Decidability of Innermost Termination and Context-Sensitive Termination for Semi-Constructor Term Rewriting Systems

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Abstract
Yi and Sakai [13] showed that the termination problem is a decidable property for the class of semi-constructor term rewriting systems, which is a superclass of the class of right-ground term rewriting systems. Decidability was shown by the fact that every non-terminating TRS in the class has a loop. In this paper we modify the proof of [13] to show that both innermost termination and $\mu$-termination are decidable properties for the class of semi-constructor TRSs.

Keywords: Context-Sensitive Termination, Dependency Pair, Innermost Termination

1 Introduction
Termination is one of the central properties of term rewriting systems (TRSs for short), where we say a TRS terminates if it does not admit any infinite reduction sequence. Since termination is undecidable in general, several decidable classes have been studied [6,8,9,12,13]. The class of semi-constructor TRSs is one of them [13], where a TRS is in this class if for every right-hand side of rules all its subterms having a defined symbol at root position are ground.

Innermost reduction, the strategy which rewrites innermost redexes, is used for call-by-value computation. Context-sensitive reduction is a strategy in which rewratable positions are indicated by specifying arguments of function symbols. Some non-terminating TRSs are terminating by context-sensitive reduction without loss of computational ability. The termination property with respect to innermost

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doi:10.1016/j.entcs.2008.03.051
(resp. context-sensitive) reduction is called innermost (resp. context-sensitive) termination. Since innermost termination and context-sensitive termination are also undecidable in general, methods for proving these terminations have been studied [2,4].

In this paper, we prove that innermost termination and context-sensitive termination for semi-constructor TRSs are decidable properties. We show that context-sensitive termination for \( \mu \)-semi-constructor TRSs having no infinite variable dependency chain is a decidable property. We also extend the classes by using dependency graphs.

2 Preliminaries

We assume the reader is familiar with the standard definitions of term rewriting systems [5], dependency pairs [4], and context-sensitive rewriting [2]. Here we just review the main notations used in this paper.

A signature \( \mathcal{F} \) is a set of function symbols, where every \( f \in \mathcal{F} \) is associated with a non-negative integer by an arity function: \( \text{arity}: \mathcal{F} \rightarrow \mathbb{N} \). The set of all terms built from a signature \( \mathcal{F} \) and a countably infinite set \( \mathcal{V} \) of variables such that \( \mathcal{F} \cap \mathcal{V} = \emptyset \), is represented by \( \mathcal{T}(\mathcal{F}, \mathcal{V}) \). The set of ground terms is \( \mathcal{T}(\mathcal{F}, \emptyset) \). The set of variables occurring in a term \( t \) is denoted by \( \text{Var}(t) \).

The set of all positions in a term \( t \) is denoted by \( \mathcal{P}(t) \) and \( \varepsilon \) represents the root position. \( \mathcal{P}(t) \) is: \( \mathcal{P}(t) = \{ \varepsilon \} \) if \( t \in \mathcal{V} \), and \( \mathcal{P}(t) = \{ \varepsilon \} \cup \{ iu \mid 1 \leq i \leq n, u \in \mathcal{P}(t_i) \} \) if \( t = f(t_1, \ldots, t_n) \). Let \( C \) be a context with a hole \( \square \). We write \( C[t]_p \) for the term obtained from \( C \) by replacing \( \square \) at position \( p \) with a term \( t \). We sometimes write \( C[t] \) for \( C[t]_p \) by omitting the position \( p \). We say \( t \) is a subterm of \( s \) if \( s = C[t] \) for some context \( C \). We denote the subterm relation by \( \sqsubseteq \), that is, \( t \sqsubseteq s \) if \( t \) is a subterm of \( s \), and \( t < s \) if \( t \sqsubseteq s \) and \( t \neq s \). The root symbol of a term \( t \) is denoted by \( \text{root}(t) \).

A substitution \( \theta \) is a mapping from \( \mathcal{V} \) to \( \mathcal{T}(\mathcal{F}, \mathcal{V}) \) such that the set \( \text{Dom}(\theta) = \{ x \in \mathcal{V} \mid \theta(x) \neq x \} \) is finite. We usually identify a substitution \( \theta \) with the set \( \{ x \mapsto \theta(x) \mid x \in \text{Dom}(\theta) \} \) of variable bindings. In the following, we write \( t \theta \) instead of \( \theta(t) \).

A rewrite rule \( l \rightarrow r \) is a directed equation which satisfies \( l \notin \mathcal{V} \) and \( \text{Var}(r) \subseteq \text{Var}(l) \). A term rewriting system TRS is a finite set of rewrite rules. A redex is a term \( \theta \) for a rule \( l \rightarrow r \) and a substitution \( \theta \). A term containing no redex is called a normal form. A substitution \( \theta \) is normal if \( x \theta \) is in normal forms for every \( x \). The reduction relation \( \xrightarrow{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V}) \) associated with a TRS \( R \) is defined as follows: \( s \xrightarrow{R} t \) if there exist a rewrite rule \( l \rightarrow r \in R \), a substitution \( \theta \), and a context \( C[p] \) such that \( s = C[l \theta]_p \) and \( t = C[r \theta]_p \), we say that \( s \) is reduced to \( t \) by contracting redex \( l \theta \). We sometimes write \( \xrightarrow{R} \) for \( \xrightarrow{L} \) by displaying the position \( p \).

A redex is innermost if all its proper subterms are in normal forms. If \( s \) is reduced to \( t \) by contracting an innermost redex, then \( s \xrightarrow{R} t \) is said to be an innermost reduction denoted by \( s \xrightarrow{in,R} t \).
Proposition 2.1 For a TRS $R$, if there is a reduction $s \xrightarrow{m,R} t$, then $C[s] \xrightarrow{m,R} C[t]$ for any context $C$.

A mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is a replacement map (or $\mathcal{F}$-map) if $\mu(f) \subseteq \{1, \ldots, \text{arity}(f)\}$. The set of $\mu$-replacing positions $\text{Pos}_\mu(t)$ of a term $t$ is: $\text{Pos}_\mu(t) = \{\varepsilon\}$, if $t \in \mathcal{V}$ and $\text{Pos}_\mu(t) = \{\varepsilon\} \cup \{iu \mid i \in \mu(f), u \in \text{Pos}_\mu(t_i)\}$, if $t = f(t_1, \ldots, t_n)$. A context $C[\cdot]_p$ is $\mu$-replacing denoted by $C[\cdot]_p^\mu$ if $p \in \text{Pos}_\mu(C)$. The set of all $\mu$-replacing variables of $t$ is $\text{Var}_\mu(t) = \{x \in \text{Var}(t) \mid \exists C, C[\cdot]_p^\mu = t\}$. The $\mu$-replacing subterm relation $\leq_\mu$ is given by $s \leq_\mu t$ if there is $p \in \text{Pos}_\mu(t)$ such that $t = C[s]_p$. A context-sensitive rewriting system is a TRS with an $\mathcal{F}$-map. If $s \xrightarrow{p,R}$ and $p \in \text{Pos}_\mu(s)$, then $s \xrightarrow{\mu,R}$ is said to be a $\mu$-reduction denoted by $s \xrightarrow{\mu,R} t$.

Let $\rightarrow$ be a binary relation on terms, the transitive closure of $\rightarrow$ is denoted by $\rightarrow^+$. The transitive and reflexive closure of $\rightarrow$ is denoted by $\rightarrow^*$. If $s \xrightarrow{\star} t$, then we say that there is a $\rightarrow$-sequence starting from $s$ to $t$ or $t$ is $\rightarrow$-reachable from $s$. We write $s \xrightarrow{\rightarrow^k} t$ if $t$ is $\rightarrow$-reachable from $s$ with $k$ steps. A term $t$ terminates with respect to $\rightarrow$ if there exists no infinite $\rightarrow$-sequence starting from $t$.

Example 2.2 Let $R_1 = \{g(x) \rightarrow h(x), h(d) \rightarrow g(c), c \rightarrow d\}$ and $\mu_1(g) = \mu_1(h) = \emptyset$. A $\mu_1$-reduction sequence starting from $g(d)$ is $g(d) \xrightarrow{\mu_1,R_1} h(d) \xrightarrow{\mu_1,R_1} g(c)$. We can not reduce $g(c)$ to $g(d)$ because $c$ is not a $\mu_1$-replacing subterm of $g(c)$.

Proposition 2.3 For a TRS $R$ and $\mathcal{F}$-map $\mu$, if there is a reduction $s \xrightarrow{\mu,R} t$, then $C[s]_p^\mu \xrightarrow{\mu,R} C[t]_p^\mu$ for any $\mu$-replacing context $C[\cdot]_p^\mu$.

For a TRS $R$ and $\mathcal{F}$-map $\mu$, we say that $R$ terminates (resp. innermost terminates, $\mu$-terminates) if every term terminates with respect to $\rightarrow_R$ (resp. $\xrightarrow{\mu,R}$).

For a TRS $R$, a function symbol $f \in \mathcal{F}$ is defined if $f = \text{root}(l)$ for some rule $l \rightarrow r \in R$. The set of all defined symbols of $R$ is denoted by $D_R = \{\text{root}(l) \mid l \rightarrow r \in R\}$. A term $t$ has a defined root symbol if $\text{root}(t) \in D_R$.

Let $R$ be a TRS over a signature $\mathcal{F}$. The signature $\mathcal{F}^\sharp$ denotes the union of $\mathcal{F}$ and $D_R^\sharp = \{f^\sharp \mid f \in D_R\}$ where $\mathcal{F} \cap D_R^\sharp = \emptyset$ and $f^\sharp$ has the same arity as $f$. We call these fresh symbols dependency pair symbols. We define a notation $t^\sharp$ by $t^\sharp = f^\sharp(t_1, \ldots, t_n)$ if $t = f(t_1, \ldots, t_n)$ and $f \in D_R$, $t^\sharp = t$ if $t \in V$. If $l \rightarrow r \in R$ and $u$ is a subterm of $r$ with a defined root symbol and $u \notin l$, then the rewrite rule $l^\sharp \rightarrow u^\sharp$ is called a dependency pair of $R$. The set of all dependency pairs of $R$ is denoted by $\text{DP}(R)$.

Example 2.4 Let $R_2 = \{a \rightarrow g(f(a)), f(f(x)) \rightarrow h(f(a), f(x))\}$. We have $\text{DP}(R_2) = \{a^\sharp \rightarrow a^\sharp, a^\sharp \rightarrow f^\sharp(a), f^\sharp(g(x)) \rightarrow a^\sharp, f^\sharp(g(x)) \rightarrow f^\sharp(a)\}$.

A rule $l \rightarrow r$ is said to be right ground if $r$ is ground. Right-ground TRSs are TRSs that consist of right-ground rules.

Definition 2.5 [Semi-Constructor TRS] A TRS $R$ is a semi-constructor system if every rule in $\text{DP}(R)$ is right ground.
Remark 2.6 The class of semi-constructor TRSs in this paper is a larger class of semi-constructor TRSs by the original definition because a rule \( l^\sharp \to u^\sharp \) is not dependency pair if \( u \not< l \). The original definition of semi-constructor TRS is as follows [11]. A term \( t \in T(F,V) \) is a semi-constructor term if every term \( s \) such that \( s \leq t \) and root\((s) \in D_R \) is ground. A TRS \( R \) is a semi-constructor system if \( r \) is a semi-constructor term for every rule \( l \to r \in R \).

Example 2.7 The TRS \( R_2 \) (in Example 2.4) is a semi-constructor TRS but not in the original definition.

3 Decidability of Innermost Termination for Semi-Constructor TRSs

Decidability of termination for semi-constructor TRSs is proved based on the observation that there exists an infinite reduction sequence having a loop if it is not terminating [13]. In this section, we prove the decidability of innermost termination in a similar way.

Definition 3.1 [loop] Let \( \to \) be a relation on terms. A reduction sequence loops if it contains \( t \to^+ C[t] \) for some context \( C \), and head-loops if containing \( t \to^+ t \).

Proposition 3.2 If there exists an innermost sequence that loops, then there exists an infinite innermost sequence.

Definition 3.3 [Innermost DP-chain] For a TRS \( R \), a sequence of the elements of \( \text{DP}(R) \) \( s_1^\sharp \to t_1^\sharp, s_2^\sharp \to t_2^\sharp, \ldots \) is an innermost dependency chain if there exist substitutions \( \tau_1, \tau_2, \ldots \) such that \( s_i^\sharp \tau_i \) is in normal forms and \( t_i^\sharp \tau_i \to^* \) in,\( R \) \( s_{i+1}^\sharp \) holds for every \( i \).

Theorem 3.4 ([4]) For a TRS \( R \), \( R \) does not innermost terminate if and only if there exists an infinite innermost dependency chain.

Let \( \mathcal{M}_{\geq} \) denote the set of all minimal non-terminating terms for a relation on terms \( \to \) and an order on terms \( \geq \).

Definition 3.5 [C-min] For a TRS \( R \), let \( C \subseteq \text{DP}(R) \). An infinite reduction sequence in \( R \cup C \) in the form \( t_1^\sharp \to_{\text{in},R \cup C} t_2^\sharp \to_{\text{in},R \cup C} t_3^\sharp \to_{\text{in},R \cup C} \cdots \) with \( t_i \in \mathcal{M}_{\geq}^{\text{in},R} \) for all \( i \geq 1 \) is called a C-min innermost reduction sequence. We use \( C_{\text{min}}(t^\sharp) \) to denote the set of all C-min innermost reduction sequences starting from \( t^\sharp \).

Proposition 3.6 ([4]) Given a TRS \( R \), the following statements hold:

(i) If there exists an infinite innermost dependency chain, then \( C_{\text{min}}(t^\sharp) \neq \emptyset \) for some \( C \subseteq \text{DP}(R) \) and \( t \in \mathcal{M}_{\geq}^{\text{in},R} \).

(ii) For any sequence in \( C_{\text{min}}(t^\sharp) \), reduction by rules of \( R \) takes place below the root while reduction by rules of \( C \) takes place at the root.
(iii) For any sequence in $C_{min}^{in}(t^\sharp)$, there is at least one rule in $C$ which is applied infinitely often.

**Lemma 3.7 ([4])** For two terms $s$ and $s'$, $s^{\sharp} \xrightarrow{in,R \cup C} s'^{\sharp}$ implies $s \xrightarrow{in,R} C[s']$ for some context $C$.

**Proof.** We use induction on the number $n$ of reduction steps in $s^{\sharp} \xrightarrow{in,R \cup C} s'^{\sharp}$. In the case that $n = 0$, $s \xrightarrow{in,R} C[s']$ holds where $C = \Box$. Let $n \geq 1$. Then we have $s^{\sharp} \xrightarrow{in,R \cup C} s''^{\sharp} \xrightarrow{in,R \cup C} s'^{\sharp}$ for some $s''^{\sharp}$. By the induction hypothesis, $s \xrightarrow{in,R} C[s'']$.

- Consider the case that $s''^{\sharp} \xrightarrow{in,R} s'^{\sharp}$. Since $s'' \xrightarrow{in,R} s'$, we have $C[s''] \xrightarrow{in,R} C[s']$ by Proposition 2.1. Hence $s \xrightarrow{in,R} C[s']$.

- Consider the case that $s''^{\sharp} \xrightarrow{in,C} s'^{\sharp}$. Since $s''$ is a normal form with respect to $\rightarrow_R$, we have $s'' \xrightarrow{in,R} C'[s']$ by the definition of dependency pairs. $C[s''] \xrightarrow{in,R} C[C'[s']]$, by Proposition 2.1. Hence $s \xrightarrow{in,R} C[C'[s']]$.

**Lemma 3.8** For a semi-constructor TRS $R$, the following statements are equivalent:

(i) $R$ does not innermost terminate.

(ii) There exists $l^\sharp \rightarrow u^\sharp \in \text{DP}(R)$ such that sq head-loops for some $C \subseteq \text{DP}(R)$ and sq $\in C_{min}^{in}(u^\sharp)$.

**Proof.** (ii) $\Rightarrow$ (i): It is obvious from Lemma 3.7, and Proposition 3.2. (i) $\Rightarrow$ (ii) : By Theorem 3.4 there exists an infinite innermost dependency chain. By Proposition 3.6(i), there exists a sequence $sq \in C_{min}^{in}(t^\sharp)$. By Proposition 3.6(ii),(iii), there exists some rule $l^\sharp \rightarrow u^\sharp \in C$, which is applied at root position in sq infinitely often. By Definition 2.5, $u^\sharp$ is ground. Thus sq contains a subsequence $u^\sharp \xrightarrow{in,R \cup \text{DP}(R)} \ast \rightarrow \{l \rightarrow u^\sharp\} \rightarrow \ast$, which head-loops.

**Theorem 3.9** Innermost termination of semi-constructor TRSs is decidable.

**Proof.** The decision procedure for the innermost termination of a semi-constructor TRS $R$ is as follows: consider all terms $u_1, u_2, \ldots, u_n$ corresponding to the right-hand sides of $\text{DP}(R) = \{l^\sharp \rightarrow u^\sharp \mid 1 \leq i \leq n\}$, and simultaneously generate all innermost reduction sequences with respect to $R$ starting from $u_1, u_2, \ldots, u_n$. The procedure halts if it enumerates all reachable terms exhaustively or it detects a looping reduction sequence $u_i \xrightarrow{in,R} C[u_i]$ for some $i$.

Suppose $R$ does not innermost-terminate. By Lemma 3.8 and 3.7, we have a looping reduction sequence $u_i \xrightarrow{in,R} C[u_i]$ for some $i$ and $C$, which we eventually detect. If $R$ innermost terminates, then the execution of the reduction sequence generation eventually stops since the reduction relation is finitely branching. In the latter case, the procedure does not detect a looping sequence, otherwise it contradicts Proposition 3.2. Thus the procedure decides innermost termination of $R$ in finitely many steps. $\square$
4 Decidability of Context-Sensitive Termination for Semi-Constructor TRSs

The proof of decidability for innermost termination is straightforward. However, the proof for context-sensitive termination is not so straightforward because of the existence of a dependency pair whose right-hand side is variable.

Definition 4.1 [μ-Loop] Let → be a relation on terms and μ be an F-map. A reduction sequence μ-loops if it contains t →+ C[μ]r for some context C[μ].

Example 4.2 Let R3 = {a → g(f(a)), f(g(x)) → h(f(a), x)}, μ2(f) = {1}, μ2(g) = ∅ and μ2(h) = {1, 2}. The μ2-reduction sequence with respect to R3 f(a) →μ2,R3 f(g(f(a))) →μ2,R3 h(f(a), f(a)) →μ2,R3 ... is μ2-looping.

Proposition 4.3 If there exists a μ-looping μ-reduction sequence, then there exists an infinite μ-reduction sequence.

Definition 4.4 [Context-Sensitive Dependency Pairs [2]] Let R be a TRS and μ be an F-map. We define DP(R, μ) = DP(F(R, μ) ∪ DPV(R, μ)) to be the set of context-sensitive dependency pairs where:

\[ DP(F(R, μ) = \{ l^d → u^d | l → r ∈ R, u ≤μ r, \text{root}(u) ∈ D_R, u ∉μ l \} \]

\[ DP_V(R, μ) = \{ l^d → x | l → r ∈ R, x ∈ \text{Var}_μ(r) \setminus \text{Var}_μ(l) \} \]

Example 4.5 Consider TRS R3 and F-map μ2 (in Example 4.2). DP(F(R3, μ2)) = {f^d(g(x)) → f^d(a)} and DPV(R3, μ2) = {f^d(g(x)) → x}.

For a given TRS R and an F-map μ, we define μ^d by μ^d(f) = μ(f) for f ∈ F, and μ^d(f^d) = μ(f) for f ∈ D_R. We write s ⪰μ t^d for s ⪰μ t.

Definition 4.6 [Context-Sensitive Dependency Chain] For a TRS R and F-map μ, a sequence of the elements of DP(R, μ) s^d_1 → t^d_1, s^d_2 → t^d_2, ... is a context-sensitive dependency chain if there exist substitutions τ_1, τ_2, ... satisfying both:

- \( t^d_i τ_i →_{μ^d,R} s^d_{i+1} τ_{i+1} \), if \( t^d_i ∉ V \)
- \( xτ_i →_{μ^d,R} s^d_{i+1} τ_{i+1} \) for some term \( u_i \), if \( t^d_i = x \).

Example 4.7 Consider TRS R3 and F-map μ2 (in Example 4.2). f(a), f(g(f(a))) ∈ M[μ2,R3] and f(f(a)), h(f(a), f(a)) ∉ M[μ2,R3].

Theorem 4.8 [2] For a TRS R and an F-map μ, there exists an infinite context-sensitive dependency chain if and only if R does not μ-terminate.

Let R be a TRS, μ be an F-map and C ⊆ DP(R, μ). We define (μ,R,C) as

\( (\rightarrow_μ, \cup (\rightarrow_{μ^d,C} \cup \rightarrow_{μ^d,R} )) \) where C_F = C ∩ DP(F(R, μ)) and C_V = C ∩ DPV(R, μ).

Definition 4.9 [μ-C-min] Let R be a TRS, μ be an F-map. An infinite sequence of terms in the form \( t^d_1 →_{μ,R,C} t^d_2 →_{μ,R,C} t^d_3 →_{μ,R,C} ... \) is called a C-min μ-sequence if
For any sequence in chain (FFIVDC) if and only if there exists no infinite context-sensitive dependency pairs and in DP

\[ \text{Context-Sensitive Semi-Constructor TRS} \]

For an \( \mathcal{F} \)-map \( \mu \), the class of \( \mu \)-semi-constructor TRSs is a superclass of the \( \mu \)-map \( \mu \)-semi-constructor TRSs.

Example 4.10 Let \( C = \text{DP}(R_3, \mu_2) \), the sequence \( f^2(a) \xrightarrow{\mu_2,R_3,C} f^2(g(f(a))) \xrightarrow{\mu_2,R_3,C} \cdots \) is a \( C \)-min \( \mu \)-sequence.

Proposition 4.11 ([2]) Given a TRS \( R \) and an \( \mathcal{F} \)-map \( \mu \), the following statements hold:

(i) If there exists an infinite context-sensitive dependency chain, then \( C_{\text{min}}(t^2) \neq \emptyset \) for some \( C \subseteq \text{DP}(R, \mu) \) and \( t \in \mathcal{M}_{\geq \mu}^{\mu,R} \).

(ii) For any sequence in \( C_{\text{min}}(t^2) \), a reduction with \( t^2 \) takes place below the root while reductions with \( \mu^t,C \) and \( \mu^t,C \) take place at the root.

(iii) For any sequence in \( C_{\text{min}}(t^2) \), there is at least one rule in \( C \) which is applied infinitely often.

Lemma 4.12 For two terms \( s \) and \( t \), \( s^2 \xrightarrow{\mu,R,C} t^2 \) implies \( s \xrightarrow{\mu,R} C_\mu[t] \) for some context \( C_\mu \).

Proof. We use induction on the length \( n \) of the sequence. In the case that \( n = 0 \), it holds trivially. Let \( n \geq 1 \). Then we have \( s^2 \xrightarrow{\mu,R,C} u^2 \xrightarrow{\mu,R,C} t^2 \) for some \( u \).

\begin{itemize}
  \item In the case that \( u^2 \xrightarrow{\mu^t,C} t^2 \), we have \( u \xrightarrow{\mu,R} C_\mu[t] \) by the definition of dependency pairs.
  \item In the case that \( u^2 \xrightarrow{\mu^t,C} v \xrightarrow{\mu^t} t^2 \), we have \( u \xrightarrow{\mu,R} C_\mu[v] \) by the definition of dependency pairs and \( v = C_\mu[t] \). Thus \( u \xrightarrow{\mu,R} C_\mu\mu[t] = C_\mu^t[t] \).
  \item In the case that \( u^2 \xrightarrow{\mu^t,R} t^2 \), we have \( u \xrightarrow{\mu,R} C_\mu[t] \) for \( C_\mu[t] = \emptyset \).
\end{itemize}

Therefore \( s \xrightarrow{\mu,R} C_\mu[u] \xrightarrow{\mu,R} C_\mu[C_\mu[t]] \) by the induction hypothesis and Proposition 2.3.

4.1 Context-Sensitive Semi-Constructor TRS

In this subsection, we discuss the decidability of \( \mu \)-termination for context-sensitive semi-constructor TRSs.

Definition 4.13 [Context-Sensitive Semi-Constructor TRS] For an \( \mathcal{F} \)-map \( \mu \), a TRS \( R \) is a context-sensitive semi-constructor (\( \mu \)-semi-constructor) TRS if all rules in \( \text{DP}_x(R, \mu) \) are right ground.

For an \( \mathcal{F} \)-map \( \mu \), the class of \( \mu \)-semi-constructor TRSs is a superclass of the class of semi-constructor TRSs from Definition 2.5 and 4.13.

For a TRS \( R \) and \( \mathcal{F} \)-map \( \mu \), we say \( R \) is free from the infinite variable dependency chain (FFIVDC) if and only if there exists no infinite context-sensitive dependency chain.
chain consisting of only elements in \( \text{DP}_V(R, \mu) \). If \( R \) is FFIVDC, then \( C_{\text{min}}(t^z) = \emptyset \) for any \( C \subseteq \text{DP}_V(R, \mu) \) and any term \( t \).

**Lemma 4.14** Let \( \mu \) be an \( \mathcal{F} \)-map. If a \( \mu \)-semi-constructor TRS \( R \) is FFIVDC, then the following statements are equivalent:

(i) \( R \) does not \( \mu \)-terminate.

(ii) There exists \( l^z \rightarrow u^z \in \text{DP}_\mathcal{F}(R, \mu) \) such that \( sq \) head-loops for \( C \subseteq \text{DP}(R, \mu) \) and some \( sq \in C_{\text{min}}(u^z) \).

**Proof.** ((ii) \( \Rightarrow \) (i)) : It is obvious from Lemma 4.12, and Proposition 4.3. ((i) \( \Rightarrow \) (ii)) : By Theorem 4.8 there exists an infinite context-sensitive dependency chain. By Proposition 4.11(i), there exists a sequence \( sq \in C_{\text{min}}(u^z) \). By Proposition 4.11(ii),(iii) and the fact that \( R \) is FFIVDC, there is some rule in \( l^z \rightarrow u^z \in C \) which is applied at the root position in \( sq \) infinitely often.

By Definition 4.13, \( u^z \) is ground. Thus \( sq \) contains a subsequence \( u^z \xrightarrow{\mu,R,C} u^z \), which head-loops and is in \( C_{\text{min}}(u^z) \).

**Theorem 4.15** Let \( \mu \) be an \( \mathcal{F} \)-map. If a \( \mu \)-semi-constructor TRS \( R \) is FFIVDC, then \( \mu \)-termination of \( R \) is decidable.

**Proof.** The decision procedure for \( \mu \)-termination of a \( \mu \)-semi-constructor TRS \( R \) is as follows: consider all terms \( u_1, u_2, \ldots, u_n \) corresponding to the right-hand sides of \( \text{DP}_\mathcal{F}(R, \mu) = \{ l^z_i \rightarrow u^z_i \mid 1 \leq i \leq n \} \), and simultaneously generate all \( \mu \)-reduction sequences with respect to \( R \) starting from \( u_1, u_2, \ldots, u_n \). The procedure halts if it enumerates all reachable terms exhaustively or it detects a \( \mu \)-looping reduction sequence \( u_i \xrightarrow{\mu,R,C} C_{\mu} [u_i] \) for some \( i \).

Suppose \( R \) does not \( \mu \)-terminate. By Lemma 4.14 and 4.12, we have a \( \mu \)-looping reduction sequence \( u_i \xrightarrow{\mu,R,C} C_{\mu} [u_i] \) for some \( i \) and \( C_{\mu} \), which we eventually detect. If \( R \) \( \mu \)-terminates, then the execution of the reduction sequence generation eventually stops since the reduction relation is finitely branching. In the latter case, the procedure does not detect a \( \mu \)-looping sequence, otherwise it contradicts to Proposition 4.3. Thus the procedure decides \( \mu \)-termination of \( R \) in finitely many steps. \( \square \)

We have to check the FFIVDC property in order to use Theorem 4.15. However, The FFIVDC property is not necessarily decidable. The following proposition provides a sufficient condition. The set \( \text{DP}^1_V(R, \mu) \) is a subset of \( \text{DP}_V(R, \mu) \) defined as follows:

\[ \text{DP}^1_V(R, \mu) = \{ f^z(u_1, \ldots, u_k) \rightarrow x \in \text{DP}_V(R, \mu) \mid \exists i, 1 \leq i \leq k, i \notin \mu(f), x \in \text{Var}(u_i) \} \]

**Proposition 4.16** ([2]) Let \( R \) be a TRS, \( \mu \) be an \( \mathcal{F} \)-map and \( C \subseteq \text{DP}^1_V(R, \mu) \).

\( C_{\text{min}}(t^z) = \emptyset \) for any term \( t \).

If \( \text{DP}^1_V(R, \mu) = \text{DP}_V(R, \mu) \) then \( R \) is FFIVDC by Proposition 4.16. Hence the following corollary directly follows from Theorem 4.15 and the fact that \( \text{DP}^1_V(R, \mu) = \text{DP}_V(R, \mu) \) is decidable.
Corollary 4.17 For an $\mathcal{F}$-map $\mu$ and a $\mu$-semi-constructor TRS $R$, $\mu$-termination of $R$ is decidable if $\text{DP}_V(R, \mu) = \text{DP}_V^1(R, \mu)$.

4.2 Semi-Constructor TRS

In this subsection, we try to remove FFIVDC condition from the results of the previous subsection. As a result, it appears that $\mu$-termination of semi-constructor TRSs (not $\mu$-semi-constructor) is decidable. The arguments of following Lemma 4.18 and 4.19 are similar to those of Lemma 3.5 and Proposition 3.6 in [3].

Lemma 4.18 Consider a reduction $s^\# = C_{\mu^3, R}^{[\theta]} t^\# = C_{\mu^3}[\theta]^p = C''[u]^q$ where $s, u \in \mathcal{M}_{\geq \mu^3}^R$ and $q \in \mathcal{P}os(t) \backslash \mathcal{P}os_\mu(t)$. Then one of the following statements holds

(i) $s \triangleright u$

(ii) $v\theta = u$ and $r = C''[v]^q$, for some $\theta$, $v \notin \mathcal{V}$, $C''$, and $q' \in \mathcal{P}os(r) \backslash \mathcal{P}os_\mu(r)$

Proof. Since $q \in \mathcal{P}os(t) \backslash \mathcal{P}os_\mu(t)$, $p$ is not below or equal to $q$. In the case that $p$ and $q$ are in parallel positions, $s \triangleright u$ trivially holds. In the case that $p$ is above $q$, it is obvious that $s \triangleright u$ holds or, $v\theta = u$ and $r = C''[v]^q$ for some $\theta$, $v \notin \mathcal{V}$, $C''$. Here the fact that $q' \in \mathcal{P}os(r) \backslash \mathcal{P}os_\mu(r)$ follows from $p \in \mathcal{P}os_\mu(t)$ and $q \notin \mathcal{P}os_\mu(t)$. □

Lemma 4.19 Let $R$ be a semi-constructor TRS, $\mu$ be an $\mathcal{F}$-map. For a $C$-min $\mu$-sequence $s_1^\# = C_{\mu^3, R}^{[\theta]} t_1^\# = C_{\mu^3}[\theta]^p, s_2^\# = C_{\mu^3, \mathcal{C}_V}^{[\theta]} t_2^\# = C_{\mu^3, \mathcal{C}_V}^{[\theta]} u_2^\# \cdots$ with no reduction by rules in $\mathcal{C}_F$, one of the following statements holds for each $i$:

(i) $s_i \triangleright s_{i+1}$

(ii) There exists $l^\# \rightarrow s_{i+1}^\# \in \text{DP}(R)$ for some $l$

Proof. Since $t_i^\# \rightarrow u_i^\# \geq \mu^3 s_{i+1}^\#$, we have $t_i^\# = C[s_{i+1}]^q$ for some $q \in \mathcal{P}os(t_i) \backslash \mathcal{P}os_\mu(t_i)$. We show (i) or the following (ii') by induction on the number $n$ of steps of $s_i^\# \rightarrow \cdots \rightarrow t_i^\# = C[s_{i+1}]^q$.

(ii') There exists a reduction by $l \rightarrow r$ in $s_i^\# \rightarrow t_i^\#$ and $t_i^\# \rightarrow s_{i+1}^\# \in \text{DP}(R)$

• In the case that $n = 0$, trivially $s_i = t_i \triangleright s_{i+1}$.

• In the case that $n > 0$, let $s_i^\# \rightarrow s_{i+1}^\# \rightarrow t_i^\# = C[s_{i+1}]^q$. By the induction hypothesis, $s_i^\# \triangleright s_{i+1}$ or the condition (ii') follows. In the former case, we have $s_i \triangleright s_{i+1}$, or, we have $v\theta = s_{i+1}$ and $r = C''[v]^q$ for some $l \rightarrow r \in R$, $\theta$, $v \notin \mathcal{V}$, $C''$ and $q' \in \mathcal{P}os(r) \backslash \mathcal{P}os_\mu(r)$ by Lemma 4.18. Hence $v\theta = v$ due to root($s_{i+1}$) $\in \mathcal{D}_R$ and Definition 2.5. Therefore (ii') follows. □

One may think that the Lemma 4.19 would hold even if $\text{DP}(R)$ were replaced with $\text{DP}(R, \mu)$. However, it does not hold as shown by the following counter example.

Example 4.20 Consider the semi-constructor TRS $R_4 = \{f(g(x)) \rightarrow x, g(b) \rightarrow g(f(g(b)))\}$, $\mu_3(f) = \{1\}$ and $\mu_3(g) = \emptyset$. There exists a $C$-min $\mu_3$-sequence

\[\underbrace{\cdots \rightarrow f(g(x)) \rightarrow \cdots}_{1} \rightarrow g(b) \rightarrow g(f(g(b))) \rightarrow \cdots}^{2}\]
However there exists no dependency pair having $f^\sharp(g(b))$ in the right-hand side in $\text{DP}(R, \mu)$.

**Lemma 4.21** For a semi-constructor TRS $R$ and an $\mathcal{F}$-map $\mu$, the following statements are equivalent:

(i) $R$ does not $\mu$-terminate.

(ii) There exists $l^\sharp \to u^\sharp \in \text{DP}(R)$ such that $sq$ head-loops for $C \subseteq \text{DP}(R, \mu)$ and some $sq \in C_{\mu_{\text{min}}}(u^\sharp)$.

**Proof.** ($\text{(ii) } \Rightarrow \text{(i)}$) : It is obvious from Lemma 4.12, and Proposition 4.3. ($\text{(i) } \Rightarrow \text{(ii)}$) : By Theorem 4.8 there exists a context-sensitive dependency chain. By Proposition 4.11(i), there exists a sequence $sq \in C_{\mu_{\text{min}}}(l^\sharp)$. By Proposition 4.11(ii),(iii), there exists a rule in $C$ applied at root position in $sq$ infinitely often.

- Consider the case that there exists a rule $l^\sharp \to r^\sharp \in C_{\mathcal{F}}$ with infinite use in $sq$. Since $u$ is ground by Proposition 4.11(ii) and $C_{\mathcal{F}} \subseteq \text{DP}(R)$, $sq$ has a subsequence $u^\sharp \hookrightarrow \mu,R,C \mu + u^\sharp$.

- Otherwise, $sq$ has an infinite subsequence without the use of the rules in $C_{\mathcal{F}}$. The subsequence is in $C_{\mu_{\text{min}}}(s^\sharp)$ for some $s^\sharp$. Then the condition (ii) of Lemma 4.19 holds for infinitely many $i$’s; otherwise, we have an infinite sequence $s_k \triangleright s_{k+1} \triangleright \cdots$ for some $k$, which is a contradiction. Hence there exists a $l^\sharp \to u^\sharp \in \text{DP}(R)$ such that $u^\sharp$ occurs more than once in $sq$. Thus the sequence $u^\sharp \hookrightarrow \mu,R,C \mu + u^\sharp$ appears in $sq$. $\square$

**Theorem 4.22** The property $\mu$-termination of semi-constructor TRSs is decidable.

**Proof.** The decision procedure for $\mu$-termination of a semi-constructor TRS $R$ is as follows: consider all terms $u_1, u_2, \ldots, u_n$ corresponding to the right-hand sides of $\text{DP}(R) = \{l_i^\sharp \to u_i^\sharp \mid 1 \leq i \leq n\}$, and simultaneously generate all $\mu$-reduction sequences with respect to $R$ starting from $u_1, u_2, \ldots, u_n$. The procedure halts if it enumerates all reachable terms exhaustively or it detects a $\mu$-looping reduction sequence $u_i \mu_{\text{R},C} \to C_{\mu}[u_i]$ for some $i$.

Suppose $R$ does not $\mu$-terminate. By Lemma 4.21 and 4.12, we have a $\mu$-looping reduction sequence $u_i \mu_{\text{R},C} \to C_{\mu}[u_i]$ for some $i$ and $C_{\mu}$, which we eventually detect. If $R$ $\mu$-terminates, then the execution of the reduction sequence generation eventually stops since the reduction relation is finitely branching. In the latter case, the procedure does not detect a $\mu$-looping sequence, otherwise it contradicts to Proposition 4.3. Thus the procedure decides $\mu$-termination of $R$ in finitely many steps. $\square$

## 5 Extending the Classes by DP-graphs

### 5.1 Innermost Termination

In this subsection, we extend the class for which innermost termination is decidable by using the dependency graph.
Lemma 5.1 Let $R$ be a TRS whose innermost termination is equivalent to the non-existence of an innermost dependency chain that contains infinite use of right-ground dependency pairs. Then innermost termination of $R$ is decidable.

Proof. We apply the procedure used in the proof of Lemma 3.9 starting with terms $u_1, u_2, \ldots, u_n$, where $u_i$'s are all ground right-hand sides of dependency pairs. Suppose $R$ is innermost non-terminating, then we have an innermost dependency chain with infinite use of a right-ground dependency pair. Similarly to the semi-constructor case, we have a looping sequence $u_i \xrightarrow{\text{in}, R} C[u_i]$, which can be detected by the procedure. \qed

Definition 5.2 [Innermost DP-Graph [4]] The innermost dependency graph (innermost DP-graph for short) of a TRS $R$ is a directed graph whose nodes are the dependency pairs and there is an arc from $s^\sharp \rightarrow t^\sharp$ to $u^\sharp \rightarrow v^\sharp$ if there exist normal substitutions $\sigma$ and $\tau$ such that $t^\sharp \sigma \xrightarrow{\text{in}, R}^* u^\sharp \tau$ and $u^\sharp \tau$ is a normal form with respect to $R$.

An approximated innermost DP-graph is a graph that contains the innermost DP-graph as a subgraph. Such computable graphs are proposed in [4], for example.

Theorem 5.3 Let $R$ be a TRS and $G$ be an approximated innermost DP-graph of $R$. If at least one node in the cycle is right-ground for every cycle of $G$, then innermost termination of $R$ is decidable.

Proof. From Lemma 5.1. \qed

Example 5.4 Let $R_5 = \{f(s(x)) \rightarrow g(x), g(s(x)) \rightarrow f(s(0))\}$. Then $\text{DP}(R_5) = \{f^\sharp(s(x)) \rightarrow g^\sharp(x), g^\sharp(s(x)) \rightarrow f^\sharp(s(0))\}$. The innermost DP-graph of $R_5$ has one cycle, which contains a right-ground node [Fig. 1]. The innermost termination of $R_5$ is decidable by Theorem 5.3. Actually we know $R_5$ is innermost terminating from the procedure in the proof of Theorem 3.9 since all innermost reduction sequences from $f(s(0))$ terminate.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (f) at (0,0) {$f^\sharp(s(x)) \rightarrow g^\sharp(x)$};
  \node (g) at (2,0) {$g^\sharp(s(x)) \rightarrow f^\sharp(s(0))$};
  \draw[->] (f) edge (g);
\end{tikzpicture}
\caption{The innermost DP-graph of $R_5$}
\end{figure}

Example 5.5 Let $R_6 = \{a \rightarrow b, f(a, x) \rightarrow x, f(x, b) \rightarrow g(x, x), g(b, x) \rightarrow h(f(a, a), x)\}$. Then $\text{DP}(R_6) = \{f^\sharp(x, b) \rightarrow g^\sharp(x, x), g^\sharp(b, x) \rightarrow f^\sharp(a, a), g^\sharp(b, x) \rightarrow a^\sharp\}$. The innermost DP-graph of $R_6$ has one cycle, which contains a right-ground node [Fig. 2]. The innermost termination of $R_6$ is decidable by Theorem 5.3. Actually we know $R_6$ is not innermost terminating from the procedure in the proof of Theorem 3.9 by detecting the looping sequence $f(a, a) \xrightarrow{\text{in}, R_6} f(b, b) \xrightarrow{\text{in}, R_6} g(b, b) \xrightarrow{\text{in}, R_6} h(f(a, a), b)$. 


5.2 Context-Sensitive Termination

We extend the class for which \( \mu \)-termination is decidable by using the dependency graph. The class extended in this subsection is the class that satisfies the condition of Corollary 4.17.

**Lemma 5.6** Let \( R \) be a TRS and \( \mu \) be an \( \mathcal{F} \)-map. If \( \mu \)-termination of \( R \) is equivalent to the non-existence of a context-sensitive dependency chain that contains infinite use of right-ground rules in \( \text{DP}_\mathcal{F}(R, \mu) \), then \( \mu \)-termination of \( R \) is decidable.

**Proof.** We apply the procedure used in the proof of Lemma 4.22 starting with terms \( u_1, u_2, \ldots, u_n \), where \( u_i \)'s are all ground right-hand sides of rules in \( \text{DP}_\mathcal{F}(R, \mu) \). Suppose \( R \) is non-\( \mu \)-terminating, then we have a context-sensitive dependency chain with infinite use of right-ground rules in \( \text{DP}_\mathcal{F}(R, \mu) \). Similar to the \( \mu \)-semi- constructor case, we have a looping sequence \( u_i \xrightarrow{\mu, R}^+ C_\mu[u_i] \), which can be detected by the procedure.

**Definition 5.7** [Context-Sensitive DP-Graph [2]] The context-sensitive dependency graph (context-sensitive DP-graph for short) of a TRS \( R \) and an \( \mathcal{F} \)-map \( \mu \) is a directed graph whose nodes are elements of \( \text{DP}(R, \mu) \):

(i) There is an arc from \( s \to t \in \text{DP}_\mathcal{F}(R, \mu) \) to \( u \to v \in \text{DP}(R, \mu) \) if there exist substitutions \( \sigma \) and \( \tau \) such that \( t\sigma \xrightarrow{\mu, R}^* u\tau \).

(ii) There is an arc from \( s \to t \in \text{DP}_\mathcal{V}(R, \mu) \) to each dependency pair \( u \to v \in \text{DP}(R, \mu) \).

Similar to the innermost case, a computable approximated context-sensitive DP-graph is proposed [2,3].

**Theorem 5.8** Let \( R \) be a TRS, \( \mu \) be an \( \mathcal{F} \)-map and \( G \) be an approximated context-sensitive DP-graph of \( R \). The property \( \mu \)-termination of \( R \) is decidable if one of the following holds for every cycle in \( G \).

(i) The cycle contains at least one node that is right-ground.

(ii) All nodes in the cycle are elements in \( \text{DP}_\mathcal{V}^1(R, \mu) \).

**Proof.** From Lemma 5.6 and Theorem 4.16.

**Example 5.9** Let \( R_7 = \{ h(x) \to g(x, x), g(a, x) \to f(b, x), f(x, x) \to h(a), a \to b \} \) and \( \mu_4(f) = \mu_4(g) = \mu_4(h) = \{1\} \) [10]. Then \( \text{DP}(R_7, \mu_4) = \{ h^\circ(x) \to g^\circ(x, x), g^\circ(a, x) \to f^\circ(b, x), f^\circ(x, x) \to h^\circ(a), f^\circ(x, x) \to a^\circ \} \). The context-sensitive DP-graph of \( R_7 \) and \( \mu_4 \) has one cycle, which contains a right-ground node [Fig.3]. The \( \mu_4 \)-termination of \( R_7 \) is decidable by Theorem 5.8. Actually we know
$R_7$ is $\mu_4$-terminating from the procedure in the proof of Theorem 4.15 since all $\mu_4$-reduction sequences from $h(a)$ terminate.

Example 5.10 Let $\mu_5(g) = \{2\}$ and $\mu_5(f) = \mu_5(h) = \{1\}$. Consider the $\mu_5$-termination of $R_7$. The context-sensitive DP-graph for $R_7$ and $\mu_4$ is the same as the one for $R_7$ and $\mu_4$ [Fig. 3]. The $\mu_5$-termination of $R_7$ is decidable by Theorem 5.8. By the decision procedure, we can detect the $\mu_5$-looping sequence $h(a) \xrightarrow{\mu_5,R_7} g(a,a) \xrightarrow{\mu_5,R_7} g(a,b) \xrightarrow{\mu_5,R_7} f(b,b) \xrightarrow{\mu_5,R_7} h(a)$. Thus $R_7$ is non-$\mu_5$-terminating.

The class of TRSs that satisfy the conditions of Theorem 5.8 is a superclass of the class of TRS that satisfy the conditions of Corollary 4.17. The class of semi-constructor TRSs and the class of TRSs that satisfy the conditions of Theorem 5.8 are not included in each other.

Example 5.11 The TRS $R_7$ with an $\mathcal{F}$-map $\mu_4$ satisfies the condition of Theorem 5.8, but is not semi-constructor TRS. On the other hand, the TRS $R_3$ with an $\mathcal{F}$-map $\mu_2$ is a semi-constructor TRS, but does not satisfy the second condition of Theorem 5.8.

6 Conclusion

We have shown that innermost termination for semi-constructor TRSs is a decidable property and $\mu$-termination for semi-constructor TRSs and $\mu$-semi-constructor TRSs are decidable properties.

It is not difficult to implement the procedures in proofs of Theorem 3.9, Theorem 4.15 and Theorem 4.22. The class of semi-constructor TRSs are a rather small class: approximately 3 % of the TRSs in the termination problem data base 4.0 [1] are in this class. We can extend the decidable classes if we succeed in developing a method for good approximated DP-graphs.

In the future we will study the decidability of innermost termination and $\mu$-termination by applying known techniques for termination results [7,13]. Currently, innermost termination for shallow TRSs is known to be decidable [7]. There are several future works, studying whether the condition FFIVDC is removed from Theorem 4.15 or not, and extending the class of semi-constructor TRSs by using notions of context-sensitive DP-graph.
Acknowledgement

We would like to thank the anonymous referees for their helpful comments and remarks. This work is partly supported by MEXT.KAKENHI #18500011 and #16300005.

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