particular $R = \bigcap_{M \in *\text{-Max}(R)} R_M$ [15].

The w-operation is the star operation defined by setting $I_w = \bigcap_{M \in t\text{-Max}(R)} IR_M$. An equivalent definition is obtained by setting $I_w = \bigcup\{(I : J);\ J$ is finitely generated and $(R : J) = R\}$. By using the latter definition, one can see that the notion of w-ideal coincides with the notion of *semi-divisorial* ideal introduced by S. Glaz and W. Vasconcelos in 1977 [14]. As a star operation, the w-operation was first considered by E. Hedstrom and E. Houston in 1980 under the name of F_∞-operation [18]. Since 1997 this star operation was intensely studied by Wang Fanggui and R. McCasland in a more general context. In particular they showed that the notion of w-closure is a very useful tool in the study of Strong Mori domains [32,33].

The w-operation is of finite type. We have w-Max$(R) = t$-Max(R) and $IR_M = I_w R_M \subseteq I_t R_M$, for each $I \in F(R)$ and $M \in t$-Max(R). Thus $I_w \subseteq I_t \subseteq I_v$.

We denote by t-Spec(R) the set of t-prime ideals of R. Each height one prime is a t-prime and each prime minimal over a t-ideal is a t-prime. We say that R has t-*dimension one* if each t-prime ideal has height one.

1. w-Divisorial domains

A *divisorial domain* is a domain such that each ideal is divisorial [3] and we say that a domain R is w-*divisorial* if each w-ideal is divisorial, that is $w = v$. Since $I_w \subseteq I_t \subseteq I_v$, for each nonzero fractional ideal I, then R is w-divisorial if and only if $w = t = v$. A domain with the property that $t = v$ is called in [20] a *TV-domain*. Mori domains (i.e., domains satisfying the ascending chain condition on proper divisorial ideals) are *TV*-domains. A domain such that $w = t$ is called a *TW-domain* [27]. An important class of *TW*-domains is the class of *PvMD*s; in fact a *PvMD* is precisely an integrally closed *TW*-domain [22, Theorem 3.1]. (Recall that a domain R is a *Prüfer v-multiplication domain*, for short a *PvMD*, if R_M is a valuation domain for each t-maximal ideal M of R.) Since a Krull domain is a Mori *PvMD*, a Krull domain is a w-divisorial domain. An example due to M. Zafrullah shows that in general $w \neq t \neq v$ [27, Proposition 1.2]. Also there exist *TV*-domains and *TW*-domains that are not w-divisorial [27, Example 2.7].

If R is a Prüfer domain, in particular a valuation domain, then w-divisoriality coincides with divisoriality, because each ideal of a Prüfer domain is a t-ideal.

Proposition 1.1. *A w-divisorial domain R is divisorial if and only if each maximal ideal of R is a t-ideal. Hence a one-dimensional w-divisorial domain is divisorial.*

Proof. If each maximal ideal of R is a t-ideal, then each ideal of R is a w-ideal by [27, Proposition 1.3]. Hence, if R is w-divisorial it is also divisorial. The converse is clear. $\square$

Following [1], we say that a nonempty family Λ of nonzero prime ideals of R is of *finite character* if each nonzero element of R belongs to at most finitely many members of Λ and we say that Λ is *independent* if no two members of Λ contain a common nonzero prime ideal. We observe that a family of primes is independent if and only if no two members of Λ contain a common t-prime ideal. In fact a minimal prime of a nonzero principal ideal is a t-ideal.

The domain R has finite character (respectively, t-finite character) if $\mathrm{Max}(R)$ (respectively, t-$\mathrm{Max}(R)$) is of finite character. If the set $\mathrm{Max}(R)$ is independent of finite character, the domain R is called by E. Matlis an *h-local* domain [26]; thus R is h-local if it has finite character and each nonzero prime ideal is contained in a unique maximal ideal. A domain R such that t-$\mathrm{Max}(R)$ is independent of finite character is called in [1] a *weakly Matlis domain*; hence R is a weakly Matlis domain if it has t-finite character and each t-prime ideal is contained in a unique t-maximal ideal.

Clearly, a domain of t-dimension one is a weakly Matlis domain if and only if it has t-finite character. A one-dimensional domain is a weakly Matlis domain if and only if it is h-local; if and only if it has finite character.

We recall that any *TV*-domain, hence any w-divisorial domain, has t-finite character by [20, Theorem 1.3]. The main result of this section shows that w-divisorial domains form a distinguished class of weakly Matlis domains.

We start by proving some technical properties of weakly Matlis domains.

Lemma 1.2. *Let R be an integral domain. The following conditions are equivalent*:

(1) *R is a weakly Matlis domain*;
(2) *For each t-maximal ideal M of R and a collection $\{I_\alpha\}$ of w-ideals of R such that $\bigcap_\alpha I_\alpha \neq 0$, if $\bigcap_\alpha I_\alpha \subseteq M$, then $I_\alpha \subseteq M$ for some α.*

Proof. (1) $\Rightarrow$ (2) follows from [1, Corollary 4.4 and Proposition 4.7], by taking $\mathcal{F} = t$-$\mathrm{Max}(R)$ and then $*_{\mathcal{F}} = w$.

(2) $\Rightarrow$ (1). First, we show that each t-prime ideal is contained in a unique t-maximal ideal. We adapt the proof of [17, Theorem 2.4]. Let P be a t-prime which is contained in

two distinct t-maximal ideals M_1 and M_2. Let $\{I_\alpha\}$ be the set of all w-ideals of R which contain P but are not contained in M_1. Such a collection is nonempty since M_2 is in it. Let $I = \bigcap I_\alpha$. Then $I \nsubseteq M_1$ and $I \subseteq M_2$. Take $x \in I \setminus M_1$. Since $x^2 \notin M_1$, then $(P + x^2R)_w \in \{I_\alpha\}$ and so $x \in (P + x^2R)_w$. Thus $x \in (P + x^2R)R_{M_2} \neq R_{M_2}$ and $sx = p + x^2r$ for some $s \in R \setminus M_2$, $p \in P$ and $r \in R$. Whence $(s - rx)x = p \in P \subseteq M_1 \cap M_2$. Now $s - rx \notin P$ because $s \notin M_2$ and $rx \in I \subseteq M_2$. But also $x \notin P$, since $x \notin M_1$; a contradiction because P is prime.

Next we show that R has t-finite character. Let $0 \neq x \in R$ and $\{M_\beta\}$ be the set of all t-maximal ideals of R which contain x. For a fixed β, let A_β be the intersection of all w-ideals of R which contain x but are not contained in M_β. By assumption $A_\beta \nsubseteq M_\beta$. Set $A = \sum_\beta A_\beta$. Then $x \in A$ and A is contained in no M_β. Hence $A_t = R$. Let $F = (a_{\beta_1}, a_{\beta_2}, \dots, a_{\beta_n})$, where $a_{\beta_i} \in A_{\beta_i}$, be a finitely generated ideal of R such that $F_t = R$. Now, if $M_\beta \notin \{M_{\beta_1}, M_{\beta_2}, \dots, M_{\beta_n}\}$, necessarily $M_\beta \supseteq F$, which is impossible because M_β is a proper t-ideal and $F_t = R$. We conclude that $\{M_\beta\} = \{M_{\beta_1}, M_{\beta_2}, \dots, M_{\beta_n}\}$ is finite. □

Lemma 1.3. *Let R be a w-divisorial domain, M a t-maximal ideal of R and $\{I_\alpha\}$ a collection of w-ideals of R such that $\bigcap_\alpha I_\alpha \neq 0$. If $\bigcap_\alpha I_\alpha \subseteq M$, then $I_\alpha \subseteq M$ for some α.*

Proof. Set $A = \bigcap_\alpha I_\alpha$. Since R is a *TW*-domain, then the I_α's and A are t-ideals. Since R is also a *TV*-domain, by [20, Lemma 1.2], if $I_\alpha \nsubseteq M$, for each α, then $A \nsubseteq M$. □

Lemma 1.4. *If R is a weakly Matlis domain, then $I_vR_M = (IR_M)_v$, for each nonzero fractional ideal I and each t-maximal ideal M.*

Proof. Apply [1, Corollary 5.3] for $\mathcal{F} = t\text{-Max}(R)$. □

We are now ready to prove the t-analogue of [3, Proposition 5.4], which states that a domain R is divisorial if and only if it is h-local and R_M is a divisorial domain, for each maximal ideal M. Local divisorial domains have been studied in [3, Section 5] and completely characterized in [4, Section 2].

Theorem 1.5. *Let R be an integral domain. The following conditions are equivalent*:

(1) *R is a w-divisorial domain*;
(2) *R is a weakly Matlis domain and R_M is a divisorial domain, for each t-maximal ideal M*;
(3) *R is a TV-domain and R_M is a divisorial domain, for each t-maximal ideal M*;
(4) *$IR_M = (IR_M)_v = I_vR_M$, for each nonzero fractional ideal I and each t-maximal ideal M.*

Proof. (1) ⇒ (2). That R is a weakly Matlis domain follows from Lemmas 1.3 and 1.2. Now let M be a t-maximal ideal of R and $I = JR_M$ a nonzero ideal of R_M, where J is an ideal of R. By Lemma 1.4, we have $I_v = (JR_M)_v = J_vR_M$. Since $J_v = J_w$, then $I_v = J_wR_M = JR_M = I$. Hence R_M is a divisorial domain.

(2) $\Rightarrow$ (4) follows from Lemma 1.4.

(4) $\Rightarrow$ (1). Let I be a nonzero fractional ideal of R. Then $I_w = \bigcap_{M \in t\text{-Max}(R)} IR_M = \bigcap_{M \in t\text{-Max}(R)} I_v R_M = I_v$. Whence R is a w-divisorial domain.

(1) $\Rightarrow$ (3) via (2).

(3) $\Rightarrow$ (4). Since $t = v$ in R and $d = t = v$ in R_M, for each nonzero fractional ideal I and each t-maximal ideal M of R, we have

$$IR_M = (IR_M)_v = (IR_M)_t = (I_t R_M)_t = I_t R_M = I_v R_M. \qquad \square$$

Any almost Dedekind domain that is not Dedekind provides an example of a locally divisorial domain that is not w-divisorial, because it is not of finite character [13, Theorem 37.2].

Corollary 1.6. *Let R be a domain of t-dimension one. Then R is w-divisorial if and only if R has t-finite character and R_P is divisorial, for each height one prime P.*

2. Localizations of w-divisorial domains

A domain whose overrings are all divisorial is called *totally divisorial* [3]. Not all divisorial domains are totally divisorial [17, Remark 5.4]; in fact a valuation domain R is divisorial if and only if its maximal ideal is principal [17, Lemma 5.2], but it is totally divisorial if and only if it is strongly discrete [3, Proposition 7.6], equivalently PR_P is a principal ideal for each prime ideal P of R [8, Proposition 5.3.8]. Since for valuation domains divisoriality coincides with w-divisoriality and each overring of a valuation domain is a localization at a certain (t-)prime, we see that w-divisoriality is not stable under localization at t-primes.

We say that an integral domain R is a *strongly w-divisorial domain* (respectively, a *strongly divisorial domain*) if R is w-divisorial (respectively, divisorial) and R_P is a divisorial domain for each $P \in t\text{-Spec}(R)$ (respectively, $P \in \text{Spec}(R)$). Note that if R is strongly w-divisorial (respectively, strongly divisorial), then R_P is strongly divisorial for each $P \in t\text{-Spec}(R)$ (respectively, for each $P \in \text{Spec}(R)$).

By Theorem 1.5 (respectively, [3, Proposition 5.4]), R is a strongly w-divisorial domain (respectively, a strongly divisorial domain) if and only if R is a weakly Matlis domain (respectively, an h-local domain) and R_P is a divisorial domain for each $P \in t\text{-Spec}(R)$ (respectively, $P \in \text{Spec}(R)$).

If R has t-dimension one, then R is w-divisorial if and only if it is strongly w-divisorial.

In this section we shall study the extension of w-divisoriality and divisoriality to distinguished classes of generalized rings of fractions such as localizations at (t-)prime ideals, (t-)flat overrings and (t-)subintersections.

We recall the requisite definitions. A nonempty family $\mathcal{F}$ of nonzero ideals of a domain R is said to be a *multiplicative system* of ideals if $IJ \in \mathcal{F}$, for each $I, J \in \mathcal{F}$. If $\mathcal{F}$ is a multiplicative system, the set of ideals of R containing some ideal of $\mathcal{F}$ is still a multiplicative system, which is called the *saturation of* $\mathcal{F}$ and is denoted by $\text{Sat}(\mathcal{F})$. A multiplicative system $\mathcal{F}$ is said to be *saturated* if $\mathcal{F} = \text{Sat}(\mathcal{F})$.

If $\mathcal{F}$ is a multiplicative system of ideals, the overring $R_{\mathcal{F}} := \bigcup\{(R : J);\ J \in \mathcal{F}\}$ of R is called the *generalized ring of fractions* of R with respect to $\mathcal{F}$. For any fractional ideal I of R, $I_{\mathcal{F}} := \bigcup\{(I : J);\ J \in \mathcal{F}\}$ is a fractional ideal of $R_{\mathcal{F}}$ and $IR_{\mathcal{F}} \subseteq I_{\mathcal{F}}$. Clearly $I_{\mathcal{F}} = I_{\mathrm{Sat}(\mathcal{F})}$.

The map $P \mapsto P_{\mathcal{F}}$ is an order-preserving bijection between the set of prime ideals P of R such that $P \notin \mathrm{Sat}(\mathcal{F})$ and the set of prime ideals Q of $R_{\mathcal{F}}$ such that $JR_{\mathcal{F}} \nsubseteq Q$ for any $J \in \mathcal{F}$, with inverse map $Q \mapsto Q \cap R$. In addition, $R_P = (R_{\mathcal{F}})_{P_{\mathcal{F}}}$ for each prime ideal $P \notin \mathrm{Sat}(\mathcal{F})$. If Q is a t-prime ideal of $R_{\mathcal{F}}$, then $Q \cap R$ is a t-prime ideal of R [10, Proposition 1.3].

If Λ is a nonempty family of nonzero prime ideals of R, the set $\mathcal{F}(\Lambda) = \{J;\ J \subseteq R$ is an ideal and $J \nsubseteq P$ for each $P \in \Lambda\}$ is a saturated multiplicative system of ideals and $I_{\mathcal{F}(\Lambda)} = \bigcap\{IR_P;\ P \in \Lambda\}$, for each fractional ideal I of R; in particular $R_{\mathcal{F}(\Lambda)} = \bigcap\{R_P;\ P \in \Lambda\}$. A generalized ring of fractions of type $R_{\mathcal{F}(\Lambda)}$ is called a *subintersection of* R; when $\Lambda \subseteq t\text{-Spec}(R)$, we say that $R_{\mathcal{F}(\Lambda)}$ is a *t-subintersection* of R.

A multiplicative system of ideals $\mathcal{F}$ of R is *finitely generated* if each ideal $I \in \mathcal{F}$ contains a finitely generated ideal J which is still in $\mathcal{F}$. As in [10], we say that $\mathcal{F}$ is a *v-finite* multiplicative system if each t-ideal $I \in \mathrm{Sat}(\mathcal{F})$ contains a finitely generated ideal J such that $J_v \in \mathrm{Sat}(\mathcal{F})$. A finitely generated multiplicative system is v-finite. If $\mathcal{F}$ is v-finite, the set Λ of t-ideals which are maximal with respect to the property of not being in $\mathrm{Sat}(\mathcal{F})$ is not empty, $\Lambda \subseteq t\text{-Spec}(R)$, $\mathcal{F}(\Lambda)$ is v-finite and $T = R_{\mathcal{F}(\Lambda)}$ [10, Proposition 1.9(a) and (b)].

An overring T of R is said to be *t-flat* over R if $T_M = R_{M\cap R}$, for each t-maximal ideal M of T [23], equivalently $T_Q = R_{Q\cap R}$, for each t-prime ideal Q of T [7, Proposition 2.6]. Flatness implies t-flatness, but the converse is not true [23, Remark 2.12]. By [7, Theorem 2.6], T is t-flat over R if and only if there exists a v-finite multiplicative system $\mathcal{F}$ of R such that $T = R_{\mathcal{F}}$. Thus T is t-flat if and only if $T = R_{\mathcal{F}(\Lambda)}$, where Λ is a family of pairwise incomparable t-primes of R and $\mathcal{F}(\Lambda)$ is v-finite. It follows that a t-flat overring of R is a t-subintersection of R.

In turn, any generalized ring of fractions is a t-linked overring; but the converse does not hold in general [5, Proposition 2.2]. We recall that an overring T of an integral domain R is *t-linked* over R if, for each nonzero finitely generated ideal J of R such that $(R : J) = R$, we have $(T : JT) = T$ [5]. This is equivalent to say that $T = \bigcap T_{R\setminus P}$, where P ranges over the t-primes of R [5, Proposition 2.13(a)].

It is well known that if P is a t-prime ideal of R, then PR_P need not be a t-ideal of R_P. When PR_P is a t-prime ideal, P is called by M. Zafrullah a *well behaved t*-prime [34, page 436]. We prefer to say that P *t-localizes* or that it is a *t-localizing prime*. Height-one prime ideals and divisorial t-maximal primes, e.g., t-invertible t-primes, are examples of t-localizing primes.

A large class of domains with the property that each t-prime ideal t-localizes is the class of v-coherent domains. We recall that a domain R is called *v-coherent* if the ideal $(R : J)$ is v-finite whenever J is finitely generated. This class of domains properly includes *PvMD*s, Mori domains and coherent domains [11,24].

If R is a w-divisorial (respectively, strongly w-divisorial) domain, then each t-maximal (respectively, t-prime) ideal t-localizes.

Lemma 2.1. *Let Λ be a set of t-localizing t-primes of R. Then*:

(1) $P_{\mathcal{F}(\Lambda)} \in t\text{-Spec}(R_{\mathcal{F}(\Lambda)})$, *for each* $P \in \Lambda$.
(2) *If* $\mathcal{F}(\Lambda)$ *is* v*-finite*, $t\text{-Max}(R_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)};\ P \text{ maximal in } \Lambda\}$.

Proof. Set $\mathcal{F} = \mathcal{F}(\Lambda)$ and $T = R_{\mathcal{F}}$.

(1) Let $P \in \Lambda$. Since $R_P = T_{P_{\mathcal{F}}}$ and by hypothesis $PR_P = P_{\mathcal{F}}T_{P_{\mathcal{F}}}$ is a t-ideal, then $P_{\mathcal{F}} = P_{\mathcal{F}}T_{P_{\mathcal{F}}} \cap T$ is a t-ideal of T.

(2) Since $P_{\mathcal{F}}$ is a t-ideal by part (1), we can apply [10, Proposition 1.9(c)]. □

Proposition 2.2. *Let Λ be a set of pairwise incomparable t-localizing t-primes of R. Then*:

(1) *Λ is independent of finite character if and only if $\mathcal{F}(\Lambda)$ is v-finite and $R_{\mathcal{F}(\Lambda)}$ is a weakly Matlis domain.*
(2) *If $R_{\mathcal{F}(\Lambda)}$ is w-divisorial, then Λ is independent of finite character.*

Proof. Set $\mathcal{F} = \mathcal{F}(\Lambda)$ and $T = R_{\mathcal{F}}$.

(1) If $\mathcal{F}$ is v-finite, by Lemma 2.1(2) we have $t\text{-Max}(T) = \{P_{\mathcal{F}};\ P \in \Lambda\}$. It follows that Λ is independent of finite character if and only if $t\text{-Max}(T) = \{P_{\mathcal{F}};\ P \in \Lambda\}$ is independent of finite character, that is T is a weakly Matlis domain. On the other hand, if Λ is of finite character, then $\mathcal{F}$ is v-finite by [10, Lemma 1.16].

(2) Since T is a weakly Matlis domain, by part (1) it suffices to show that Λ is of finite character.

By Lemma 2.1(1), $P_{\mathcal{F}}$ is a t-prime of T, for each $P \in \Lambda$. We show that each proper divisorial ideal of T is contained in some $P_{\mathcal{F}}$. We have $T = \bigcap_{P \in \Lambda} R_P = \bigcap_{P \in \Lambda} T_{P_{\mathcal{F}}}$. If I is a proper divisorial ideal of T, there is $x \in K \setminus T$ (where K is the quotient field of R) such that $I \subseteq x^{-1}T \cap T$. Since $x \notin T$, there exists $P \in \Lambda$ such that $x \notin T_{P_{\mathcal{F}}}$, equivalently $x^{-1}T \cap T \subseteq P_{\mathcal{F}}$.

Since $t = v$ on T, we conclude that $t\text{-Max}(T) = \{P_{\mathcal{F}};\ P \in \Lambda\}$. Since T has t-finite character, it follows that Λ is of finite character. □

Theorem 2.3. *Let R be a w-divisorial domain. If $\Lambda \subseteq t\text{-Max}(R)$, then $R_{\mathcal{F}(\Lambda)}$ is a t-flat w-divisorial overring of R.*

Proof. Since R is a weakly Matlis domain (Theorem 1.5), $t\text{-Max}(R)$ is independent of finite character; thus Λ has the same properties. In addition, each t-maximal ideal is a t-localizing prime ideal. It follows that $\mathcal{F}(\Lambda)$ is v-finite and $T := R_{\mathcal{F}(\Lambda)}$ is a t-flat weakly Matlis domain (Proposition 2.2(1)). By Lemma 2.1(2), for each $N \in t\text{-Max}(T)$, there exists $M \in \Lambda$ such that $N = M_{\mathcal{F}(\Lambda)}$. It follows that $T_N = R_M$ is divisorial and so T is w-divisorial by Theorem 1.5. □

As we have mentioned above, the localization of a w-divisorial domain at a t-prime need not be a (w-)divisorial domain. Thus Theorem 2.3 does not hold for an arbitrary $\Lambda \subseteq t\text{-Spec}(R)$. However, under the hypothesis that R is strongly w-divisorial, we have a satisfying result.

Theorem 2.4. *Let R be a strongly w-divisorial domain and Λ a set of pairwise incomparable t-primes of R. The following conditions are equivalent*:

(1) $R_{\mathcal{F}(\Lambda)}$ *is w-divisorial*;
(2) $R_{\mathcal{F}(\Lambda)}$ *is strongly w-divisorial*;
(3) $R_{\mathcal{F}(\Lambda)}$ *is a t-flat weakly Matlis domain*;
(4) $R_{\mathcal{F}(\Lambda)}$ *is a t-flat TV-domain*;
(5) Λ *is independent of finite character.*

Proof. Set $\mathcal{F} = \mathcal{F}(\Lambda)$ and $T = R_{\mathcal{F}}$. Since R is strongly w-divisorial, each $P \in \Lambda$ t-localizes.

(1) $\Rightarrow$ (5) by Proposition 2.2(2).

(5) $\Rightarrow$ (3) by Proposition 2.2(1).

(3) $\Rightarrow$ (2). If Q is a t-prime of T, then $P = Q \cap R \in t\text{-Spec}(R)$ and $T_Q = R_P$ is divisorial. Whence T is strongly w-divisorial.

(3) $\Leftrightarrow$ (4). By t-flatness, T_M is divisorial for each t-maximal ideal M. Thus we can apply Theorem 1.5.

(2) $\Rightarrow$ (1) is obvious. □

Divisorial flat overrings of a strongly divisorial domain have a similar characterization. Recall that an overring T of R is flat if $T_M = R_{M\cap R}$, for each maximal ideal M of T; in this case $T = R_{\mathcal{F}(\Lambda)}$, where Λ is a set of pairwise incomparable prime ideals of R.

Corollary 2.5. *Let R be a strongly divisorial domain and $T = R_{\mathcal{F}(\Lambda)}$ a flat overring, where Λ is a set of pairwise incomparable prime ideals of R. The following conditions are equivalent*:

(1) T *is divisorial*;
(2) T *is strongly divisorial*;
(3) T *is h-local*;
(4) Λ *is independent of finite character.*

Proof. (1) $\Leftrightarrow$ (3). By [3, Proposition 5.4], T is divisorial if and only if it is h-local and locally divisorial. But, since T is flat and R is strongly divisorial, for each maximal ideal M of T, $T_M = R_{M\cap R}$ is divisorial.

(1) $\Rightarrow$ (2). Since T is flat and R is strongly divisorial, then $T_Q = R_{Q\cap R}$ is divisorial, for each prime ideal Q of T.

(2) $\Rightarrow$ (4). Since R and T are divisorial, then $d = w = t = v$ in R and T. Thus we can apply Theorem 2.4 ((2) $\Rightarrow$ (5)).

(4) $\Rightarrow$ (1). Since $d = w = t = v$ in R, by Theorem 2.4 ((5) $\Rightarrow$ (1)), T is w-divisorial. To prove that T is divisorial, we show that each maximal ideal of T is a t-ideal (Proposition 1.1). If M is a maximal ideal of T, by flatness we have $T_M = R_{M\cap R}$. Since R is strongly divisorial, MT_M is a t-ideal and so $M = MT_M \cap T$ is a t-ideal. □

Corollary 2.6. *Let R be an integral domain. The following conditions are equivalent*:

(1) *Each t-flat overring of R is strongly w-divisorial*;
(2) *R is strongly w-divisorial and each t-flat overring is a weakly Matlis domain*;
(3) *R is strongly w-divisorial and each t-flat overring is a TV-domain*;
(4) *R is strongly w-divisorial and each family Λ of pairwise incomparable t-primes of R such that $\mathcal{F}(\Lambda)$ is v-finite is independent of finite character.*

Proof. By Theorem 2.4, recalling that an overring T is t-flat over R if and only if $T = R_{\mathcal{F}(\Lambda)}$, where Λ is a family of pairwise incomparable t-primes of R and $\mathcal{F}(\Lambda)$ is v-finite. □

In order to study t-subintersections, we need the following technical lemma.

Lemma 2.7. *Let R be an integral domain and $\mathcal{C}$ an ascending chain of t-localizing t-primes of R. If $R_{\mathcal{F}(\mathcal{C})}$ is a TV-domain, then $\mathcal{C}$ is stationary.*

Proof. Let $\mathcal{C} = \{P_\alpha\}$ and set $\mathcal{F} = \mathcal{F}(\mathcal{C})$ and $T = R_{\mathcal{F}}$. By Lemma 2.1(1), $(P_\alpha)_{\mathcal{F}}$ is a t-prime ideal of T, for each α. It follows that $M = \bigcup_\alpha (P_\alpha)_{\mathcal{F}}$ is a proper t-prime ideal of T (since it is an ascending union of t-primes) and so M is divisorial (because T is a *TV*-domain). We have $T = \bigcap_\alpha T_{R \setminus P_\alpha}$; thus the map $I \mapsto I^\star = \bigcap_\alpha I T_{R\setminus P_\alpha}$ defines a star operation on T. Since M is divisorial, we have $M^\star \subseteq M$; so that $M^\star$ is a proper ideal. It follows that there exists α such that $M \cap R \subseteq P_\alpha$. Hence $M \cap R = P_\alpha$ and so $P_\beta = P_\alpha$ for $\beta \geqslant \alpha$. □

Theorem 2.8. *Let R be an integral domain. The following conditions are equivalent*:

(1) *Each t-subintersection of R is strongly w-divisorial*;
(2) *R is a strongly w-divisorial domain which satisfies the ascending chain condition on t-prime ideals and each family Λ of pairwise incomparable t-primes of R is independent of finite character.*

Proof. (1) $\Rightarrow$ (2). Clearly R is a strongly w-divisorial domain. If Λ is a set of pairwise incomparable t-prime ideals, then by assumption $R_{\mathcal{F}(\Lambda)}$ is strongly w-divisorial. Hence Λ is independent of finite character, by Theorem 2.4. It remains to show that R has the ascending chain condition on t-prime ideals. This follows from Lemma 2.7. In fact, if $\mathcal{C}$ is an ascending chain of t-prime ideals of R, $R_{\mathcal{F}(\mathcal{C})}$ is strongly w-divisorial. Hence each t-prime in $\mathcal{C}$ t-localizes and it follows that $\mathcal{C}$ is stationary.

(2) $\Rightarrow$ (1). Let $R_{\mathcal{F}(\Lambda)}$ be a t-subintersection of R. By the ascending chain condition on t-prime ideals, Λ has maximal elements; thus we can assume that Λ is a set of pairwise incomparable t-primes. The conclusion follows from Theorem 2.4. □

Corollary 2.9. *Let R be a domain. If each t-subintersection of R is strongly w-divisorial, then each t-subintersection of R is t-flat.*

Proof. If each t-subintersection of R is strongly w-divisorial, then R satisfies the ascending chain condition on t-primes (Theorem 2.8). Thus each t-subintersection is of type

$R_{\mathcal{F}(\Lambda)}$, where Λ is a family of pairwise incomparable t-primes. By Theorem 2.4, $R_{\mathcal{F}(\Lambda)}$ is t-flat. $\square$

Remark 2.10. If each subintersection of the domain R is strongly divisorial, then clearly R is strongly divisorial. In addition, since $d = w = t = v$ on R, then R satisfies the ascending chain condition on prime ideals and each family Λ of pairwise incomparable prime ideals of R is independent of finite character (Theorem 2.8).

Conversely, assume that R is a strongly divisorial domain satisfying the ascending chain condition on prime ideals and that each family Λ of pairwise incomparable prime ideals of R is independent of finite character.

Then each subintersection T of R is of type $R_{\mathcal{F}(\Lambda)}$, where Λ is a family of pairwise incomparable prime ideals independent of finite character. Thus $\mathcal{F}(\Lambda)$ is finitely generated [10, Lemma 1.16] and T is strongly w-divisorial and t-flat by Theorem 2.4. We conclude that T is (strongly) divisorial if and only if each maximal ideal of T is a t-ideal (Proposition 1.1) if and only if T is flat.

We observe that in general, if $\mathcal{F}$ is a finitely generated multiplicative system of ideals, then $R_{\mathcal{F}}$ need not be a flat extension of R [9, page 32]. On the other hand, we do not know any example of a strongly divisorial domain R with a finitely generated multiplicative system $\mathcal{F}$ such that $R_{\mathcal{F}}$ is not flat.

If R is any domain, we say that Spec(R) (respectively, t-Spec(R)) is *treed* (under inclusion) if any maximal (respectively, t-maximal) ideal of R cannot contain two incomparable primes (respectively, t-primes). The Spectrum of a Prüfer domain and the t-Spectrum of a *PvMD* are treed. If Spec(R) is treed, then Spec$(R) = t$-Spec(R) [23, Proposition 2.6]; in particular each maximal ideal is a t-ideal and so w-divisoriality coincides with divisoriality by Proposition 1.1.

If t-Spec(R) is treed and t-Max(R) is independent of finite character, then each family Λ of pairwise incomparable t-prime ideals of R is independent of finite character. Hence the next results are easy consequences of Theorems 2.4 and 2.8 respectively.

Corollary 2.11. *Let R be an integral domain such that t-*Spec(R) *is treed. The following conditions are equivalent*:

(1) *R is strongly w-divisorial*;
(2) *$R_{\mathcal{F}(\Lambda)}$ is a t-flat w-divisorial domain, for each set Λ of pairwise incomparable t-primes*;
(3) *$R_{\mathcal{F}(\Lambda)}$ is a t-flat strongly w-divisorial domain, for each set Λ of pairwise incomparable t-primes.*

If R has t-dimension one, then clearly t-Spec(R) is treed. In this case, the conditions stated in Corollary 2.11 are all satisfied if R is w-divisorial (cf. Theorem 2.3).

Corollary 2.12. *Let R be an integral domain such that t-*Spec(R) *is treed. The following conditions are equivalent*:

(1) *R is a strongly w-divisorial domain which satisfies the ascending chain conditions on t-prime ideals*;
(2) *Each t-subintersection of R is t-flat and strongly w-divisorial.*

3. Integrally closed w-divisorial domains

W. Heinzer proved in [17] that an integrally closed domain is divisorial if and only if it is an h-local Prüfer domain with invertible maximal ideals. We start this section by showing that integrally closed w-divisorial domains have a similar characterization among *PvMD*s. Note that a divisorial *PvMD* is a Prüfer domain.

Lemma 3.1. *Let R be a w-divisorial domain and $M \in t\text{-Max}(R)$. The following conditions are equivalent*:

(1) *M is t-invertible*;
(2) *MR_M is a principal ideal*;
(3) *R_M is a valuation domain.*

Proof. $(1) \Leftrightarrow (2)$. Since $t\text{-}\mathrm{Max}(R)$ has t-finite character (Theorem 1.5), we can apply [34, Theorem 2.2 and Proposition 3.1].

$(2) \Rightarrow (3)$ follows from [31, Lemme 1, Section 4], because R_M is a divisorial domain (Theorem 1.5), and $(3) \Rightarrow (2)$ follows from [17, Lemma 5.2]. □

Proposition 3.2. *Let R be a w-divisorial domain. Then R is a PvMD if and only if each t-maximal ideal of R is t-invertible.*

Theorem 3.3. *Let R be an integral domain. The following conditions are equivalent*:

(1) *R is an integrally closed w-divisorial domain*;
(2) *R is a weakly Matlis PvMD and each t-maximal ideal of R is t-invertible.*

Proof. $(1) \Rightarrow (2)$. A domain R is a *PvMD* if and only if R is an integrally closed *TW*-domain [22, Theorem 3.5]. Hence an integrally closed w-divisorial domain is a *PvMD*. By Theorem 1.5, R is a weakly Matlis domain and by Proposition 3.2 each t-maximal ideal is t-invertible.

$(2) \Rightarrow (1)$. A t-maximal ideal M of a *PvMD* is t-invertible if and only if MR_M is a principal ideal [19]. Since R_M is a valuation domain, this means that R_M is divisorial [17, Lemma 5.2]. Now we can apply Theorem 1.5. □

The previous theorem can be proved also by using the fact that a domain R is a *PvMD* if and only if R is an integrally closed *TW*-domain [22, Theorem 3.5] and the characterization of *PvMD*s which are *TV*-domains given in [20, Theorem 3.1].

Recall that a Prüfer domain R is strongly discrete if $P^2 \neq P$ for each nonzero prime ideal P of R [8, Section 5.3] and that a generalized Dedekind domain is a strongly discrete

Prüfer domain with the property that each ideal has finitely many minimal primes [30]. We say that a *PvMD* R is *strongly discrete* if $(P^2)_t \neq P$, for each $P \in t\text{-Spec}(R)$ [7, Remark 3.10]. If R is a strongly discrete *PvMD* and each t-ideal of R has only finitely many minimal primes, then R is called a *generalized Krull domain* [7].

The next theorem shows that the class of strongly w-divisorial domains and the class of strongly discrete *PvMD*s are strictly related to each other.

Lemma 3.4. *Let R be a domain. The following conditions are equivalent*:

(1) *R is a strongly discrete PvMD*;
(2) *R_M is a strongly discrete valuation domain, for each $M \in t\text{-Max}(R)$*;
(3) *R_P is a strongly discrete valuation domain, for each $P \in t\text{-Spec}(R)$*;
(4) *R_P is a valuation domain and PR_P is a principal ideal, for each $P \in t\text{-Spec}(R)$*;
(5) *R_P is a divisorial valuation domain, for each $P \in t\text{-Spec}(R)$.*

Proof. (1) $\Leftrightarrow$ (4). For each t-prime ideal P of R, we have $(P^2)_t = P^2R_P \cap R$ [19, Proposition 1.3]. Hence $(P^2)_t \neq P$ if and only if $P^2R_P \neq PR_P$. Now recall that a maximal ideal of a valuation domain is not idempotent if and only if it is principal.

(2) $\Leftrightarrow$ (3) because each overring of a strongly discrete valuation domain is a strongly discrete valuation domain [8, Proposition 5.3.1(3)].

(3) $\Leftrightarrow$ (4) by [8, Proposition 5.3.8 ((2) $\Leftrightarrow$ (6))].

(4) $\Leftrightarrow$ (5) by [17, Lemma 5.2]. □

Theorem 3.5. *Let R be an integral domain. The following conditions are equivalent*:

(1) *R is a strongly discrete PvMD and a weakly Matlis domain*;
(2) *R is an integrally closed strongly w-divisorial domain*;
(3) *R is integrally closed and each t-flat overring of R is w-divisorial*;
(4) *R is integrally closed and each t-linked overring of R is w-divisorial*;
(5) *R is a w-divisorial generalized Krull domain*;
(6) *R is a generalized Krull domain and each t-prime ideal of R is contained in a unique t-maximal ideal.*

Proof. (1) $\Rightarrow$ (2). Clearly R is integrally closed. In addition, by Lemma 3.4, R_P is a divisorial domain, for each $P \in t\text{-Spec}(R)$. Hence R is a strongly w-divisorial domain.

(2) $\Rightarrow$ (3). By Theorem 3.3, R is a *PvMD*; in particular $t\text{-Spec}(R)$ is treed. Thus we can apply Corollary 2.11.

(3) $\Rightarrow$ (1). By Theorem 3.3, R is a weakly Matlis *PvMD*. Now, given $P \in t\text{-Spec}(R)$, R_P is a divisorial valuation domain. Hence R is a strongly discrete *PvMD* by Lemma 3.4.

(3) $\Leftrightarrow$ (4). By Theorem 3.3, statements (3) and (4) imply that R is a *PvMD*. The conclusion now follows from the fact that each t-linked overring of a *PvMD* R is t-flat [23, Proposition 2.10].

(1) $\Rightarrow$ (5). By (1) $\Rightarrow$ (2), R is a w-divisorial domain. To show that R is a generalized Krull domain, let I be a t-ideal of R. Since R has t-finite character, then I is contained in only finitely many t-maximal ideals. Furthermore, each t-prime ideal is contained in

a unique t-maximal ideal. Thus I has just finitely many minimal (t-)prime ideals. We conclude by using [7, Theorem 3.9].

(5) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (1). It is enough to show that R has t-finite character. This follows from the fact that each nonzero principal ideal has finitely many minimal (t-)primes. □

As a consequence of Theorem 3.5, we obtain the following characterization of integrally closed totally divisorial domains (see also [28]).

Corollary 3.6. *Let R be an integral domain. The following conditions are equivalent*:

(1) *R is an integrally closed totally divisorial domain*;
(2) *R is integrally closed and each flat overring of R is divisorial*;
(3) *R is an integrally closed strongly divisorial domain*;
(4) *R is an h-local strongly discrete Prüfer domain*;
(5) *R is a divisorial generalized Dedekind domain*;
(6) *R is a generalized Dedekind domain and each nonzero prime ideal is contained in a unique maximal ideal.*

Proof. This follows from the fact that in a Prüfer domain the d- and t-operation coincide, that each overring of a Prüfer domain is a flat Prüfer domain, and that a Prüfer domain is a generalized Krull domain if and only if it is a generalized Dedekind domain [7]. □

Recall that the *complete integral closure* of R is the overring $\widetilde{R} := \bigcup\{(I : I);\ I$ nonzero ideal of $R\}$. If $R = \widetilde{R}$, we say that R is *completely integrally closed*.

Proposition 3.7. *Let R be an integral domain. The following conditions are equivalent*:

(1) *R is an integrally closed w-divisorial domain of t-dimension one*;
(2) *R is an integrally closed domain of t-dimension one and each t-linked overring of R is w-divisorial*;
(3) *R is a completely integrally closed w-divisorial domain*;
(4) *R is a Krull domain.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4). Clearly a w-divisorial domain of t-dimension one is strongly w-divisorial. Since a generalized Krull domain of t-dimension one is a Krull domain [7, Theorem 3.11], we can conclude by applying Theorem 3.5.

(3) $\Leftrightarrow$ (4) because a completely integrally closed *TV*-domain is Krull [20, Theorem 2.3]. □

It is well known that a divisorial Krull domain is a Dedekind domain; hence by the previous proposition we recover that a completely integrally closed divisorial domain is a Dedekind domain [17, Proposition 5.5].

Remark 3.8. Recall that, for any domain R, $\widetilde{R}$ is integrally closed and t-linked over R [5, Corollary 2.3]. Since each localization of a t-linked overring of R is still t-linked over R, if each t-linked overring of R is w-divisorial, we have that $\widetilde{R}$ is an integrally closed strongly w-divisorial domain. In this case, by Theorem 3.5, $\widetilde{R}$ is a weakly Matlis strongly discrete *PvMD*. If in addition $\widetilde{R}$ is completely integrally closed, for example if $(R : \widetilde{R}) \neq 0$, by Proposition 3.7 $\widetilde{R}$ is a Krull domain.

In a similar way, by using Corollary 3.6, we see that if R is totally divisorial, the integral closure of R is an h-local strongly divisorial Prüfer domain.

4. Mori w-divisorial domains

We start by recalling some properties of Noetherian divisorial domains proved in [17, 31].

Proposition 4.1. *Let R be a domain. The following conditions are equivalent*:

(1) *R is a one-dimensional w-divisorial Mori domain*;
(2) *R is a divisorial Mori domain*;
(3) *R is a divisorial Noetherian domain*;
(4) *R is a Mori domain and each two generated ideal of R is divisorial*;
(5) *R is a one-dimensional Mori domain and $(R : M)$ is a two generated ideal, for each $M \in \mathrm{Max}(R)$*;
(6) *R is a one-dimensional Noetherian domain and $(R : M)$ is a two generated ideal, for each $M \in \mathrm{Max}(R)$.*

Proof. (1) $\Rightarrow$ (2) by Proposition 1.1.

(2) $\Rightarrow$ (3) because each v-ideal of a Mori domain is v-finite.

(3) $\Rightarrow$ (1) because Noetherian divisorial domains are one-dimensional [17, Corollary 4.3].

(3) $\Leftrightarrow$ (6) and (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) by [31, Theorem 3, Section 2]. □

An integrally closed w-divisorial Mori domain is a Krull domain. In fact it has to be a *PvMD* (Theorem 3.3). By Proposition 4.1, any Noetherian integrally closed domain of dimension greater than one is a w-divisorial Noetherian domain that is not divisorial.

We say that a nonzero fractional ideal I of R is a *w-divisorial ideal* if $I_v = I_w$. With this notation, a w-divisorial domain is a domain in which each nonzero ideal is w-divisorial. We also say that, for $n \geqslant 1$, I is n w-generated if $I_w = (a_1R + \cdots + a_nR)_w$, for some $a_1, \ldots, a_n$ in the quotient field of R.

Theorem 4.2. *Let R be a Mori domain. The following conditions are equivalent*:

(1) *R is a w-divisorial domain*;
(2) *Each two generated nonzero ideal is w-divisorial*;

(3) *R has t-dimension one and $(R:M)$ is a two w-generated ideal, for each $M \in$ t-*Max(R).

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Let $M \in t$-Max(R). Since R is a Mori domain, then M is a divisorial ideal. Let $x \in (R:M) \setminus R$, then $(R:M) = (R + Rx)_v$. So that by assumption $(R:M) = (R+Rx)_w$. To conclude, we show that R_M is one-dimensional. Let I be a nonzero two generated ideal of R_M. Then, we can assume that $I = (a,b)R_M$ for some $a, b \in I \cap R$. Since R is a Mori domain, then $I_v = ((a,b)R_M)_v = (a,b)_v R_M$. Hence $I_v = (a,b)_w R_M = (a,b)R_M = I$. Thus each two generated ideal of R_M is divisorial. It follows from Proposition 4.1 that R_M is one-dimensional.

(3) $\Rightarrow$ (1). Since R is a *TV*-domain, by Theorem 1.5, it is enough to show that R_M is a divisorial domain for each $M \in t$-Max(R). This follows again from Proposition 4.1. In fact, by assumption R_M is a Mori domain of dimension one. Let $(R:M) = (a,b)_w$ for some $a, b \in (R:M)$. Then $(R_M : MR_M) = (R:M)R_M = (a,b)_w R_M = (a,b)R_M$ is two generated (the first equality holds because M is v-finite). □

Recall that a *Strong Mori domain* is a domain satisfying the ascending chain condition on w-ideals. A domain R is a Strong Mori domain if and only if it has t-finite character and R_M is Noetherian, for each t-maximal ideal M [33, Theorem 1.9]. Thus a Mori domain is Strong Mori if and only if R_M is Noetherian, for each t-maximal ideal M.

Corollary 4.3 [27, Corollary 2.5]. *A w-divisorial Mori domain is a Strong Mori domain of t-dimension one.*

Proof. A w-divisorial Mori domain is Strong Mori (because $w = v$) and has t-dimension one by Theorem 4.2. □

We next investigate w-divisoriality of overrings of Mori domains. Our first result in this direction shows that, if R is Mori, w-divisoriality is inherited by generalized ring of fractions. This improves [27, Theorem 2.4].

We observe that a Mori domain is a v-coherent *TV*-domain, because each t-ideal of a Mori domain is v-finite. We also recall that if R is v-coherent, we have $I_t R_S = (IR_S)_t$, for each nonzero fractional ideal I and each multiplicative set S.

Proposition 4.4. *Let R be a v-coherent domain. The following conditions are equivalent*:

(1) *R is a TW-domain*;
(2) *All the nonzero ideals of R_M are t-ideals, for each $M \in t$-*Max(R);
(3) *All the nonzero ideals of R_P are t-ideals, for each $P \in t$-*Spec(R);
(4) *Each t-flat overring of R is a TW-domain.*

Proof. (1) $\Leftrightarrow$ (2). Let I be a nonzero ideal and M a t-maximal ideal of R. If $t = w$ on R, then $IR_M = I_w R_M = I_t R_M = (IR_M)_t$.

Conversely, we have $IR_M = (IR_M)_t = I_t R_M$. Thus

$$I_w = \bigcap_{M \in t\text{-Max}(R)} IR_M = \bigcap_{M \in t\text{-Max}(R)} I_t R_M = I_t.$$

$(2) \Rightarrow (3)$. Let I be a nonzero ideal of R, P a t-prime of R and M a t-maximal ideal containing P. Then

$$IR_P = (IR_M)R_P = (IR_M)_t R_P = (I_t R_M)R_P = I_t R_P = (IR_P)_t.$$

$(3) \Rightarrow (4)$. Let T be a t-flat overring of R. Then T is a v-coherent domain [10, Proposition 3.1]. If N is a t-maximal ideal of T, then $P = N \cap R$ is a t-prime of R and $T_N = R_P$. Hence, if (3) holds, each nonzero ideal of T_N is a t-ideal and T is a *TW*-domain by $(2) \Rightarrow (1)$.

$(4) \Rightarrow (1)$ is clear. □

Theorem 4.5. *Let R be a Mori domain. The following conditions are equivalent*:

(1) *R is w-divisorial*;
(2) *R is strongly w-divisorial*;
(3) *Each t-flat overring of R is w-divisorial*;
(4) *Each generalized ring of fractions of R is w-divisorial*;
(5) *R_M is a divisorial domain, for each $M \in t$-*Max(R).

Proof. Each generalized ring of fractions of a Mori domain is Mori [31, Corollaire 1, Section 3]; thus it is a *TV*-domain. In addition, each generalized ring of fractions of a Mori domain is t-flat, because each t-ideal is v-finite and so each multiplicative system of ideals is v-finite. Hence we can apply Proposition 4.4. □

t-Linked overrings of Mori domains do not behave as well as generalized rings of fractions. In fact a Mori non-Krull domain has t-linked overrings which are not t-flat [6, Corollary 2.10]. Also, if each t-linked overring of a Mori domain R is Mori, then R has t-dimension one [5, Proposition 2.20]. The converse holds if R is a Strong Mori domain; precisely, we have the following result.

Proposition 4.6. *Each t-linked overring of a Strong Mori domain of t-dimension one is either a field or a Strong Mori domain of t-dimension one.*

Proof. It follows from [33, Theorem 3.4] recalling that an overring of a domain is a w-module if and only if it is t-linked [5, Proposition 2.13(a)]. □

Corollary 4.7. *If R is a w-divisorial Mori domain, then each t-linked overring of R is either a field or a Strong Mori domain of t-dimension one.*

Proof. It follows from Corollary 4.3 and Proposition 4.6. □

Our next purpose is to improve and generalize to Mori domains some results proved in [3] for Noetherian totally divisorial domains.

Proposition 4.8. *Let R be a domain. The following conditions are equivalent*:

(1) *R is a one-dimensional domain and each t-linked overring of R is w-divisorial*;
(2) *R is a one-dimensional totally divisorial domain*;
(3) *R is a Noetherian totally divisorial domain*;
(4) *Each ideal of R is two generated.*

Proof. (1) $\Rightarrow$ (2). Since $\dim(R) = 1$, each overring of R is t-linked over R [5, Corollary 2.7(b)]. Hence each overring T of R is w-divisorial. Assume that T is not a field. To prove that T is divisorial it suffices to check that $\dim(T) = 1$ (Proposition 1.1). Let R' be the integral closure of R and T' that of T. Since R' is one-dimensional and w-divisorial, then R' is divisorial. Thus R', being integrally closed, is a Prüfer domain [17, Theorem 5.1]. It follows that the extension $R' \subseteq T'$ is flat, and so $\dim(T') \leqslant \dim(R') = 1$. Hence $\dim(T) = \dim(T') = 1$. We conclude that T is divisorial and therefore R is totally divisorial.

(2) $\Rightarrow$ (3) by [3, Proposition 7.1].

(3) $\Rightarrow$ (1) by Proposition 4.1.

(3) $\Leftrightarrow$ (4) by [3, Theorem 7.3], because in the Noetherian case a domain is totally divisorial if and only if it is totally reflexive [29, Section 3]. □

Lemma 4.10 below is similar to [26, Theorem 26(2)]. We will need the following version of Chinese Remainder Theorem, whose proof is straightforward.

Lemma 4.9. *Let R be an integral domain, I an ideal of R, $P_1, \dots, P_n$ a set of pairwise incomparable prime ideals and $S = R \setminus (P_1 \cup \cdots \cup P_n)$. If $x_1, \dots, x_n \in I$, there exists $x \in IR_S$ such that $x \equiv x_i \pmod{IP_iR_{P_i}}$, for each $i = 1, \dots, n$.*

Lemma 4.10. *Let R be an integral domain which has t-finite character and I a nonzero ideal of R. Let n be a positive integer and assume that, for each $M \in t\text{-Max}(R)$, a minimal set of generators of IR_M has at most n elements. Then I is w-generated by a number of generators $m \leqslant \max(2, n)$.*

Proof. If I is not contained in any t-maximal ideal, then $I_w = R$. Otherwise, let $M_1, \dots, M_r$ be the t-maximal ideals of R which contain I. For $i = 1, \dots, r$, let $a_{1i}, \dots, a_{ni} \in I$ be such that $IR_{M_i} = (a_{1i}, \dots, a_{ni})R_{M_i}$. By Lemma 4.9, if $S = R \setminus (M_1 \cup \cdots \cup M_r)$, for each $j = 1, \dots, n$, there exists $a_j \in IR_S \subseteq IR_{M_i}$ such that $a_j \equiv a_{ji} \pmod{IM_iR_{M_i}}$, for each $i = 1, \dots, r$. By going modulo $IM_iR_{M_i}$ and using Nakayama's Lemma, we get $IR_{M_i} = (a_1, \dots, a_n)R_{M_i}$ for each $i = 1, \dots, r$. We can assume that the a_j's are in I and $a_1 \neq 0$. Let $N_1, \dots, N_s$ be the set of t-maximal ideals which contain a_1, with $N_1 = M_1, \dots, N_r = M_r$. Let $b \in I \setminus \bigcup_{j=r+1}^{s} M_j$. Then $IR_{N_j} = (a_1, \dots, a_n)R_{N_j}$ for $j = 1, \dots, r$ and $IR_{N_j} = (a_1, b)R_{N_j} = R_{N_j}$ for $j = r+1, \dots, s$. By arguing as above, there exist $b_1 = a_1, b_2, \dots, b_n \in I$ such that $IR_{N_j} = (b_1, \dots, b_n)R_{N_j}$ for each $j = 1, \ldots, s$. We claim that $I_w = (b_1, \ldots, b_n)_w$. Let M be a t-maximal ideal of R. If $M = N_j$ for some j, then $IR_M = (b_1, \ldots, b_n)R_M$. If $M \neq N_j$ for $j = 1, \ldots, s$, then $IR_M = R_M = (b_1, \ldots, b_n)R_M$, since $b_1 = a_1 \notin M$. □

Theorem 4.11. *Let R be a domain. The following conditions are equivalent*:

(1) *R has t-dimension one and each t-linked overring of R is w-divisorial*;
(2) *R is a Mori domain and each t-linked overring of R is w-divisorial*;
(3) *R is a Mori domain and R_M is totally divisorial, for each $M \in t\text{-Max}(R)$*;
(4) *Each nonzero ideal of R is a two w-generated w-divisorial ideal*;
(5) *Each nonzero ideal of R is two w-generated.*

Proof. (1) ⇒ (2). R has t-finite character, because it is w-divisorial (Theorem 1.5). We now show that, for each $M \in t\text{-Max}(R)$, R_M is Noetherian. Since R_M is a one-dimensional t-linked overring of R, then R_M is divisorial (Proposition 1.1). In addition, each overring T of R_M is t-linked over R_M [5, Corollary 2.7] and so it is t-linked over R. Thus T is a w-divisorial domain. By Proposition 4.8, R_M is Noetherian. We conclude that R is a (Strong) Mori domain.

(2) ⇒ (3). R is clearly w-divisorial. Hence R_M is a one-dimensional Noetherian domain (Corollary 4.3). Let T be a t-linked overring of R_M. Hence T is t-linked over R and so by assumption it is w-divisorial. By Proposition 4.8 R_M is totally divisorial.

(3) ⇒ (4). R is w-divisorial by Theorem 4.5. Hence R_M is one-dimensional and Noetherian by Corollary 4.3. Thus, for each $M \in t\text{-Max}(R)$, each ideal of R_M is two generated by Proposition 4.8. By using Lemma 4.10, we conclude that every nonzero ideal of R is a two w-generated w-divisorial ideal.

(4) ⇒ (5) is clear.

(5) ⇒ (3). If (5) holds, R is a Strong Mori domain and so R_M is a Noetherian domain, for each $M \in t\text{-Max}(R)$. Let IR_M be a nonzero ideal of R_M, where I is an ideal of R. By assumption, $I_w = (a, b)_w$ for some $a, b \in R$. Thus $IR_M = (a, b)_w R_M = (a, b)R_M$ is a two generated ideal. It follows from Proposition 4.8 that R_M is a totally divisorial domain.

(3) ⇒ (2). R is w-divisorial by Theorem 4.5. Let T be a t-linked overring of R, $T \neq K$. By Corollary 4.7, T is a Mori domain. To show that T is w-divisorial, by Theorem 4.5, we have to prove that T_N is a divisorial domain, for each $N \in t\text{-Max}(T)$. Since $R \subseteq T$ is t-linked, then $Q = (N \cap R)_t \neq R$ [5, Proposition 2.1]; but as R has t-dimension one (Corollary 4.3), then Q is a t-maximal ideal of R. Since R_Q is totally divisorial and $R_Q \subseteq T_N$, then T_N is a divisorial domain.

(2) ⇒ (1) by Corollary 4.3. □

Corollary 4.12. *Let R be a domain and assume that each t-linked overring of R is w-divisorial. Then R is a Mori domain if and only if it has t-dimension one.*

Example 4.13. Mori non-Krull and non-Noetherian domains satisfying the equivalent conditions of Theorem 4.11 can be constructed by using pullbacks, as the following example shows.

Let T be a Krull domain having a maximal ideal M of height one and assume that the residue field $K = T/M$ has a subfield k such that $[K : k] = 2$. Let $R = \varphi^{-1}(k)$ be the pullback of k with respect to the canonical projection $\varphi : T \to K$.

The domain R is Mori and it is Noetherian if and only if T is Noetherian [11, Theorems 4.12 and 4.18]. M is a maximal ideal of R that is divisorial; thus $M \in t\text{-Max}(R)$. Since R_M is the pullback of k with respect to the natural projection $T_M \to K$, R_M is divisorial by [27, Corollary 3.5]. In addition T_M is the only overring of R_M. In fact each overring of R_M is comparable with T_M under inclusion; but T_M is a *DVR* and $[K : k] = 2$. Thus R_M is totally divisorial.

If N is a t-maximal ideal of R and $N \neq M$, there is a unique t-maximal ideal N' of T such that $N' \cap R = N$ [12, Theorem 2.6(1)] and for this prime $T_{N'} = R_N$. Thus R_N is a *DVR*. It follows that R_N is totally divisorial, for each $N \in t\text{-Max}(R)$.

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