Arithmetic Properties of Bernoulli–Padé Numbers and Polynomials

Karl Dilcher

Department of Mathematics and Statistics, Dalhousie University,
Halifax, Nova Scotia, B3H 3J5, Canada
E-mail: dilcher@mathstat.dal.ca

and

Louise Malloch

1991 Brunswick Street, Apt. 205, Halifax, Nova Scotia, B3J 2G9, Canada
E-mail: az360@chebucto.ns.ca

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A class of generating functions based on the Padé approximants of the exponential function gives a doubly infinite class of number and polynomial sequences. These generalize the Bernoulli numbers and polynomials, as well as other sequences found in the literature. We derive analogues of the Kummer congruences, the von Staudt–Clausen Theorem, and other properties also satisfied by the ordinary Bernoulli numbers and polynomials.

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1. INTRODUCTION

The Bernoulli numbers are usually defined by the generating function

\[
\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi).
\] (1.1)

Various different generalizations, most notably Bernoulli numbers of higher orders, and generalized Bernoulli numbers belonging to a Dirichlet character, arise from changing the generating function (1.1) in certain ways.

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The topic of this paper is related to the following generalizations, namely

the numbers $A_{k,n}$ defined by

$$\frac{1}{k!} t^k \frac{e^t - \left(1 + \frac{t^2}{2!} + \cdots + \frac{t^{k-1}}{(k-1)!}\right)}{t^n = \sum_{n=0}^{\infty} A_{k,n} \frac{t^n}{n!}},$$

and the numbers $V_{k,n}$ defined by

$$\frac{k!}{(2k+1)!} t^{2k+1} \frac{t^k \left(y_k(-2/t) e^t - (-t)^k y_k(2/t)\right)}{t^n = \sum_{n=0}^{\infty} V_{k,n} \frac{t^n}{n!}},$$

where $y_k(x)$ is the Bessel polynomial of degree $k$,

$$y_k(x) = \sum_{j=0}^{k} \frac{(k+j)!}{j!(k-j)!} \frac{x^j}{2^j}. \quad (1.4)$$

Obviously, $A_{1,n} = V_{0,n} = B_n$. Both these generalizations, along with the corresponding polynomial sequences, were studied by Howard in [7] and [9], respectively. The sequences in (1.3) also generalize the “van der Pol numbers” defined by

$$\frac{-1}{12} t^3 \frac{e^t - \left(1 + \frac{t}{2}\right)}{t^n = \sum_{n=0}^{\infty} V_{n} \frac{t^n}{n!}},$$

so that $V_{1,n} = V_n$ (see [6]). For further related references, see the introduction of [5].

Our main definition is based on the observation that the sequence of quotients

$$\frac{(-t)^k y_k(2/t)}{t^k y_k(-2/t)} \quad (k = 0, 1, 2, \ldots)$$

related to (1.3) consists of the diagonal entries in the Padé table for the exponential function $e^t$ (see, e.g., [2]) and that the polynomials

$$1 + t + \frac{t^2}{2!} + \cdots + \frac{t^{k-1}}{(k-1)!}$$

occurring in (1.2) form the first row of the same Padé table, as Maclaurin polynomials for $e^t$. 
This suggests an extension of both (1.2) and (1.3) by considering sequences of numbers (and polynomials) generated by arbitrary entries in the Padé table of \( e^t \).

Given a function \( f \) with a Maclaurin expansion

\[
f(t) = \sum_{j=0}^{\infty} c_j t^j
\]

in a neighborhood of the origin, a Padé approximant \([r/s]\) to \( f(t) \) is a ratio of two polynomials

\[
[r/s] = \frac{a_0 + a_1 t + \cdots + a_r t^r}{b_0 + b_1 t + \cdots + b_s t^s}
\]

which has a Maclaurin expansion that agrees with (1.6) “as far as possible.” \([r/s]\) is normalized by setting \( b_0 = 1 \); this leaves the \( r+s+1 \) independent coefficients \( a_0, a_1, \ldots, a_r, b_1, \ldots, b_s \) to be determined in such a way that

\[
\sum_{j=0}^{\infty} c_j t^j - \frac{a_0 + a_1 t + \cdots + a_r t^r}{b_0 + b_1 t + \cdots + b_s t^s} = t^{r+s+1}(d_0 + d_1 t + d_2 t^2 + \cdots).
\]

The resulting linear systems normally have unique solutions; see, e.g., [2]. The doubly infinite array that has \([r/s]\) in the \( s \)th row and \( r \)th column (counting from 0) is called the Padé table for \( f \).

In the case \( f(t) = e^t \) the Padé table can be determined explicitly: If we denote \([r/s] = P^{(r,s)}(t)/Q^{(r,s)}(t)\), then

\[
P^{(r,s)}(t) = \sum_{j=0}^{r} \binom{r}{j} \frac{(r+s-j)!}{(r+s)!} t^j,
\]

\[
Q^{(r,s)}(t) = \sum_{j=0}^{s} \binom{s}{j} \frac{(r+s-j)!}{(r+s)!} (-t)^j
\]

(see, e.g., [2, p. 8ff.], where the first few entries of the Padé table for \( e^t \) are also given explicitly). By comparing (1.4) with (1.7) it is easy to see that

\[
P^{(r,r)}(t) = \frac{r!}{(2r)!} t^r y_r \left( \frac{2}{t} \right).
\]

Also, obviously,

\[
P^{(r,0)}(t) = Q^{(0)}(-t).
\]

We can now define the main objects of study of this paper.
Definition 1.1. Given the nonnegative integers \( r, s \), we define the \( n \)th Bernoulli–Padé numbers of order \((r, s)\) by the generating function

\[
\frac{(-1)^s r! s! t^{r+s+1}}{(r+s)! (r+s+1)!} \left[ Q^{(r, s)}(t) e^t - P^{(r, s)}(t) \right] = \sum_{n=0}^\infty B_n^{(r, s)} \frac{t^n}{n!}
\]  

(1.11)

and the \( n \)th Bernoulli–Padé polynomials of order \((r, s)\) by

\[
\frac{(-1)^s r! s! t^{r+s+1} e^{xt}}{(r+s)! (r+s+1)!} \left[ Q^{(r, s)}(t) e^t - P^{(r, s)}(t) \right] = \sum_{n=0}^\infty B_n^{(r, s)}(x) \frac{t^n}{n!}
\]  

(1.12)

It is clear from this definition and from (1.7)–(1.9) that

\[
A_{k+1, n} = B_n^{(k, 0)}, \quad V_{k, n} = B_n^{(k, k)},
\]

(1.13)

thus the Bernoulli–Padé numbers generalize both classes of sequences defined in (1.2) and (1.3).

In the recent paper [5] the above definition was extended, with the help of confluent hypergeometric functions, to arbitrary real or complex parameters \( r, s \), with the exception of certain cases where the two parameters add up to a negative integer. Here we will not be concerned with the general situation, but rather restrict our attention to the numbers \( B_n^{(r, s)} \) and the polynomials \( B_n^{(r, s)}(x) \) as defined above.

The main purpose of this paper is to study properties of the Bernoulli–Padé numbers and polynomials that are analogous to some of the more important properties of the ordinary Bernoulli numbers and polynomials.

In Section 2 we derive some basic properties and then study analogues of the fundamental difference equation. In Sections 3 and 5 we prove various results that are analogous to the von Staudt–Clausen Theorem, and in Section 4 we obtain congruences that resemble the Kummer congruences satisfied by the ordinary Bernoulli numbers. Finally, in Section 7, we prove an irreducibility result for an infinite class of Bernoulli–Padé polynomials.

An analogue of another important property of the Bernoulli numbers, namely Euler's formula which links them with the Riemann zeta function, can be found in [5].

2. RECURRENCE RELATIONS AND FUNCTIONAL EQUATIONS

One of the most basic and important properties of the Bernoulli–Padé numbers is the following recurrence relation which generalizes that of the ordinary Bernoulli numbers and those for the sequences defined in (1.2) and (1.3).
Proposition 2.1. For fixed nonnegative integers \( r \) and \( s \), the Bernoulli–Padé numbers satisfy \( B_0^{(r,s)} = 1 \), and for \( n \geq 1 \),
\[
\sum_{k=0}^{n} \binom{n+r+s+1}{k} \binom{n+s-k}{s} B_k^{(r,s)} = 0,
\]
or equivalently
\[
\frac{B_n^{(r,s)}}{n!} = -(r+s+1)! \sum_{k=0}^{n-1} \binom{n+s-k}{s} B_k^{(r,s)} k!(n+r+s+1-k)!.
\]

This is straightforward but tedious to prove by multiplying both sides of (1.11) by the denominator of the left-hand side, using (1.7) and (1.8). Another equivalent form of the recurrence is given in [5]. It follows from (2.1) that the Bernoulli–Padé numbers are all rational numbers. The first few, in addition to \( B_0^{(r,s)} = 1 \), are

\[
B_1^{(r,s)} = \frac{s+1}{r+s+2},
\]
\[
B_2^{(r,s)} = \frac{(s+1)(s^2+rs+4s+2)}{(r+s+2)^2(r+s+3)},
\]
\[
B_3^{(r,s)} = \frac{(s+1)(s^2+2s^2r+s^2r^2+9s^3+8s^2r^2+24s^2r-4sr^2+14s^2-6r)}{(r+s+2)^3(r+s+3)(r+s+4)}.
\]

We continue with some basic properties of the Bernoulli–Padé polynomials. Most of the identities (2.2)–(2.6) were already given in [5]; we repeat them here for completeness and easy reference.

Proposition 2.2. For all nonnegative integers \( r, s, \) and \( n \), we have

\[
B_n^{(r,s)}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(r,s)} x^{n-k},
\]

\[
B_n^{(r,s)}(x+h) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(r,s)}(x) h^{n-k},
\]

\[
\frac{d}{dx} B_n^{(r,s)}(x) = nB_n^{(r,s)}(x),
\]

\[
B_n^{(r,s)}(1-x) = (-1)^n B_n^{(r,s)}(x).
\]
Proof. Since the left-hand side of (1.12) is of the form \( F(t) e^{xt} \), where \( F(t) \) is the generating function in (1.11), the sequences \( \{B_n^{(\nu, \gamma)}(x)\}_{n=0}^{\infty} \) are Appell polynomials (see, e.g., [11]), and as such they satisfy the first three properties. Identity (2.5) follows easily from (1.12) and (1.10).

Note that in property (2.5) the orders are interchanged. When \( r = s \), we obtain a direct analogue to the corresponding well-known identity for the ordinary Bernoulli polynomials, and \( x = 1/2 \) gives, as in the ordinary case,

\[
B_{2n+1}^{(\nu, \gamma)}(0) = 0 \quad (n = 0, 1, 2, \ldots).
\]

(2.6)

This complements the obvious relation \( B_n^{(\nu, \gamma)}(0) = B_n^{(\nu, \gamma)} \).

One of the most important properties of the ordinary Bernoulli polynomials \( B_n(x) \) is the difference equation

\[
B_n(x+1) - B_n(x) = nx^{n-1} \quad (n \geq 1).
\]

(2.7)

This has been generalized to the polynomials \( V_n(x) \) associated with the van der Pol numbers defined in (1.5), namely

\[
(n+1)[V_n(x+1) + V_n(x)] - 2[V_{n+1}(x+1) - V_{n+1}(x)] = \binom{n+1}{3} x^{n-2}
\]

(2.8)

for \( n \geq 2 \) (see [8]), and also to the polynomials \( A_{k,n}(z) \) associated with the numbers defined in (1.2): For \( n \geq k \),

\[
A_{k,n}(x+1) - A_{k,n}(x) - A'_{k,n}(z) - \cdots - \frac{1}{(k-1)!} A_{k,n}^{(k-1)}(x) = \binom{n}{k} x^{n-k}
\]

(2.9)

(see [7]). With (2.4) this can be rewritten as

\[
A_{k,n}(x+1) - A_{k,n}(x) - nA_{k,n-1}(z) - \frac{n}{2} A_{k,n-2}
\]

\[
- \cdots - \binom{n}{k} A_{k,n-k+1}(x) = \binom{n}{k} x^{n-k}.
\]

(2.10)

The form (2.9) now suggests the following generalization: Let \( D \) be the usual derivative operator, with \( D^0f(x) = f(x) \) and \( D^k f(x) = f^{(k)}(x) \) for \( k \geq 1 \).
Proposition 2.3. For any nonnegative integers \(r\), \(s\), and \(n \geq r+s+1\) we have

\[
Q^{(r,s)}(D) B_n^{(r,s)}(x+1) - P^{(r,s)}(D) B_n^{(r,s)}(x) = (-1)^s \left( \frac{r+s+1}{r+s} \right) x^{n-(r+s+1)}.
\]  

(2.11)

Proof. We begin with a general observation. If \(p_n(x)\) is an Appell polynomial of degree \(n\), i.e., \(\frac{d}{dx} p_n(x) = np_{n-1}(x)\), then

\[
C_{n=0}^\infty t^n p_n(x) \frac{t^n}{n!} = \sum_{n=0}^\infty \frac{n!}{(n-k)!} p_{n-k}(x) \frac{t^n}{n!},
\]

so that

\[
\sum_{n=0}^\infty t^n p_n(x) \frac{t^n}{n!} = \sum_{n=0}^\infty D^n p_n(x) \frac{t^n}{n!}.
\]  

(2.12)

To simplify notation, we set

\[
c^{(r,s)} := (-1)^s \frac{r!s!}{(r+s)! (r+s+1)!}.
\]

Then with (1.12) and (2.12) we get

\[
c^{(r,s)} t^{r+s+1} e^{xt} = \frac{c^{(r,s)} t^{r+s+1}}{Q^{(r,s)}(t) e^{t} - P^{(r,s)}(t) e^{t^2}} \left[ Q^{(r,s)}(t) e^{t(x+1)} - P^{(r,s)}(t) e^{tx} \right]
\]

\[
= \sum_{n=0}^\infty \left[ Q^{(r,s)}(t) B_n^{(r,s)}(x+1) - P^{(r,s)}(t) B_n^{(r,s)}(x) \right] \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^\infty \left[ Q^{(r,s)}(D) B_n^{(r,s)}(x+1) - P^{(r,s)}(D) B_n^{(r,s)}(x) \right] \frac{t^n}{n!}.
\]

On the other hand, we have

\[
c^{(r,s)} t^{r+s+1} e^{xt} = c^{(r,s)} \sum_{n=0}^\infty x^n \frac{t^{n+r+s+1}}{n!}
\]

\[
= \frac{(-1)^s \frac{r!s!}{(r+s)! (r+s+1)!}}{x^{n-(r+s+1)}} \sum_{n=0}^\infty \frac{n! x^{n-(r+s+1)}}{(n-r-s+1)! (n-(r+s+1))!} \frac{t^n}{n!}.
\]

Now (2.11) follows by equating coefficients of \(t^n\).
It is easy to see that (2.7), (2.8), and (2.9) are special cases of (2.11). Using (1.7), (1.8), and (2.4), we can rewrite (2.11) in a more explicit form:

**Corollary 2.1.** For nonnegative integers \( r, s, \) and \( n \geq r+s+1 \) we have

\[
\sum_{j=0}^{s} (-1)^j \binom{s}{j} \binom{j}{r+s} B_{n-j}^{(r,j)}(x+1) - \sum_{j=0}^{r} \binom{r}{j} \binom{j}{r+s} B_{n-j}^{(s,j)}(x) \\
= (-1)^r \binom{n}{r+s+1} \binom{r+s+1}{s} x^{n-(r+s+1)}. \tag{2.13}
\]

We state the special case \( r = s \) separately:

**Corollary 2.2.** For nonnegative integers \( r \) and \( n \geq 2r+1 \) we have

\[
\sum_{j=0}^{r} \binom{n}{j} \binom{2r-j}{r} \left[ (-1)^j B_{n-j}^{(r,j)}(x+1) - B_{n-j}^{(r,r)}(x) \right] = (-1)^r \binom{n}{2r+1} x^{n-2r-1}. \tag{2.14}
\]

We see again that this extends (2.7) and (2.8).

We close this section with a consequence of (2.13). An interesting relation among van der Pol numbers that has no analogue with the Bernoulli numbers is the following (see [8]):

\[
V_{2n+1} + \frac{2n+1}{2} V_{2n} = 0 \quad (n > 1). \tag{2.15}
\]

This and its generalization to the numbers \( V_{r,n} = B_{n}^{(r,r)} \) are easily obtained from (2.14) by setting \( x = 0 \) and observing that we have \( B_{n-j}^{(r,j)}(1) = (-1)^{n-j} B_{n-j}^{(r,r)} \) by (2.5). Then, for even \( n \), (2.14) becomes meaningless, while in the odd case (replacing \( n \) by \( 2n+1 \)) we get

\[
\sum_{j=0}^{r} \binom{2n+1}{j} \binom{2r-j}{r} B_{2n+1-j}^{(r,j)} = 0 \quad (n > r). \tag{2.16}
\]

For \( r = 1 \) this clearly reduces to (2.15), while for \( r = 0 \) we obtain the well-known fact that \( B_{2n+1} = 0 \) for \( n > 0 \). We note that (2.16) is not new; it was derived in [9], in a somewhat different notation.
In the more general case of (2.13), we obtain the following identity, again having set \(x = 0\) and using (2.5).

**Corollary 2.3.** For nonnegative integers \(r, s, \text{ and } n > r + s + 1\) we have

\[
(-1)^s \sum_{j=0}^{s} \binom{n}{j} \binom{r+s-j}{r} B_{n-j}^{(r,s)} = \sum_{j=0}^{r} \binom{n}{j} \binom{r+s-j}{s} B_{n-j}^{(r,s)}. \tag{2.17}
\]

As a particular example, we obtain

\[
B_n^{(0,1)} = (-1)^s \left[ B_n^{(1,0)} + nB_n^{(1,0)} \right] \quad (n > 2).
\]

3. THE DENOMINATORS OF BERNOULLI-PADÉ NUMBERS

We know from the recurrence relations in Proposition 2.1 that the numbers \(B_n^{(r,s)}\) are rational numbers. In the case \(B_n = B_n^{(0,0)}\), the ordinary Bernoulli numbers, the denominators are completely determined by the von Staudt–Clausen theorem which states that the denominator of \(B_n\) is the product of all primes \(p\) such that \(p - 1\) divides \(n\). The theorem is actually somewhat stronger; see, e.g., [10, p. 233].

In the general case the situation is more complicated, but one can observe some strong patterns, most of which we will prove in this and the next two sections. We first introduce the following notation.

Given a prime \(p\), let \(\alpha_p^{(r,s)}(n)\) denote the highest power of \(p\) dividing the denominator of \(B_n^{(r,s)}\).

Examples indicate the following behaviour as \(n\) grows, with \(p, r, \text{ and } s\) fixed: \(\alpha_p^{(r,s)}(n)\) usually increases by 1 when \(p - r - s - 1 \mid n\); decreases by \(m\) when \(p^m \mid n\) (i.e., \(m\) is the highest power of \(p\) dividing \(n\)); and otherwise remains unchanged, although there are some exceptions. Thus, if \(p^m \mid n\) then we would usually have, for \(p > r + s + 1\),

\[
\alpha_p^{(r,s)}(n) = \begin{cases} 
\alpha_p^{(r,s)}(n-1) - m + 1, & \text{if } p - r - s - 1 \mid n, \\
\alpha_p^{(r,s)}(n-1) - m, & \text{otherwise}.
\end{cases}
\]

This would imply that \(\alpha_p^{(r,s)}(n)\) is approximately equal to

\[
\frac{n}{p - (r+s+1)} - v_p(n),
\]

where

\[
v_p(n) := \sum_{m=1}^{\infty} \left\lfloor \frac{n}{p^m} \right\rfloor = \text{ord}_p(n!); \tag{3.1}
\]
here \( \text{ord}_p(a) \) is the highest power of \( p \) that divides \( a \). Indeed, we have

**Proposition 3.1.** For \( p > r+s+1 \),

\[
\alpha_p^{(r,s)}(n) \leq \left\lfloor \frac{n}{p-(r+s+1)} \right\rfloor - v_p(n). \tag{3.2}
\]

The proof is based on that of an analogous result of Howard [7] for the numbers \( A_{k,v} \) defined in (1.2). We use the following lemma from [7].

**Lemma 3.1.** Given a prime \( p \) and integers \( m \geq 0 \) and \( 0 < v < p \), we have

\[
\frac{m}{p-v} \geq v_p(m+v). \tag{3.3}
\]

We will also make use of the inequality

\[
\left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \geq \left\lfloor \frac{n-k}{m} \right\rfloor \tag{3.4}
\]

for integers \( n > k \geq 0 \) and \( m > 0 \), which is not difficult to verify. To simplify notation, we set \( w := r+s+1 \) here and for the remainder of this paper.

**Proof of Proposition 3.1.** We prove the result by induction on \( n \). Since \( B_0^{(r,s)} = 1 \), (3.2) is certainly true for \( n = 0 \). Suppose now that it holds for \( 0, 1, \ldots, n-1 \) and rewrite the recurrence relation (2.1) as

\[
p \left[ \frac{s}{x} \right] B_s^{(r,s)} = -w! \sum_{k=0}^{n-1} \binom{n+s-k}{s} \frac{p \left[ \frac{s}{x} \right] B_k^{(r,s)}}{(n-w-k)!}. \]

Now take \( \text{ord}_p \) on both sides,

\[
\left\lfloor \frac{n}{p-w} \right\rfloor - \alpha_p^{(r,s)}(n) - v_p(n) \geq \text{ord}_p w! + \min_{0 \leq k \leq n-1} \left\{ \text{ord}_p \left( \frac{n+s-k}{s} \right) \right\} + \left\lfloor \frac{n}{p-w} \right\rfloor - \frac{k}{p-w} - v_p(n+w-k) + \left\lfloor \frac{k}{p-w} \right\rfloor - \alpha_p^{(r,s)}(k) - v_p(k). \]

Since \( p > r+s+1 \), we have \( \text{ord}_p w! = 0 \). Also clearly \( \text{ord}_p \left( \frac{n+s-k}{s} \right) \geq 0 \), and by the induction hypothesis we have

\[
\left\lfloor \frac{k}{p-w} \right\rfloor - \alpha_p^{(r,s)}(k) - v_p(k) \geq 0, \quad 0 \leq k < n.
\]
Hence
\[
\frac{n}{p-w} - \sigma_p^{(r, s)}(n) - v_p(n) \\
\geq \min_{0 \leq k \leq n-1} \left\{ \left\lfloor \frac{n}{p-w} \right\rfloor - \left\lfloor \frac{k}{p-w} \right\rfloor - v_p(n+k) \right\}. \tag{3.5}
\]

Now by (3.3) and (3.4) we have
\[
\left\lfloor \frac{n}{p-w} \right\rfloor - \left\lfloor \frac{k}{p-w} \right\rfloor \geq \left\lfloor \frac{n-k}{p-w} \right\rfloor \geq v_p(n+k) = v_p(n+w-k). \tag{3.6}
\]

This means that the right-hand side of (3.5) is nonnegative, which implies (3.2).

For certain values of \( n \) the inequality in (3.2) can be shown to be an equality. To prove this fact, we require some results from the following section.

4. KUMMER-TYPE CONGRUENCES

Among the most important and useful arithmetic properties of the ordinary Bernoulli numbers are the Kummer congruence
\[
B_{n+p-1} \equiv B_n \pmod{p} \tag{4.1}
\]
\((p \text{ a prime and } p-1 \nmid n)\) and its many variants; see, e.g., [10, p. 239]. Here we will prove an analogue and some consequences; these will be used in later sections. The results and proofs are again similar to analogous results on the numbers \( A_{k, n} \) in (1.2) due to Howard [7].

Recall that we have set \( w = r+s+1 \). To simplify notation, we set for primes \( p \geq w \) and integers \( n \) and \( m \), \( 0 \leq m < p-w \),
\[
b_p^{(r, s)}(n, m) := \frac{p^n}{(n(p-w)+m)!} B_{n(p-w)+m}^{(r, s)} \quad (n > 0) \tag{4.2}
\]
and \( b_p^{(r, s)}(n, m) := 0 \) for \( n \leq 0 \).
Proposition 4.1. Let $p$ be a prime, $p > w > 1$, and $n > 0$ be an integer. Then

$$b_p^{(r,s)}(n, 0) \equiv w! \left( \frac{p-r-1}{s} \right) b_p^{(r,s)}(n-1, 0) \pmod{p},$$

and for $0 < m < p-w$ we have

$$b_p^{(r,s)}(n, m) \equiv w! \sum_{j=0}^{m} \sum_{k=0}^{\left\lfloor \frac{n}{p-w} \right\rfloor} (-1)^{k+1} \left( \frac{k(p-w)+j+s}{s} \right) b_p^{(r,s)}(n-k, m-j) \frac{k!(j+(1-k)w)!}{k!(j+(1-k)w)!} \pmod{p}.$$
Considering (3.6) again, we see that for (4.7) to hold we also need 
\[ \left\lfloor \frac{j}{(p-w)} \right\rfloor = v_p(j+w). \]
Writing \( \frac{j+w}{n} p = \sum_{i=0}^{n} a_i < (p-1)p \), we can use Lemma 4.1 with \( m = 1 \) and see that we need only consider those \( j \) for which

\[ (w-1)(j+w) + (p-w) \sum_{i=0}^{n} a_i < (p-1)p. \]  

(4.9)

Now, if \( k \geq p+1 \) then the left-hand side of (4.9) is at least \( p^2 \), which is a contradiction. If \( k = p, \) then \( j+w = p(p-w)+l-w < p^2-pw+p \leq p^2 \) since \( w \geq 1 \); therefore by (3.1),

\[ v_p(j+w) = \left\lfloor \frac{p(p-w)+j+w}{p} \right\rfloor = p-w, \]

which by (4.8) means that (4.7) cannot hold. Hence we must have \( k < p \).

Another condition is that \( kw \leq l+w \), for otherwise we would have \( v_p(j+w) = v_p(kp-kw+l+w) < v_p(kp) = k \), again by (3.1), which once again contradicts (4.7). Also, \( j+w = kp+w+l-kw \), with \( 0 \leq w+l-kw < p \) and \( k < p \), and therefore \( v_p(j+w) = k \). Thus we have shown that (4.7) holds if and only if \( j = k(p-w)+l \) is such that either \( l = 0 \) and \( k = 1 \), or else \( 1 \leq l \leq m \) and \( 0 \leq k \leq \lfloor (l+w)/w \rfloor \).

The result now follows by rewriting the sum in (4.6) according to the above cases, then using (4.2) and, where required, Wilson’s theorem \((p-1)! \equiv -1 \pmod{p}\).

We can rewrite (4.3) in a more explicit form, using (4.2) and the congruence

\[ \binom{p-r-1}{s} \equiv (-1)^r \binom{r+s}{s} \pmod{p} \]

which follows from Wilson’s theorem.

**Corollary 4.1.** Let \( p \) be a prime, \( p > w = r+s+1 > 1 \), and \( n > 0 \). Then

\[ \frac{p^r}{(n(p-w))!} B^{(r,s)}_{n(p-w)} \equiv (-1)^r \frac{w!(r+s)}{s!} p^{r-1} \frac{B^{(r,s)}_{(n-1)(p-w)}}{((n-1)(p-w))!} \pmod{p}. \]  

(4.10)

We prove two more consequences of Proposition 4.1. The first one is just an iteration of congruence (4.10), using the fact that \( B^{(r,s)}_{0} = 1 \). Both are needed in the following section.
Corollary 4.2. Given a prime $p > w = r + s + 1 > 1$ and an integer $n \geq 0$, we have

$$\frac{p^n}{(n(p-w))!} B_{n(p-w)}^{(r, s)} \equiv \left[w!(-1)^r \binom{r+s}{s}\right]^n \pmod{p} \quad (4.11)$$

and, in particular,

$$p B_{p-r-s-1}^{(r, s)} \equiv (-1)^{r+1} \frac{(r+s+1)!}{r!s!} \pmod{p}. \quad (4.12)$$

The congruence (4.12) follows immediately from (4.11) if we note that $(p-r-s-1)! \equiv (-1)^{r+s+1} / (r+s)! \pmod{p}$ by Wilson’s theorem. Congruence (4.12) can be seen as a partial analogue of a variant of the von Staudt–Clausen theorem, namely

$$p B_{2n} \equiv -1 \pmod{p}$$

if $p-1 | 2n$.

Corollary 4.3. Given a prime $p > w + 1$ and an integer $n \geq 0$, we have

$$\frac{p^n}{(n(p-w)+1)!} B_{n(p-w)+1}^{(r, s)} \equiv \left[w!(-1)^r \binom{r+s}{s}\right]^n \times \left(\frac{nw(r-s)}{w^2-1} - \frac{s+1}{w+1}\right) \pmod{p}. \quad (4.13)$$

Proof. We prove this by induction on $n$. The statement is true for $n = 0$ since $B_0^{(r, s)} = -(s+1)/(r+s+2)$. Suppose that the congruence holds for $0, 1, \ldots, n-1$. By (4.4) we have

$$b_p^{(r, s)}(n, 1) \equiv w!\left[-\binom{1+s}{s} b_p^{(r, s)}(n, 0) + \binom{p-r}{s} b_p^{(r, s)}(n-1, 0) + \binom{p-r-1}{s} b_p^{(r, s)}(n-1, 0) \right] \pmod{p}.$$ 

Since the left-hand sides of (4.9) and (4.10) are $b_p^{(r, s)}(n, 0)$, resp. $b_p^{(r, s)}(n, 1)$, we get from (4.9), (4.10), and the induction hypothesis that

$$b_p^{(r, s)}(n, 1) \equiv \left[w!(-1)^r \binom{r+s}{s}\right]^n \left(\frac{s+1}{w+1} + \frac{p-r}{p-r-s} + \frac{(n-1) w(r-s)}{w^2-1} - \frac{s+1}{w+1}\right) \pmod{p}.$$
It is now easy to verify that the term in parentheses above is congruent (mod \( p \)) to that in (4.13).

5. FURTHER RESULTS ON THE DENOMINATOR

The congruences of the previous section can be used to show that for \( n \) in certain residue classes, the inequality in (3.2) becomes an equality.

**Proposition 5.1.** If \( p \) is a prime, \( p > w = r + s + 1 > 1 \), and \( n \geq 0 \), then

\[
\alpha_p^{(r,s)}(n(p-w)) = n - v_p(n(p-w)).
\]  

(5.1)

If \( p \neq w+1 \) and either \( r \equiv s \pmod{p} \), or \( n \not\equiv \frac{(r+s)(r+1)}{(r+s+1)(r-s)} \pmod{p} \), then

\[
\alpha_p^{(r,s)}(n(p-w)+1) = n - v_p(n(p-w)+1).
\]  

(5.2)

In other words, (3.2) becomes an equality if \( n \equiv 0 \) or \( 1 \pmod{p-w} \).

**Proof.** By definition and by (4.2) we have

\[
-\alpha_p^{(r,s)}(n(p-w)) + n - v_p(n(p-w)) = \text{ord}_p(\alpha_p^{(r,s)}(n, 0)).
\]

Since \( p > w \), the right-hand side of (4.10) is not divisible by \( p \), and this proves (5.1). Similarly,

\[
-\alpha_p^{(r,s)}(n(p-w)+1) + n - v_p(n(p-w)+1) = \text{ord}_p(\alpha_p^{(r,s)}(n, 1)).
\]

Here we use (4.12), the right-hand side of which is not divisible by \( p \), which follows again from \( p > w \) and from the extra conditions. This completes the proof.

Next we investigate the question of when a given prime \( p > r+s+1 > 1 \) first appears in the denominator of \( B_n^{(r,s)} \), as \( n \) grows. In the case of the ordinary Bernoulli numbers, the von Staudt–Clausen theorem tells us that \( p \) divides the denominator of \( B_n \) if and only if \( p-1 \mid n \), so the first appearance is at \( n = p-1 \). In the case of the van der Pol numbers \( V_n = B_n^{(1,1)} \), Howard [8] showed that the first appearance is at \( n = p-3 \). This leads to the conjecture that in general, the first appearance of a prime \( p > r+s+1 > 1 \) in the denominator is at \( n = p-(r+s+1) \).

We will show that this is indeed the case. We begin with a preliminary result which can also be seen as an analogue to the von Staudt–Clausen Theorem.
Proposition 5.2. Let \( p \) be a prime with \( p > r + s + 1 > 1 \) and \( h \) be a positive integer such that \( hw < \frac{p}{p-(h+1)w} \). If \( n \) is in one of the intervals \([kp, (k+1)(p-w))\), then \( B^{(r,s)}_n \) is \( p \)-integral. Furthermore, if \( n \) is in the interval \([0, 2(p-w))\), then \( p^2 B^{(r,s)}_n \equiv 0 \) (mod \( p \)).

Proof. Let \( n \) be in one of the intervals \([kp, (k+1)(p-w))\), with \( 0 \leq k < h \). Then \( n = kp + m \) for some \( 0 \leq m < p-(k+1)w \). Now \( n < h(p-w) < hp \), and also \( h < p \) since by hypothesis we have \( hw < p \). Hence \( n < p^2 \) and therefore

\[
v_r(n) = \left\lfloor \frac{n}{p^3} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots = \left\lfloor \frac{kp+m}{p} \right\rfloor = k.
\]

Next, since \( n = kp + m = k(p-w) + kw + m \) and \( kw + m < kw + p - (k+1)w = p - w \), we have \( |n/(p-w)| = k \). Hence, by (3.2), \( a^{(r,s)}_p(n) \leq 0 \), i.e., \( B^{(r,s)}_n \) is \( p \)-integral.

For the second statement, suppose that \( 0 \leq n < 2(p-w) \). Then \( |n/(p-w)| < 2 \), and with \( n = kp + m \), \( 0 \leq k \leq 1 \) and \( 0 \leq m < p \), we have, as before, \( v_r(n) = k \) and thus, again by (3.2), \( a^{(r,s)}_p(n) \leq 1 \), which implies the second statement.

Proposition 5.3. The first \( n \) for which the prime \( p > r + s + 1 \) appears in the denominator of \( B^{(r,s)}_n \) is \( n = p - (r+s+1) \), and we have \( a^{(r,s)}_p(p - (r+s+1)) = 1 \).

Proof. By Proposition 5.2 we know that \( p \) does not appear in the denominator for \( n < p-w \). On the other hand, (4.11) shows that \( pB^{(r,s)}_p(2(p-w)) \equiv 0 \) (mod \( p \)), while by the second part of Proposition 5.2 we have \( p^2 B^{(r,s)}_p(2(p-w)) \equiv 0 \) (mod \( p \)); hence \( a^{(r,s)}_p(p - w) = 1 \).

So far we have only considered the case where \( p > r + s + 1 \). The final two results in this section, which we give without proof, deal with primes \( p \leq r + s + 1 \).

Proposition 5.4. Let \( p \leq r + s + 1 \) be a prime and \( n \geq 0 \). If \( r + s + 1 = a_0 + a_1 + \cdots + a_i + p \) (0 \( a_i < p \)) and \( m = a_0 + a_1 + \cdots + a_i + a_0 \), then

\[
a^{(r,s)}_p(n) \leq \left\lfloor \frac{mn}{p-1} \right\rfloor - v_r(n).
\]

The proof is similar to that of Proposition 3.1 and is given in [7] for \( s = 0 \). While it can be shown that (5.3) is an equality for certain pairs \( p \) and \( r + s \) (this is done in [7] for the special case \( s = 0 \)), in other cases the inequality (5.3) is far from the best possible. For instance, when \( p = r + s + 1 \) (and thus \( m = 1 \)), the right-hand side of (5.3) is unbounded as
\( n \) grows. In fact, it is easy to see that for \( n = p^a - 1 \) the right-hand side of (5.3) is equal to \( a \). On the other hand, we have the following result:

**Proposition 5.5.** If \( p = r + s + 1 \) is a prime, then \( B_n^{(r, s)} \) is \( p \)-integral for all \( n \geq 0 \).

The proof, by induction on \( n \), is again similar to that of Proposition 3.1.

6. AN IRREDUCIBILITY RESULT

The question of the irreducibility of Bernoulli and related polynomials has attracted a good deal of attention. Irreducibility results have been published for the Bernoulli and Euler polynomials, for Bernoulli polynomials of higher orders, and for Bernoulli polynomials associated with Dirichlet characters. A partial bibliography of such results can be found in [1]; see also [4]. Most of the published results use in their proofs the Eisenstein criterion or one of its variants, along with the von Staudt–Clausen Theorem or an analogue, and divisibility properties of binomial coefficients.

Since we now have analogues of the von Staudt–Clausen Theorem for the Bernoulli–Padé numbers, we can expect this approach to work for various classes of Bernoulli–Padé polynomials \( B_n^{(r, s)}(x) \). We restrict ourselves to one easy class, although it appears clear that other cases can be dealt with in a similar way.

**Proposition 6.1.** For any pair of nonnegative integers \( r \) and \( s \), the polynomial \( B_n^{(r, s)}(p^w-r-s-1)(x) \) is irreducible for all primes \( p > r+s+1 \).

**Proof.** For \( r = s = 0 \), this is a result of Carlitz [3]. Let now \( w = r + s + 1 > 1 \), and let \( d^{(r, s)}_p \) denote the least common multiple of the denominators of \( B_k^{(r, s)} \), ..., \( B_{p-w}^{(r, s)} \). By the relation (2.2) we have

\[
\frac{1}{x} \sum_{k=0}^{p-w} \binom{p-w}{k} d^{(r, s)}_p B_k^{(r, s)} x^k = d^{(r, s)}_p x^{p-w} B_{p-w}^{(r, s)} \left( \frac{1}{x} \right),
\]

and clearly this polynomial has integer coefficients. Now, by Proposition 5.3 we know that \( B_k^{(r, s)} \) is \( p \)-integral for \( 0 \leq k < p-w \) and that \( \alpha_p^{(r, s)}(p-w) = 1 \). Hence \( p \parallel d^{(r, s)}_p \). Since \( p \) does not divide the binomial coefficients in (6.1), we see that the leading coefficient of the polynomial is not divisible by \( p \), all the other coefficients are divisible by \( p \), but the constant coefficient is not divisible by \( p^2 \). Hence Eisenstein’s irreducibility criterion applies, and the polynomial in (6.1) and consequently \( B_{p-w}^{(r, s)}(x) \) are irreducible.
REFERENCES