A Cell Decomposition of the Space of Real Hankel Matrices of Rank $\leq n$ and Some Applications

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ABSTRACT

We construct a cellular decomposition of the space of all infinite real Hankel matrices of rank $\leq n$. It induces cell decompositions on each manifold $\mathcal{H}(m)$ of Hankel matrices of fixed rank $m < n$. The relationship between the cellular decomposition of $\mathcal{H}(n)$ and the decomposition of Rat($n$) into continued fraction cells is determined. Finally, the cell decomposition is applied to derive some topological properties of the manifolds $\mathcal{H}(n)$ and Rat($n$).

1. INTRODUCTION

In one of the first topological studies in linear systems theory Brockett (1976) has shown that the space Rat($n$) of real rational transfer functions of degree $n$ consists of $n + 1$ connected components Rat$_k(n)$, $k = 0, \ldots, n$, classified by the Cauchy index $n - 2k$. These connected components are analytical manifolds of dimension $2n$. One knows that Rat$_0(n)$ and Rat$_n(n)$ are cells and that Rat$_1(n)$, Rat$_{n-1}(n)$ are diffeomorphic to $S^1 \times \mathbb{R}^{2n-1}$. However, the other components are more complicated, and so it seems
reasonable to decompose $\text{Rat}(n)$ even further, into simpler pieces. In this paper, we investigate cell decompositions of $\text{Rat}(n)$ and associated Hankel spaces.

The decomposition of spaces into cells is a well-established technique in topology [cf. e.g. Whitehead (1949), Massey (1978)] and has recently been successfully applied to orbit manifolds of linear systems; see Helmke (1986).

For a survey about cell decompositions in linear systems theory, their role in the parametrization of system spaces, and their relationship to canonical forms, see Helmke and Hinrichsen (1986) and Hinrichsen (1985).

Fuhrmann and Krishnaprasad (1986) were the first to propose a decomposition of $\text{Rat}(n)$ into cells defined by continued fraction expansion. They conjectured that this decomposition was indeed a cellular decomposition in a strong topological sense, but failed to give a proof.

In the present paper we proceed from a Hankel point of view. We consider the space $\mathcal{H}(n)$ of all infinite real Hankel matrices of rank $\leq n$. This space is of special interest in areas like partial realization and model reduction, where systems of different orders have to be regarded as elements of the same space. Our aim is to construct a cell decomposition of $\mathcal{H}(n)$ which induces cellular decompositions of each analytical manifold $\mathcal{H}(m) \cong \text{Rat}(m)$ of Hankel matrices of rank $m \leq n$.

The construction is based on the Bruhat decomposition of finite Hankel matrices introduced in Hinrichsen et al. (1986). The main results about the Bruhat decomposition of $\mathcal{H}(n)$ (Sections 2 to 4) have been described without proof in Hinrichsen and Manthey (1986). It is remarkable that this cellular subdivision has a close relationship to the cell decomposition of $\text{Rat}(n)$ suggested by Fuhrmann and Krishnaprasad. Since essentially the same cellular decomposition arises in apparently quite different contexts, it seems to be intimately related to the topology of the homeomorphic manifolds $\text{Rat}(n) \cong \mathcal{H}(n)$.

The organization of the paper is as follows. In Section 2 the Bruhat decomposition of $\mathcal{H}(n)$ is defined. In Section 3 the Bruhat strata are parametrized. Moreover, it is shown that their connected components yield a cellular subdivision of $\mathcal{H}(n) = \bigcup_{0 \leq m \leq n} \mathcal{H}(m)$ with the desired properties. In Section 4 the relationship between the Bruhat cells and the continued fraction cells is made precise, and the conjecture of Fuhrmann and Krishnaprasad is partly confirmed. The Bruhat cells of Hankel matrices with given rank and signature are characterized. Finally, in Section 5 we illustrate how various topological properties of $\mathcal{H}(n)$ and of $\text{Rat}(n)$ can be derived by means of the constructed cell decompositions. A concise new proof of Brockett's theorem is presented, a conjectured combinatorial formula of Fuhrmann and Krishnaprasad is proved, and some of the mod 2 Betti numbers of the connected components $\text{Rat}_k(n)$ are determined.
2. BRUHAT DECOMPOSITION OF HANKEL MATRICES

Let $\mathbb{N}$ be the set of nonzero natural numbers and $\mathcal{H} \cong \mathbb{R}^\mathbb{N}$ denote the vector space of all infinite real Hankel matrices provided with the pointwise topology (or, equivalently, the product topology), and let

$$\mathcal{H}(n) := \{ H \in \mathcal{H} ; \text{rk } H = n \}, \quad n \in \mathbb{N}.$$ 

$\mathcal{H}(n)$ is an analytical manifold of dimension $2n$ and analytically diffeomorphic to the space $\text{Rat}(n)$ of all strictly proper transfer functions of degree $n$ (see e.g. Byrnes and Duncan, 1981).

The union

$$\mathcal{H}(n) = \bigcup_{0 \leq m \leq n} \mathcal{H}(m)$$

can be viewed as the space of all real rational transfer functions of degree $\leq n$. $\mathcal{H}(n)$ is a closed, hence complete subspace of $\mathcal{H}$ and coincides with the topological closure of $\mathcal{H}(n)$ in $\mathcal{H}$; see Helmke and Manthey (1984).

In this section, we will construct the Bruhat decomposition of $\mathcal{H}(n)$ which induces cell decompositions in each of the analytical manifolds $\mathcal{H}(m)$, $m \leq n$. For any $H = (h_{i+j-1})^c_{1} \in \mathcal{H}$ and $k \geq 1$ we denote by $H_k$ the $k \times k$ Hankel matrix $(h_{i+j-1})_k^c$. For any subset $\mu \subseteq n = \{1, \ldots, n\}$, consider

$$\mathcal{H}(\mu) := \{ H \in \mathcal{H} ; \det H_i \neq 0 \iff i \in \mu \}. \quad (2.1)$$

We call $\mathcal{H}(\mu)$ a Bruhat stratum of $\mathcal{H}(n)$. Clearly,

$$\mathcal{H}(n) = \bigcup_{\mu \subseteq \mathbb{N}} \mathcal{H}(\mu) \quad (2.2)$$

is a finite partition of $\mathcal{H}(n)$ into disjoint subsets which induces a decomposition

$$\mathcal{H}(m) = \bigcup_{\mu \subseteq m} \mathcal{H}(\mu) \quad (2.3)$$

of the Hankel manifold $\mathcal{H}(m)$ for each $0 \leq m \leq n$. The decompositions (2.2) and (2.3) will be called the Bruhat decomposition of $\mathcal{H}(n)$ and of $\mathcal{H}(m)$, respectively, and the elements $\mu_1 < \mu_2 < \cdots < \mu_r$ of $\mu \subseteq \mathbb{N}$ are called the
Bruhat indices of $H \in \mathcal{H}(\mu)$. The system theoretic meaning of the Bruhat indices is most easily explained in the context of partial realizations. Consider the underlying sequence $h = (h_1, h_2, h_3, \ldots)$ of $H \in \mathcal{H}(\mu)$. For any $k \in \mathbb{N}$ let $d_k$ denote the dimension of a minimal realization of the subsequence $h^k := (h_1, \ldots, h_k)$, $d_0 := 0$. Then the integers $t_i \in \mathbb{N}$ for which $d_i \neq d_{i-1}$ are called the jump points of $h$, and the differences $q_i = d_i - d_{i-1}$, $t_0 := 0$ the associated jump sizes (Kalman, 1979). Jump points and jump sizes are determined by the Bruhat indices $\mu_i \in \mu$ via

$$
t_i = \mu_i + \mu_{i-1}, \quad q_i = \mu_i - \mu_{i-1}, \quad \mu_0 := 0; \quad (2.4)
$$

see also Gragg and Lindquist (1983).

**Remark 2.1.** In Himichsen et al. (1986) an extended Bruhat decomposition of the set $\text{Hank}(n) \equiv \mathbb{R}^{2n-1}$ of $n \times n$ Hankel matrices was derived from the classical Bruhat decomposition of the general linear group $\text{GL}(n)$. $\mathcal{H}(n)$ is embedded homeomorphically as a constructible subset in $\text{Hank}(n+1)$ via the projection $H \rightarrow H_{n+1}$. The subdivision (2.2) of $\mathcal{H}(n)$ coincides with the partition induced by the Bruhat decomposition of $\text{Hank}(n+1)$, which explains our terminology.

3. PARAMETRIZATION OF THE BRUHAT STRATA

For any $m \geq 1$ and $H \in \mathcal{H}$ with $\det H_m \neq 0$ the following system of linear equations admits a unique solution $c_m = (c_{m,1}, \ldots, c_{m,m})^T \in \mathbb{R}^m$:

$$
c_{m,1}h_1 + \cdots + c_{m,m}h_m = h_{m+1},
$$

$$
\vdots
$$

$$
c_{m,1}h_m + \cdots + c_{m,m}h_{2m-1} = h_{2m}.
$$

(3.1)

By Cramer's rule, the components

$$
c_m, j = c_m, j(h_1, \ldots; h_{2m}), \quad j \in m, \quad m \in \mathbb{N},
$$

are rational functions of the data $(h_1, \ldots, h_{2m})$.

The following lemma is taken from Hinrichsen et al. (1986), we include the proof for completeness.
Lemma 3.1. Let \( m, v \in \mathbb{N} \) be given and \( H \in \mathbb{K} \) with \( \det H_m \neq 0 \). Then \( H \) satisfies

\[
\det H_{m+1} = \cdots = \det H_{m+v-1} = 0
\]  

if and only if

\[
h_{2m+j} = \sum_{k=1}^{m} c_{m,k} h_{m+k+j-1}, \quad j = 1, \ldots, v-1,
\]  

where the coefficients \( c_{m,k} \) are defined by (3.1). In this case

\[
\det H_{m+v} = (-1)^\binom{v}{2} (\det H_m) \left( h_{2m+v} - \sum_{k=1}^{m} c_{m,k} h_{m+k+v-1} \right)^v.
\]

Proof. By (3.2) and the Hankel structure of \( H_{m+v-1} \), the \((m+1)\)th row vector \((h_{m+1}, \ldots, h_{2m+v-1})\) of \( H_{m+v-1} \) is a linear combination of the first \( m \) row vectors \((h_k, \ldots, h_{m+k+v-2})\), \( k = 1, \ldots, m \), of \( H_{m+v-1} \), with uniquely determined coefficients defined by (3.1). This shows (3.3). The reverse direction is obvious. To prove (3.4) let \( H(k) \) denote the \( k \)th column of \( H_{m+v} \). By subtracting \( \sum_{k=1}^{m} c_{m,k} H(i+k-1) \) from \( H(m+i) \), for \( i = 1, \ldots, v \), \( H_{m+v} \) is transformed into a block triangular matrix

\[
\tilde{H} = \begin{bmatrix}
    h_1 & \cdots & h_m & 0 & \cdots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    h_m & \cdots & h_{2m-1} & 0 & \cdots & 0 \\
    h_{m+1} & \cdots & h_{2m} & 0 & \cdots & \tilde{h}_{2m+v} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    h_{m+v} & \cdots & h_{2m+v-1} & 0 & \cdots & \tilde{h}_{2m+2v-2} \\
    \tilde{h}_{2m+v} & \cdots & \tilde{h}_{2m+2v-2} & \tilde{h}_{2m+2v-1}
\end{bmatrix}
\]

with

\[
\tilde{h}_{2m+v} = h_{2m+v} - \sum_{k=1}^{m} c_{m,k} h_{m+k+v-1}.
\]
Thus

\[ \det H_{m+r} = \det H = (-1)^{\frac{n(n-1)}{2}} (\det H_m)^{\binom{r}{2}}. \]

**Remark 3.2.** Lemma 3.1 also holds for arbitrary finite \( n \times n \) Hankel matrices \( H \).

Now suppose \( \mu = \{ \mu_1, \ldots, \mu_r \} \subset \mathbb{N}, \mu_1 < \cdots < \mu_r, \nu_i = \mu_{i+1} - \mu_i, 0 \leq i \leq r-1 \), \( \mu_0 = 0, \nu_r := \infty \). Applying Lemma 3.1 successively for \( m = \mu_i, \nu = \nu_i, i = 1, \ldots, r \), one obtains the following characterization of the Bruhat strata \( \mathcal{H}(\mu) \): \( H \in \mathcal{H}(\mu) \) if and only if

\[ h_1 = \cdots = h_{\mu_1} = 0 \quad (3.5a) \]

and

\[ d_i := h_{\mu_i+1} - \sum_{k=1}^{\mu_i-1} c_{\mu_i-k, h_{\mu_i+k}} \neq 0, \]

\[ h_{2\mu_i+j} = \sum_{k=1}^{\mu_i} c_{\mu_i-k, h_{\mu_i+k+j-1}}, \quad j = 1, \ldots, \nu_i-1, \quad i = 1, \ldots, r, \quad (3.5b) \]

where the \( c_{i,j} \) are defined by (3.1).

Thus the mapping

\[ \Phi^\mu: H \to (d_1, \ldots, d_r; h_{\mu_1+1}, \ldots, h_{2\mu_1}; \ldots; h_{\mu_{r-1}+\mu_r+1}, \ldots, h_{2\mu_r}) \quad (3.6) \]

is a continuous rational bijection of \( \mathcal{H}(\mu) \) onto \( (\mathbb{R}^*)^r \times \mathbb{R}^{\mu_r} \) with a continuous rational inverse, \( \mathbb{R}^* := \mathbb{R} \setminus \{0\} \). This shows:

**Theorem 3.3.** \( \Phi^\mu: \mathcal{H}(\mu) \to (\mathbb{R}^*)^r \times \mathbb{R}^{\mu_r}, \) defined by (3.5), (3.6), is a birational homeomorphism. Moreover, for any \( H \in \mathcal{H}(\mu) \) and all \( k = 1, \ldots, r \),

\[ \det H_{\mu_k} = (-1)^{\binom{\nu_k}{2}} \cdots \binom{\nu_{k-1}}{2} d_1^{\nu_0} \cdots d_k^{\nu_{k-1}}. \quad (3.7) \]
Corollary 3.4. For every \( \mu = \{\mu_1, \ldots, \mu_r\} \subset \mathbb{n} \), the Bruhat stratum \( \mathcal{H}(\mu) \) is an analytical manifold of pure dimension \( r + \mu_r \). \( \mathcal{H}(\mu) \) has \( 2^r \) connected components, each of which is a cell, i.e. is homeomorphic to some \( \mathbb{R}^k \), \( k \in \mathbb{N} \).

Altogether the Bruhat decomposition (2.2) of \( \overline{\mathcal{H}}(n) \) contains exactly

\[
\sum_{r=0}^{n} \binom{n}{r} 2^r = 3^n
\]
cells. To specify them, let \( \Phi^\mu_i \) denote the \( i \)th component mapping of the rational bijection \( \Phi^\mu \), defined by (3.5), (3.6):

\[
\Phi^\mu_i(H) = d_i, \quad i = 1, \ldots, r.
\]

Let \( \mu = \{\mu_1, \ldots, \mu_r\} \subset \mathbb{n} \), \( \mu_1 < \cdots < \mu_r \), and \( \sigma = (\sigma_1, \ldots, \sigma_r) \in \{-1, 1\}^r \) be given. The subset

\[
\mathcal{H}(\mu, \sigma) := \{ \mathcal{H} \in \mathcal{H}(\mu); \text{sign} \Phi^\mu_i(H) = \sigma_i, i = 1, \ldots, r \}, \quad (3.8)
\]

where

\[
\text{sign} \Phi^\mu_i(H) = \begin{cases} 
1 & \text{if } d_i = \Phi^\mu_i(H) > 0, \\
-1 & \text{if } d_i < 0
\end{cases} \quad (3.9)
\]
is called the Bruhat cell with symbol \( (\mu, \sigma) \). By Theorem 3.3 each \( \mathcal{H}(\mu, \sigma) \) is a cell of dimension \( r + \mu_r \). More precisely, the map \( \Phi^\mu \) restricts to a homeomorphism of the cell \( \mathcal{H}(\mu, \sigma) \) onto

\[
\Phi^\mu(\mathcal{H}(\mu, \sigma)) = \prod_{i=1}^{r} \mathbb{R}_{\sigma_i} \times \mathbb{R}^{\mu_r}, \quad (3.10)
\]
where \( \mathbb{R}_{\sigma_i} = (0, \infty) \) for \( \sigma_i = 1 \), \( \mathbb{R}_{\sigma_i} = (-\infty, 0) \) for \( \sigma_i = -1 \). Given integers \( m, n \geq 1 \) let

\[
\mathcal{P}(n) = \{(\mu, \sigma); \mu \subset \mathbb{n}, \sigma \in \{-1, 1\}^{\lfloor n \rfloor}\} \quad (3.11)
\]
and

$$\mathcal{D}(m) = \left\{ (\mu, \sigma); \mu \subseteq m, \max \mu = m, \sigma \in \{-1, 1\}^{\lvert \mu \rvert} \right\}, \quad (3.12)$$

where $\lvert \mu \rvert$ is the cardinality of the set $\mu$.

**Definition 3.5.** Let $X$ be a locally compact topological space. A finite decomposition $(X_i | i \in I)$ of $X$ into disjoint subsets is called a cell decomposition provided one has

(a) each $X_i$ is homeomorphic to some $\mathbb{R}^n$, $n_i \in \mathbb{N}$;
(b) the boundary $\partial X_i = \overline{X}_i - X_i$ of $X_i$ is contained in the union of the cells $X_j$ with $\dim X_j < \dim X_i$.

**Theorem 3.6.**

(a) For every $n \geq 1$, the family $(\mathcal{H}(\mu, \sigma) | (\mu, \sigma) \in \mathcal{D}(n))$ of Bruhat cells is a cell decomposition of $\mathcal{H}(n)$.

(b) For each $m \leq n$, $(\mathcal{H}(\mu, \sigma) | (\mu, \sigma) \in \mathcal{D}(m))$ is a cell decomposition of $\mathcal{H}(m)$.

**Proof.** Since the minors map $H \mapsto \det H_k$ is continuous on $\mathcal{H}(n)$ for $k = 1, \ldots, n$, it is clear that

$$\partial \mathcal{H}(\mu) \subseteq \bigcup_{\lambda \subsetneq \mu} \mathcal{H}(\lambda), \quad \mu \subset n.$$ 

By Corollary 3.4, $\dim \mathcal{H}(\lambda) < \dim \mathcal{H}(\mu)$ for all $\lambda \subsetneq \mu$. The result follows. ■

We refer to $(\mathcal{H}(\mu, \sigma) | (\mu, \sigma) \in \mathcal{D}(m))$ as the Bruhat cell decomposition of $\mathcal{H}(m)$. Note that the number of Bruhat cells $\mathcal{H}(\mu, \sigma)$ of $\mathcal{H}(m)$ of dimension $m + r$ is equal to $\binom{m - 1}{r - 1} 2^r$. Hence $(\mathcal{H}(\mu, \sigma) | (\mu, \sigma) \in \mathcal{D}(m))$ contains

$$\sum_{r=1}^{m} \binom{m - 1}{r - 1} 2^r = 2 \times 3^{m-1}$$

Bruhat cells.

**Remark 3.7.** A basic problem is to characterize the boundary of a given Bruhat stratum $\mathcal{H}(\mu)$ in the manifold $\mathcal{H}(n)$. This problem has been solved
For complex Hankel matrices in Hinrichsen et al. (1986), where it was shown that the inclusion in the above proof is in fact an equality:

$$\partial H(\mu) = \bigcup_{\lambda \subseteq \mu} H(\lambda).$$  \hspace{1cm} (3.13)

For some time it was an open question whether the formula also holds over the field of real numbers. That this is indeed the case was recently shown by Manthey; details will appear in his Ph.D. thesis.

4. CONTINUED FRACTION CELLS AND BRUHAT CELLS

Let $\text{Rat}(n)$ denote the set of all strictly proper transfer functions

$$g(s) = \frac{p(s)}{q(s)} = \frac{p_{n-1}s^{n-1} + \cdots + p_0}{s^n + q_{n-1}s^{n-1} + \cdots + q_0}$$

with real coefficients, $p$ and $q$ coprime. We consider $\text{Rat}(n)$ as the Zariski-open subset of $\mathbb{R}^{2n}$ defined by the $2n$ coefficients of the numerator and the denominator of $g(s)$. Thus $\text{Rat}(n)$ is a real algebraic manifold of dimension $2n$ and is endowed with the standard metric defined by

$$d(g, \tilde{g})^2 = \sum_{j=0}^{n-1} (p_j - \tilde{p}_j)^2 + (q_j - \tilde{q}_j)^2.$$  

Following Fuhrmann and Krishnaprasad (1986), we consider the continued fraction decomposition of $\text{Rat}(n)$, which is defined as follows.

Given a coprime factorization $g = p/q \in \text{Rat}(n)$, an application of the Euclidean algorithm gives

$$q = \frac{\alpha_1}{\lambda_1} p - \rho_2, \quad \deg \rho_2 < \deg p$$

$$p = \frac{\alpha_2}{\lambda_2} \rho_2 - \rho_3, \quad \deg \rho_3 < \deg \rho_2$$

$$\vdots$$

$$\rho_{r-1} = \frac{\alpha_r}{\lambda_r} \rho_r, \quad \deg \rho_r = 0, \quad \rho_r \neq 0.$$  \hspace{1cm} (4.1)
Here, for \( i = 1, \ldots, r \), \( \alpha_i \in \mathbb{R}[s] \) are monic polynomials and \( \lambda_i \) are uniquely determined nonzero scaling factors. The monic polynomials \( \alpha_i \) are called the atoms of \((p, q)\). With

\[
\beta_0 := \lambda_1 \\
\beta_j := \lambda_j \lambda_{j+1}, \quad j = 1, \ldots, r - 1,
\]

(4.1) is equivalent to the continued fraction expansion of \( g(s) \):

\[
g(s) = \frac{\beta_0}{\alpha_1(s) - \frac{\beta_1}{\alpha_2(s) - \frac{\beta_2}{\cdots - \frac{\beta_{r-1}}{\alpha_r(s)}}}}. \tag{4.2}
\]

Given any partition \( \nu = (\nu_1, \ldots, \nu_r) \in \mathbb{N}^r \), \( \nu_1 + \cdots + \nu_r = n \), of the integer \( n \), let \( \omega = (\omega_1, \ldots, \omega_r) \in \{-1, 1\}^r \) denote an arbitrary sign vector.

**Definition 4.1.**

(a) The continued fraction stratum \( \Gamma(\nu) \) is the set of all transfer functions \( g(s) \in \text{Rat}(n) \) whose continued fraction expansion (4.2) satisfies \( \deg \alpha_i = \nu_i \) for \( i = 1, \ldots, r \).

(b) The continued fraction cell \( \Gamma(\nu, \omega) \) is defined as the set of all \( g(s) \in \text{Rat}(n) \) satisfying \( \deg \alpha_i = \nu_i \), \( \text{sign} \beta_{i-1} = \omega_i \) for \( i = 1, \ldots, r \).

The corresponding partition of \( \text{Rat}(n) \) into disjoint subsets

\[
\text{Rat}(n) = \bigcup_{\nu} \Gamma(\nu) \tag{4.3}
\]

is called the continued fraction decomposition of \( \text{Rat}(n) \).

Fuhrmann and Krishnaprasad have noted that \( \Gamma(\nu, \omega) \) is a cell in \( \text{Rat}(n) \). More formally, we show:

**Proposition 4.2.** Let \( \nu = (\nu_1, \ldots, \nu_r) \in \mathbb{N}^r \), \( \nu_1 + \cdots + \nu_r = n \), \( \omega = (\omega_1, \ldots, \omega_r) \in \{-1, 1\}^r \) be given. Then

(a) \( \Gamma(\nu) \) is quasiaffine subvariety of \( \text{Rat}(n) \) of pure dimension \( n + r \).

(b) \( \Gamma(\nu) = \bigcup_{\omega \in \{-1, 1\}^r} \Gamma(\nu, \omega) \) decomposes into \( 2^r \) connected components, and each such component \( \Gamma(\nu, \omega) \) is a cell homeomorphic to \( \mathbb{R}^{n+r} \).
Proof. We proceed by induction on the number \( r \) of atoms. For \( r = 1 \),
\[ \Gamma(v) = \Gamma(v_1) \cong \mathbb{R}^n \times \mathbb{R}^* \]
is clearly quasiaffine and splits into two connected components, according to the sign of \( \lambda_1 = \beta_0 \). Let \( \Gamma_i(v_1) \) denote the set of all \( g(s) \in \text{Rat}(n) \) with the first atom \( \alpha_1 \) of degree \( \nu_1 \). Consider the algebraic morphism

\[ \varphi : \mathbb{R}^{\nu_1} \times \mathbb{R}^* \times \text{Rat}(n - \nu_1) \to \text{Rat}(n), \]

\[ \left( \alpha_1, \lambda, \frac{u}{v} \right) \to \frac{\lambda v}{\alpha_1 v - u}. \quad (4.4) \]

By (4.1), \( \varphi \) is an algebraic isomorphism of the product space \( \mathbb{R}^{\nu_1} \times \mathbb{R}^* \times \text{Rat}(n - \nu_1) \) onto the quasiaffine subvariety \( \Gamma_i(v_1) \) of \( \text{Rat}(n) \). Applying the induction argument, we suppose that \( \Gamma(v_1, \ldots, v_r) \subseteq \text{Rat}(n - \nu_1) \) is quasiaffine and homeomorphic to \( \mathbb{R}^{\nu_2 + \cdots + \nu_r} \times (\mathbb{R}^*)^{r-1} \). Using the algebraic isomorphism \( \varphi \), it follows that \( \Gamma(v_1, \ldots, v_r) = \varphi(\mathbb{R}^{\nu_1} \times \mathbb{R}^* \times \Gamma(v_2, \ldots, v_r)) \) is quasiaffine and homeomorphic to \( \mathbb{R}^{\nu_1} \times \mathbb{R}^* \times \Gamma(v_2, \ldots, v_r) \cong \mathbb{R}^n \times (\mathbb{R}^*)^r \). The result follows. \[ \blacksquare \]

Remark 4.3. For \( \nu_1 = 1 \), \( \alpha_1(s) = s \), \( \lambda_1 = 1 \), the algebraic map \( \varphi \) in (4.4) induces an algebraic imbedding

\[ i : \text{Rat}(n - 1) \to \text{Rat}(n), \]

\[ g(s) \to \frac{1}{s - g(s)}. \quad (4.5) \]

This mapping \( i \) respects the continued fraction decompositions of \( \text{Rat}(n - 1) \) and \( \text{Rat}(n) \):

\[ i(\Gamma(\tilde{\nu}_1, \ldots, \tilde{\nu}_r)) \subseteq \Gamma(1, \tilde{\nu}_1, \ldots, \tilde{\nu}_r) \quad (4.6a) \]

\[ i(\Gamma(\tilde{\nu}, \omega)) \subseteq \Gamma((1, \tilde{\nu}), (1, \omega)). \quad (4.6b) \]

The following theorem clarifies the relationship between the subdivision of \( \text{Rat}(n) \) into continued fraction cells and the Bruhat cell decomposition of \( \mathcal{H}(n) \). In a special case, with all \( \nu_i = 1 \), the theorem has also appeared in Byrnes and Lindquist (1982).
To state the theorem, let

$$L: \mathbb{R}(s) \to \mathcal{H}, \ g \to H(g)$$

be the Laurent map which associates with any rational function $g(s) = \sum_{j=-\infty}^{\infty} h_j s^{-j}$ the infinite Hankel matrix $H(g) = (h_{i+j-1})_{i,j}^\infty$. $L$ is linear and maps $\text{Rat}(n)$ onto $\mathcal{H}(n)$, by Kronecker’s theorem. It is known from realization theory (see e.g. Byrnes and Duncan, 1981) that the restriction $L_n = L | \text{Rat}(n)$ is a bianalytical homeomorphism.

**Theorem 4.4.** For any $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{N}^r$, $\nu_1 + \cdots + \nu_r = n$, $\omega = (\omega_1, \ldots, \omega_r) \in \{-1, 1\}^r$ define

$$\mu = \{\mu_1, \ldots, \mu_r\}, \quad \mu_j = \nu_1 + \cdots + \nu_j,$$

$$\sigma = (\sigma_1, \ldots, \sigma_r), \quad \sigma_j = \omega_1 \cdots \omega_j,$$

$j = 1, \ldots, r$. Then the Laurent map $L_n$ yields a bianalytical homeomorphism of the

(a) continued fraction stratum $\Gamma(\nu)$ onto the Bruhat stratum $\mathcal{H}(\mu)$;

(b) continued fraction cell $\Gamma(\nu, \omega)$ onto the Bruhat cell $\mathcal{H}(\mu, \sigma)$.

**Proof.** It suffices to prove $L_n(\Gamma(\nu, \omega)) = \mathcal{H}(\mu, \sigma)$, with $(\mu, \sigma)$ defined by (4.7). Consider $g(s) \in \Gamma(\nu, \omega)$ with continued fraction expansion (4.2). For $i = 1, \ldots, r$ we call

$$N_i \begin{array}{l} \beta_0 \\ \alpha_1 \end{array} \begin{array}{c} \beta_1 \\ \vdots \\ \beta_{i-1} \\ \alpha_i \end{array}$$

the $i$th approximant, and $N_i$ ($D_i$) the $i$th numerator (ith denominator) of the continued fraction expansion (4.2). Set

$$N_{-1} := 1, \quad N_0 := 0, \quad D_{-1} := 0, \quad D_0 := 1.$$
Then the recurrence relations
\[ N_i = \alpha_i N_{i-1} + \beta_{i-1} N_{i-2}, \]
\[ D_i = d_i D_{i-1} + \beta_{i-1} D_{i-2}, \quad i = 1, \ldots, r, \quad (4.9) \]
hold with \( \beta_0 := \beta_0, \quad \beta_j := -\beta_j, 1 \leq j \leq 1. \) Applying these formulas recursively for \( i, \ i-1, \ldots, 2, 1 \) one obtains
\[ N_i D_{i-1} - N_{i-1} D_i = \beta_0 \cdots \beta_{i-1} \quad \text{for} \quad i = 1, \ldots, r. \quad (4.10) \]
Since the right hand side of (4.10) is a nonzero constant, \( N_i \) and \( D_i \) are coprime and hence by (4.8)
\[ \frac{N_i}{D_i} \in \text{Rat}(\mu_i) \quad \text{for} \quad i = 1, \ldots, r. \quad (4.11) \]
Denote by
\[ H^{(i)} := \mathcal{L} \left( \frac{N_i}{D_i} \right), \quad i = 1, \ldots, r, \]
the corresponding Hankel matrix, \( H^{(i)} = (h_1^{(i)}, h_2^{(i)}, \ldots) \in \mathbb{R}^N. \) Since \( \mathcal{L} \) is a bijection of \( \text{Rat}(k) \) onto \( \mathcal{X}(k), \) (4.11) is equivalent to
\[ \det H^{(i)}_{\mu_i} \neq 0 \]
\[ \det H^{(i)}_j = 0 \quad \text{for} \quad j > \mu_i. \quad (4.12) \]
Moreover, since \( D_{i-1}D_i \) has degree \( \mu_{i-1} + \mu_i, \) we have by (4.10)
\[ H^{(i)} = \mathcal{L} \left( \frac{N_i}{D_i} \right) = \mathcal{L} \left( \frac{N_{i-1}}{D_{i-1}} \right) + \mathcal{L} \left( \frac{N_{i-1}}{D_{i-1}} - \frac{N_{i-1}}{D_{i-1}} \right) \]
\[ = H^{(i-1)} + \beta_0 \cdots \beta_{i-1} \mathcal{L} \left( \frac{1}{D_{i-1}D_i} \right) \]
\[ = \left( h_1^{(i-1)}, h_2^{(i-1)}, \ldots \right) + (0, \ldots, 0, \beta_0 \cdots \beta_{i-1}, \ldots). \]
where $\beta_0 \cdots \beta_{i-1}$ is the $(\mu_{i-1} + \mu_i)$th term of the sequence. Thus, for $i = 1, \ldots, r$,

$$h_j^{(i)} = h_j^{(i-1)} \quad \text{for} \quad 1 \leq j < \mu_{i-1} + \mu_i$$

(4.13)

Comparing (4.13) with (3.5), one obtains that

$$\sigma_i = \text{sign } d_i = \text{sign}(\beta_0 \cdots \beta_{i-1}) = \omega_1 \cdots \omega_i.$$

Proceeding by induction on $r$, it is seen that (4.12) and (4.13) imply $L(g) = H(g) \in H(\mu, \sigma)$, hence $L_n(\Gamma(v, \omega)) \subset H(\mu, \sigma)$. Since $L_n : \text{Rat}(n) \to H(n)$ is a bijection and the Bruhat cells $H(\mu, \sigma)$, $\mu \subset n$, $\sigma \in \{-1, 1\}^r$ are naturally disjoint, we conclude that $L_n(\Gamma(v, \omega)) = H(\mu, \sigma)$. Finally, it follows from Proposition 4.2(b) and the one-to-one correspondence between $\omega \in \{-1, 1\}^p$ and $\sigma \in \{-1, 1\}^r$ that

$$L_n(\Gamma(v)) = \bigcup_{\omega} L_n(\Gamma(v, \omega)) = H(\mu).$$

**Remark 4.5.** The relationship between continued fractions and Hankel matrices has been extensively studied, and parts of the above theorem can be found in the literature; compare e.g. Magnus (1962), Gragg (1974), Kalman (1979). A. Magnus (1962) derives the following useful formula for the $i$th approximant $N_i/D_i$ of $g(s) \in \text{Rat}(n)$ in terms of the associated Hankel matrix $H(g) = (h_{i+j-1})$:

$$N_i(s) = \frac{\det \begin{bmatrix} 0 & h_1 & h_1s + h_2 & \cdots & h_1s^{\mu_i-1} + \cdots + h_{\mu_i} \\ h_1 & h_2 & h_3 & \cdots & h_{\mu_i+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\mu_i} & h_{\mu_i+1} & h_{\mu_i+2} & \cdots & h_{2\mu_i} \end{bmatrix}}{\det \begin{bmatrix} 1 & s & s^2 & \cdots & s^{\mu_i} \\ h_1 & h_2 & h_3 & \cdots & h_{\mu_i+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\mu_i} & h_{\mu_i+1} & h_{\mu_i+2} & \cdots & h_{2\mu_i} \end{bmatrix}}.$$  

(4.14)
This equation yields an explicit formula for minimal partial realizations of an arbitrary Hankel matrix $H \in \mathcal{H}$.

As a consequence of Theorems 3.6, 4.4 we obtain the following partial confirmation of a conjecture of Fuhrmann and Krishnaprasad (1986).

**Corollary 4.6.** *The continued fraction cells form a cellular decomposition of $\text{Rat}(n)$.***

**Remark 4.7.** The concept of "cell decomposition" employed in Fuhrmann and Krishnaprasad (1986) requires that the cells $X_i$ satisfy additionally the frontier condition

$$X_i \cap \bar{X}_j \neq \emptyset \Rightarrow X_i \subset \bar{X}_j.$$

It is still an open problem whether the Bruhat cells or the continued fraction cells satisfy this condition.

We conclude this section with some comments concerning the relationship between the continued fraction cells and the Cauchy index. Let

$$\text{Rat}_k(n) = \{ g \in \text{Rat}(n); \text{CI}(g) = n - 2k \}, \quad k = 0, \ldots, n, \quad (4.15)$$

where $\text{CI}(g)$ is the Cauchy index of $g \in \text{Rat}(n)$; see Brockett (1976), Cantmacher (1959). Then

$$\text{Rat}(n) = \bigcup_{k=0}^{n} \text{Rat}_k(n). \quad (4.16)$$

Let

$$\mathcal{H}(n) = \bigcup_{k=0}^{n} \mathcal{H}_k(n) \quad (4.17)$$

denote the corresponding partition of $\mathcal{H}(n)$, where

$$\mathcal{H}_k(n) = \{ H \in \mathcal{H}(n); \text{signature } H = n - 2k \}, \quad k = 0, \ldots, n. \quad (4.18)$$

By the theorem of Hermite and Hurwitz, the Cauchy index $\text{CI}(g)$ of
g ∈ \text{Rat}(n) and the signature of the associated Hankel matrix \( H(g) \in \mathcal{H}(n) \) coincide:

\[
\text{CI}(g) = \text{signature } H(g).
\]

As a consequence we note

**Corollary 4.8.** The Laurent map \( \mathcal{L}_n : \text{Rat}(n) \to \mathcal{H}(n) \) induces a bianalytical diffeomorphism between \( \text{Rat}_k(n) \) and \( \mathcal{H}_k(n), k = 0, \ldots, n \).

By (3.4) and the signature rule of Frobenius [see Frobenius (1895), Gantmacher (1959)],

\[
\text{Signature } H = \sum_{\mu_i - \mu_{i-1} \text{ odd}}^{r} \sigma_i \quad \text{for } H \in \mathcal{H}(\mu).
\]  

(4.19)

Applying Theorem 4.4 and the theorem of Hermite and Hurwitz, one obtains from (4.19) the following expression for the Cauchy index of \( g \in \text{Rat}(n) \) in terms of the continued fraction expansion (4.2) of \( g(s) \):

\[
\text{CI}(g) = \sum_{\nu_i \text{ odd}}^{r} \text{sign}(\beta_0 \cdots \beta_{i-1}).
\]

(4.20)

In particular, we see that the Cauchy index is constant on the continued fraction cells. Altogether, we have the following characterization of the cells in \( \text{Rat}_k(n) \) and \( \mathcal{H}_k(n) \).

**Corollary 4.9.** Let \( \nu \in \mathbb{N}^r, \omega, \sigma \in \{-1,1\}^r \), and \( \mu = \{\mu_1, \ldots, \mu_r\} \subset \mathbb{N} \). Then, for \( k = 0, \ldots, n \),

\[
\Gamma(\nu, \omega) \subset \text{Rat}_k(n) \iff \sum_{i=1}^{r} \nu_i = n \quad \text{and} \quad \sum_{\nu_i \text{ odd}}^{r} \omega_1 \cdots \omega_i = n - 2k,
\]

(4.21)

\[
\mathcal{H}(\mu, \sigma) \subset \mathcal{H}_k(n) \iff \mu_r = n \quad \text{and} \quad \sum_{\mu_i - \mu_{i-1} \text{ odd}}^{r} \sigma_i = n - 2k.
\]

(4.22)
By this corollary, $\mathcal{H}_0(n)$ and $\mathcal{H}_n(n)$ contain only one cell, namely $\mathcal{H}(n,(1,\ldots,1))$ and $\mathcal{H}(n,(-1,\ldots,-1))$, respectively. Hence $\mathcal{H}_0(n) \cong \text{Rat}_0(n)$ and $\mathcal{H}_n(n) \cong \text{Rat}_n(n)$ are both homeomorphic to $\mathbb{R}^{2n}$.

5. TOPOLOGICAL CONSEQUENCES

In this section we illustrate by various examples how the Bruhat cell decomposition or the continued fraction cells can be used to derive topological properties of $\mathcal{H}(n)$ or $\text{Rat}(n)$, respectively.

5.1. Brockett's Theorem

First we rederive Brockett's well-known theorem that the $n + 1$ subsets $\text{Rat}_k(n)$, $k = 0,\ldots,n$, are the connected components of $\text{Rat}(n)$. We prove this theorem in the Hankel context by analyzing under which conditions two top dimensional Bruhat cells touch each other in $\mathcal{H}(n)$. Since the signature of a real symmetric matrix remains invariant under a continuous change of the entries if the rank does not change [see e.g. Gantmacher (1959), p. 309], the subsets $\mathcal{H}_k(n)$, $k = 0,\ldots,n$, are open in $\mathcal{H}(n)$. It remains to show that the sets $\mathcal{H}_k(n)$ are connected. We will prove a slightly stronger statement, namely that the union $\mathcal{U}_k(n)$ of all the cells $\mathcal{H}(\mu, \sigma)$ of dimensions $2n$ and $2n - 1$ in $\mathcal{H}_k(n)$ is connected.

A cell $\mathcal{H}(\mu, \sigma)$ is a $2n$-dimensional cell in $\text{Rat}_k(n)$ if and only if

$$\mu = n, \quad \sigma \in \{-1,1\}^n, \quad \text{and} \quad \sigma_1 + \cdots + \sigma_n = n - 2k. \quad (5.1)$$

To analyze how these cells are pasted together in $\text{Rat}_k(n)$ we need some combinatorial preparations. Consider the lexicographic order $\prec$ on the set

$$I_k = \{ \sigma \in \{-1,1\}^n; \sigma_1 + \cdots + \sigma_n = n - 2k \}. \quad (5.2)$$

Obviously,

$$\tau^k := \max I_k = (1,1,\ldots,1,-1,\ldots,-1)$$

with the entry $-1$ appearing $k$ times.

For any $\sigma \in I_k$, $\sigma \neq \tau^k$, let $i = i(\sigma)$ denote the largest index $i \leq n - 1$ such that $\sigma_i = -1$, $\sigma_{i+1} = 1$, and define

$$f(\sigma) = (\sigma_1,\ldots,\sigma_{i-1},1,-1,\sigma_{i+2},\ldots,\sigma_n). \quad (5.3)$$
For any $\sigma \in I_k$, $\sigma \neq \tau^k$ there exists an integer $l \geq 1$ such that

$$f^l(\sigma) = f(f^{l-1}(\sigma)) = \tau^k.$$  

We now show that every top dimensional cell $\mathcal{H}_\sigma := \mathcal{H}(n, \sigma)$, $\sigma \in I_k$, $\sigma \neq \tau^k$ is adherent in $\mathcal{H}_k(n)$ to the cell $\mathcal{H}_{f(\sigma)}$. Note that nothing is to be shown for $k = 0$ and $k = n$, since the components $\mathcal{H}_0(n)$ and $\mathcal{H}_n(n)$ consist only of one Bruhat cell each.

**Lemma 5.1.** Let $1 \leq k \leq n - 1$, $\sigma \in I_k$, $\sigma \neq \tau^k$, $i = i(\sigma)$, and

$$\mu := n \setminus \{i\}, \quad \tau = (\tau_1, \ldots, \tau_{n-1}) := (\sigma_1, \ldots, \sigma_{i-1}, \tau_i, \sigma_{i+2}, \ldots, \sigma_n),$$

where $\tau_i \in \{-1, 1\}$ arbitrary. Then

$$\mathcal{H}(\mu, \tau) \subseteq \overline{\mathcal{H}_\mu \cap \mathcal{H}_{f(\sigma)}},$$

where the closures are taken in $\mathcal{H}_k(n)$.

**Proof.** It follows from (4.22) and (5.2) that $\mathcal{H}(\mu, \tau) \subset \mathcal{H}_k(n)$. Suppose that $H = (h_{i+j-1})^\epsilon \in \mathcal{H}(\mu, \tau)$. For any $\epsilon \in \mathbb{R}$, $|\epsilon|$ sufficiently small, there exists a unique Hankel matrix $H(\epsilon) = (h_{i+j-1}(\epsilon)) \in \mathcal{H}(n)$ such that

$$(h_1(\epsilon), \ldots, h_{2n}(\epsilon)) := (h_1, \ldots, h_{2i-2}, h_{2i-1} + \epsilon, h_{2i}, \ldots, h_{2n}).$$

[cf. Iohvidov (1982), 2nd extension theorem]. The entries $h_{i+j-1}(\epsilon)$ depend continuously on $\epsilon$; hence

$$\lim_{\epsilon \to 0} H(\epsilon) = H \quad \text{and} \quad H(\epsilon) \in \mathcal{H}(n) \quad \text{for } |\epsilon| \text{ small, } \epsilon \neq 0.$$

Adopting the notation of (3.5) for $H$ and $H(\epsilon)$, we have by construction

$$d_j(\epsilon) = d_j \quad \text{for } j = 1, \ldots, i - 1,$$

$$d_i(\epsilon) = \epsilon.$$

(5.5)  

(5.6)
Moreover, we have by the continuity of the determinant and formula (3.7) for \( i + 1 \leq j \leq n \)
\[
d_i(\varepsilon) \cdots d_j(\varepsilon) = \det H_i(\varepsilon) \to \det H_j = -d_1 \cdots d_{i-1}d_i^2d_{i+1} \cdots d_j(\varepsilon)
\]
if \( \varepsilon \to 0 \). This together with (5.5), (5.6) implies
\[
d_i(\varepsilon)d_{i+1}(\varepsilon) = \varepsilon d_{i+1}(\varepsilon) \to -d_i^2 < 0 \quad \text{if} \quad \varepsilon \to 0, \quad (5.8)
\]
\[
d_j(\varepsilon) \to d_{j-1} \quad \text{if} \quad \varepsilon \to 0, \quad j \geq i + 2. \quad (5.9)
\]
We conclude from (5.5)-(5.9) and (5.3) that for \( |\varepsilon| \) sufficiently small
\[
H(\varepsilon) \in \mathcal{H}_o \quad \text{if} \quad \varepsilon < 0 \quad \text{and} \quad H(\varepsilon) \in \mathcal{H}_{f(\sigma)} \quad \text{if} \quad \varepsilon > 0. \quad (5.10)
\]
This proves the lemma.

**PROPOSITION 5.2.** For any \( k = 0, \ldots, n \) the union \( \mathcal{U}_k(n) \) of all Bruhat cells \( \mathcal{H}(\mu, \sigma) \) in \( \mathcal{H}_k(n) \) of dimension \( \geq 2n - 1 \) is arcwise connected.

**Proof.** We may suppose \( 1 \leq k \leq n - 1 \). Using the notation of the previous lemma, we have just proved that there is an arc \( \varepsilon \to H(\varepsilon), \varepsilon \in [-\delta, \delta] \), \( \delta > 0 \) within \( \mathcal{H}_o \cup \mathcal{H}(\mu, \tau) \cup \mathcal{H}_{f(\sigma)} \subset \mathcal{U}_k(n) \) which joins a point in \( \mathcal{H}_o \) to a point in \( \mathcal{H}_{f(\sigma)} \) via some point in \( \mathcal{H}(\mu, \tau) \). Since these three sets are cells, the union \( \mathcal{H}_o \cup \mathcal{H}(\mu, \tau) \cup \mathcal{H}_{f(\sigma)} \) is arcwise connected.

Note that for a given \( \sigma \in I_k, \sigma \neq \tau_k \), the corresponding \( \mu \) is uniquely determined by (5.4), whereas there are two choices for \( \tau \). Conversely, if \( \mathcal{H}(\mu, \tau) \) is a Bruhat cell of dimension \( 2n - 1 \) in \( \mathcal{H}_k(n) \), then \( \mu = n \setminus \{i\} \) for some \( i \leq n - 1 \) and there exists \( \sigma \in I_k \) such that \( \tau \) is of the form described in (5.4). Thus, proceeding as above from \( \sigma \) in \( I_k \) to \( f(\sigma), f^2(\sigma), \ldots, f^i(\sigma) = \tau^k \), we see that any two points in \( \mathcal{U}_k(n) \) can be connected by an arc in \( \mathcal{U}_k(n) \).

Since \( \mathcal{H}(n) \) is dense in \( \mathcal{H}(n) \), the union of top dimensional cells in \( \mathcal{H}_k(n) \) is dense in \( \mathcal{H}_k(n) \). This implies \( \mathcal{U}_k(n) = \mathcal{H}_k(n) \), and we obtain Brockett’s result as an immediate consequence of the previous proposition:

**COROLLARY 5.3.** For any \( n \geq 1 \), \( \text{Rat}(n) \) has \( n + 1 \) connected components and these are \( \text{Rat}_0(n), \ldots, \text{Rat}_n(n) \).
Remark 5.4. Proceeding in a similar way, Ober (1987) has proved an analogous connectivity result for the subset of stable transfer functions in $\text{Rat}(n)$.

5.2 The Euler Characteristic of $\text{Rat}_k(n)$

Let $N(n, k; m), 0 \leq k \leq n,$ denote the number of continued fraction cells of $\text{Rat}_k(n)$ of dimension $m.$ Although Corollary 4.9 gives a complete combinatorial characterization of the cells $\Gamma(\nu, \omega)$ in $\text{Rat}_k(n)$, it is difficult to derive explicit formulas for $N(n, k; m).$ A full list of these numbers for $n = 1, \ldots, 6$ can be found in Fuhrmann and Krishnaprasad (1986, Table 4.1). Computing the alternating sum

$$\chi(\text{Rat}_k(n)) = \sum_{m=0}^{2n} (-1)^m N(n, k; m) \quad (5.11)$$

for $k = 1, 2,$ the two authors obtained zero and conjectured that the same holds true for all $k = 1, \ldots, n - 1.$ Since the continued fraction cells form a cellular decomposition of $\text{Rat}_k(n), the alternating sum (5.11) is the Euler characteristic of $\text{Rat}_k(n)$; see Massey (1978, p. 61). Thus the conjecture of Fuhrmann and Krishnaprasad has an interesting topological interpretation. It says that the Euler characteristics of the “nontrivial” connected components $\text{Rat}_k(n), 1 \leq k \leq n - 1,$ of $\text{Rat}(n)$ are all zero.

That this is indeed the case can be derived from the following reformulation of a result of Segal (1979); see Proposition 7.1.

Theorem 5.5 (Segal). Suppose $0 \leq k \leq n/2.$ Then $\text{Rat}_k(n)$ is homeomorphic to the product space

$$\mathbb{C}^{n-2k} \times \text{Rat}(k, \mathbb{C}),$$

where $\text{Rat}(k, \mathbb{C})$ denotes the space of all strictly proper complex rational transfer functions of McMillan degree $k.$

By means of this result we can prove the conjecture mentioned above.

Proposition 5.6. Suppose $n \geq 2.$ For $k = 1, 2, \ldots, n - 1$ the Euler characteristic of $\text{Rat}_k(n)$ is zero:

$$\chi(\text{Rat}_k(n)) = \sum_{m=0}^{2n} (-1)^m N(n, k; m) = 0.$$
Proof. Since Rat\(k(\mathbb{n})\) is homeomorphic to Rat\(_{n-k}(\mathbb{n})\), we may assume \(1 \leq k \leq n/2\). By Theorem 5.5 it only remains to verify that the Euler characteristic of Rat\((k, \mathbb{C})\) is zero. But this has been shown in Helmke (1986).

5.3. Betti Numbers of Rat\(n\)

Let

\[ b_q(\text{Rat}_k(n), \mathbb{Z}/2) = \text{rank } H_q(\text{Rat}_k(n), \mathbb{Z}/2) \quad (5.12) \]

denote the \(q\)th mod 2 Betti number of Rat\(_k(n)\). [Here \(H_q(X, \mathbb{Z}/2)\) is the \(q\)th singular homology group with coefficients in the field \(\mathbb{Z}/2\).] The following proposition yields upper bounds for the Betti numbers in terms of the continued fraction cell decomposition.

**Proposition 5.7.** For \(n > 1\), \(k = 0, \ldots, n\), the mod 2 Betti numbers of Rat\(_k(n)\) are bounded by

\[ b_q(\text{Rat}_k(n), \mathbb{Z}/2) \leq N(n, k; 2n - q), \quad 0 \leq q \leq 2n. \quad (5.13) \]

In particular,

\[ H_q(\text{Rat}_k(n), \mathbb{Z}/2) = \{0\}, \quad q \geq n. \quad (5.14) \]

**Proof.** Since the partition of Rat\(_k(n)\) into continued fraction cells \(\Gamma(\nu, \omega)\) defines a cell decomposition, the weak Morse inequalities yield

\[ \text{rank } H_c^m(\text{Rat}_k(n), \mathbb{Z}/2) \leq N(n, k; m), \quad m \in \mathbb{N}, \]

where \(H_c^m(X, \mathbb{Z}/2)\) denotes the \(m\)th Alexander-Spanier cohomology group with compact support and coefficients in \(\mathbb{Z}/2\); see Helmke and Hinrichsen (1986). By Poincaré duality,

\[ H_q(\text{Rat}_k(n), \mathbb{Z}/2) \cong H_c^{2n-q}(\text{Rat}_k(n), \mathbb{Z}/2), \quad 0 \leq q \leq 2n; \]

hence

\[ b_q(\text{Rat}_k(n), \mathbb{Z}/2) = \text{rank } H_c^{2n-q}(\text{Rat}_k(n), \mathbb{Z}/2). \]
Thus (5.13) follows. Since $\text{Rat}_n(n)$ has no continued fraction cells of dimension $\leq n$, (5.14) is an immediate consequence of (5.13).

In view of Theorem 5.5, it is of equal interest to derive bounds for the Betti numbers of the manifold of complex transfer functions of fixed degree $n$.

**Corollary 5.8.** For $n \geq 1$

$$H_q(\text{Rat}(n, \mathbb{C}), \mathbb{Z}/2) = \{0\}, \quad q \geq 2n,$$

$$b_{2n-1}(\text{Rat}(n, \mathbb{C}), \mathbb{Z}/2) \leq 2.$$  

**Proof.** By Segal's Theorem

$$\text{Rat}(n, \mathbb{C}) \cong \text{Rat}_n(2n),$$

so that (5.15) follows from (5.14). Moreover $\text{Rat}_n(2n)$ has exactly two continued fraction cells of dimension $2n + 1$, namely

$$\Gamma(2n, \pm 1) = \left\{ \frac{\beta_0}{\alpha_1(s)} \mid \pm \beta_0 > 0, \alpha_1 \text{ monic}, \deg \alpha_1 = 2n \right\}.$$ 

Therefore (5.16) is a consequence of (5.13).

**Remark 5.9.**

(i) Note that the equalities (5.15) hold for an arbitrary Abelian group $G$ of coefficients, instead of $\mathbb{Z}/2$. For $q > 2n + 1$ the equations (5.15) hold for more general reasons. In fact, $\text{Rat}(n, \mathbb{C})$ is homeomorphic to the smooth affine variety $\mathcal{H}(n \times n, \mathbb{C}) \times \mathbb{C}$, where $\mathcal{H}(n \times n, \mathbb{C})$ is the manifold of invertible $n \times n$ complex Hankel matrices. By a theorem of Lefschetz, the singular homology groups $H_q(X, \mathbb{Z})$ of a smooth affine variety $X$ of complex dimension $k$ disappear for $q > k$. Hence by the universal coefficient theorem the Betti numbers $b_q(\text{Rat}(n, \mathbb{C}), \mathbb{Z}/2)$ vanish for $q > 2n + 1$.

(ii) Segal (1979) has shown that

$$H_q(\text{Rat}(n, \mathbb{C}), \mathbb{Z}) \cong H_q(\Omega^2_n(S^2), \mathbb{Z}), \quad 0 \leq q \leq n,$$

where $\Omega^2_n(S^2)$ is the (second loop) space of base-point-preserving continuous
maps $S^2 \to S^2$ of degree $n$. In view of Corollary 5.8, these equalities determine half of the possibly nontrivial homology groups of $\text{Rat}(n, \mathbb{C})$. In fact, Boyer and Mann (1987) have recently shown that for $n = 2^k$

$$b_{2n-1}(\text{Rat}(n, \mathbb{C}), \mathbb{Z}/2) \neq 0.$$  

Thus the vanishing result (5.15) cannot be improved.

We conclude this paper by determining the mod 2 Betti numbers $b_q(\text{Rat}(n), \mathbb{Z}/2)$ for $n \leq 5$. By the theorems of Brockett and Segal,

$$b_q(\text{Rat}(n), \mathbb{Z}/2) = \sum_{k=0}^{n} b_q(\text{Rat}_k(n), \mathbb{Z}/2) = \sum_{k=0}^{n} b_q(\text{Rat}(\text{min}(k, n-k), \mathbb{C}), \mathbb{Z}/2). \quad (5.18)$$

Clearly

$$\text{Rat}(1, \mathbb{C}) \simeq \mathbb{C}^* \times \mathbb{C}$$

has the homotopy type of a circle $S^1$. The mod 2 Betti numbers of $\text{Rat}(2, \mathbb{C})$ were determined in Byrnes and Helmke (1986). It was shown that $\text{Rat}(2, \mathbb{C})$ has the same mod 2 homology groups as the real projective 3-space $\mathbb{RP}^3$, i.e.

$$b_q(\text{Rat}(2, \mathbb{C}), \mathbb{Z}/2) = 1 \quad \text{for} \quad q = 0, 1, 2, 3.$$  

Altogether, we obtain by (5.18) the complete table of mod 2 Betti numbers of $\text{Rat}(n)$, $n \leq 5$, which is given in Table 1.

<table>
<thead>
<tr>
<th>TABLE 1</th>
<th>MOD 2 BETTI NUMBERS OF RAT(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$q = 0$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
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<td>3</td>
<td>4</td>
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<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>
It is interesting to compare Table 1 with the numbers of continued fraction cells of codimension $q$ in $\text{Rat}(n)$ (see Table 4.1 in Fuhrmann and Krishnaprasad (1986)):

$$N(n; q) = \sum_{k=0}^{n} N(n, k; 2n-q),$$

which are shown in Table 2. A comparison between Tables 1 and 2 shows that the bounds for the mod 2 Betti numbers of $\text{Rat}(n)$ obtained from the weak Morse inequalities (5.13) are rather poor.

REFERENCES


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