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SHRINKING WITHOUT LENGTHENING

R. H. Bing

With great sadness and a profound sense of loss we must relate that shortly after submission of this manuscript to Topology Professor Bing passed away. The journal is grateful to his wife Mary Bing and to his colleague and admirer Bob Edwards for their help with final editorial details.

It was shown in [2] that the union of two solid Alexander horned balls sewed together along their boundaries with the identity map is homeomorphic to a 3-sphere. (The Alexander horned ball is the closed *exterior* complement in S^3 of the strangely embedded 2-sphere described by Alexander; see Fig. 1.) This led to the conclusion that there is a homeomorphism of period two (an involution) of a 3-sphere onto itself having as its fixed point set a wild 2-sphere—one simply swaps the two halves of the above union. Such an involution is not equivalent to any PL or differential homeomorphism and is considered wild. (Two homeomorphisms h_1, h_2 are equivalent if there is a homeomorphism h such that $h_1 = h^{-1} h_2 h$.) Later it was discovered [1, 3, 5] that there are uncountably many mutually inequivalent wild involutions and indeed there are wild periodic homeomorphisms of S^3 onto itself of all periods.

The original description in [2] of the union of the Alexander horned balls considered each to appear as shown in Fig. 1. It was noted that this Alexander horned ball is the decomposition space of a 3-cell I^3 as shown in Fig. 2 where the nondegenerate elements of the decomposition are tame arcs. Each of these arcs is the intersection of a decreasing sequence of folded solid cylinders C, C_i, C_{ij}, . . . with bases on the base of I^3 as shown in Fig. 3, but each arc has just one end on the base of I^3 and each horizontal cross section of each arc is just one point.

It is easier to visualize the union of two copies of I^3 with their associated decompositions than it is to picture the union of two Alexander horned balls. The union of the decomposition spaces would be a decomposition of $I_1^3 \cup I_2^3 = S^3$ whose nondegenerate

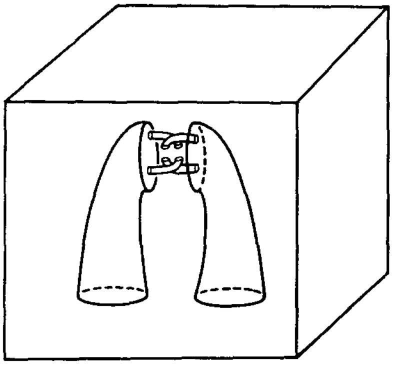

Fig. 1.

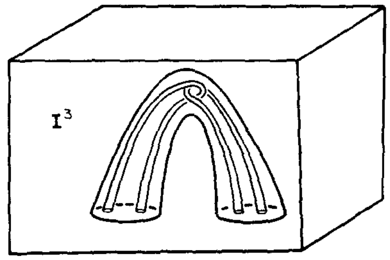

Fig. 2.

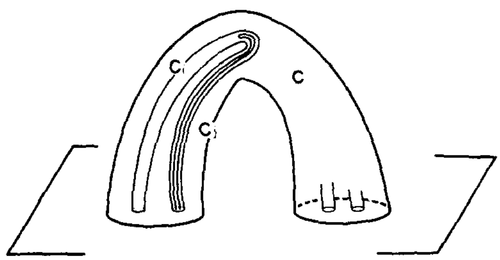

Fig. 3.

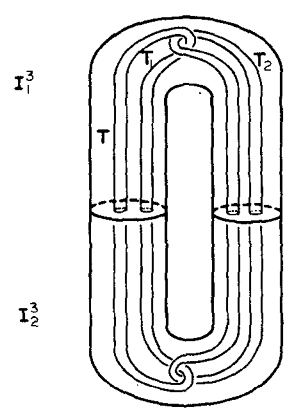

Fig. 4.

elements are tame arcs each of which is the intersection of solid tori T, T_i, T_{ij}, . . . as shown in Fig. 4. We say that T, T_i, T_{ij}, . . . are tori at the 0th, 1st, 2nd, . . . stages respectively.

It was shown in [2] that the decomposition space for Fig. 4 is a 3-sphere by showing that for each $\varepsilon_1 > 0$, there is an integer n_1 such that there is a homeomorphism h_1 of S^3 onto itself fixed outside T such that the images of the solid tori at the n_1th stage have diameters less than ε_1. (See Theorem 1 below.) Similarly one can construct a homeomorphism h_2 that agrees with h_1 outside the solid tori at the n_1th stage but shrinks solid tori at some n_2th stage to images with diameters less than $\varepsilon_2 = \varepsilon_1/2$. Iterating the procedure gives a decomposition map (lim $h_1, h_2, \ldots$) showing that the decomposition space is a 3-sphere and indeed, the union of two Alexander horned balls sewed together on their boundaries with the identity map is a 3-sphere.

In getting the homeomorphism h_1 to shrink the tori at the n_1th stage, no effort was made to decrease the lengths of their centerlines. Let us consider the case where T is long and thin (so that we can ignore vertical distances) and the length of the centerline of T is 2ε, so that the horizontal size of T as in Fig. 5 is approximately ε. Suppose we want to get a homeomorphism h that is fixed on BdT and takes the inside T's at some future stage to images with diameters less than $\varepsilon/4$. Let us consider several ways to proceed.

One such way might be to move T_1 and T_2 about as shown in Fig. 5 so that the new T_i's have diameters less than $\varepsilon/2$. This method is sometimes called the "standard mistake" because there is no way to move the new T_{i1} and T_{i2} about in the new T_i to significantly reduce their diameters.

To avoid this apparent trap a different procedure was followed in [2]. Rather than get new T_i's at the first step with diameters less than $\varepsilon/2$, we contented ourselves to get them with diameter less than $3\varepsilon/4$ as shown in Fig. 6.

A vertically enlarged copy of the new T_i appears in Fig. 7 with the way that the new T_{i1} and T_{i2} might appear in it. Each of the new T_{i1} and T_{i2} has a diameter approximately $\varepsilon/2$ (even though the length of the centerline of T_{i1} is more than 2ε). It is an easy task to place a new T_{i21} and a new T_{i22} in the new T_{i2} (using the "standard mistake") to make them have diameters approximately $\varepsilon/4$ and Fig. 8 shows how a new T_{i11} and T_{i12} could be placed in the new T_{i1} to make them have diameters approximately $\varepsilon/4$. (We hesitate to show all of T_{i11} since it is so long.)

By using an arbitrary number n in place of 4 and following a similar procedure we can get new T's so that for $m=1, 2, \ldots, n-1$ the new T's at the mth stage have diameters less than $(1-(m/n))\varepsilon$. Hence, we have the following result from [2].

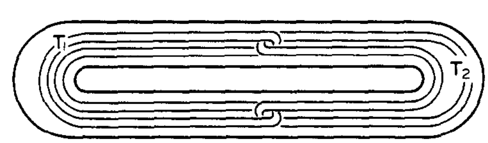

Fig. 5.

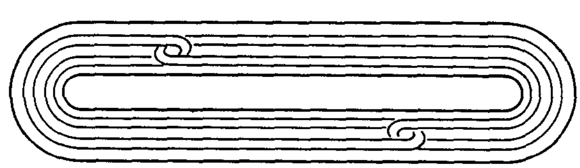

Fig. 6.

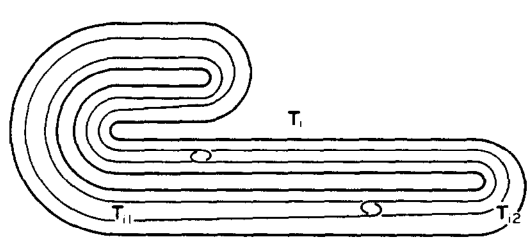

Fig. 7.

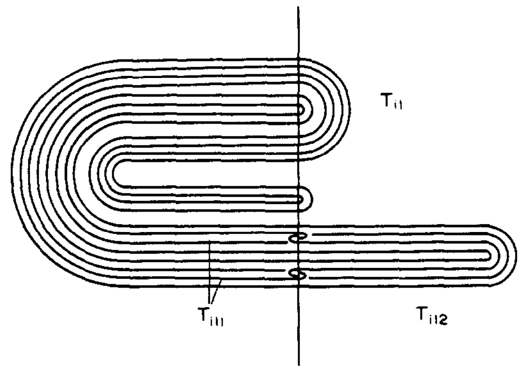

Fig. 8.

THEOREM 1. *Suppose the length of the centerline $T \leq 2\varepsilon$ and n is a positive integer. Then there is a homeomorphism h of T onto itself that is fixed on BdT such that for each $T_{ij...k}$ at the $(n-1)$st stage*

$$\textit{diameter } h(T_{ij...k}) < \varepsilon/n.$$

A homeomorphism h such as we have described to prove Theorem 1 would have increased the lengths of the images of centerlines of some of the $T_{1ij...k}$'s drastically. Michael Freedman asked if there is a homeomorphism h that does not. We give an affirmative answer in Theorem 2 in the case where n is of form 2^m. Theorem 2 has had applications in [4].

THEOREM 2. *Suppose the length of the centerline of $T \leq 2\varepsilon$ and n is a nonnegative integer. Then there is a homeomorphism h of T onto itself that is fixed on BdT and such that for each integer m with $0 \leq m \leq n$ and for each $T_{ij...k}$ at the 2^mth stage*

$$\textit{diameter of } h(T_{ij...k}) < \varepsilon/2^m \textit{ and}$$

$$\textit{length of centerline of } h(T_{ij...k}) < 2\varepsilon.$$

Description of h. By reselecting the centerline of T we can suppose that the reselected centerline S has length less than 2ε. Each of the centerlines S_1, S_2 of the adjusted T_1, T_2 runs parallel to one half of S, makes a turn, then runs back parallel to the same half, and finally makes a turn to connect to the start. We say that S_1 and S_2 "hook elbows" at the turns. The lengths of the turns are so short that they will be ignored. We suppose that the halves of S_1 and S_2 are so close and parallel to halves of S that the length of each S_i is about twice the length of one half of S. By cutting corners we could make the lengths of S_1 and S_2 less than the length of S but shall not concern ourselves with this. We note that S_1 and S_2 are each so close to one half of S that their diameters are less than ε. Hence, we suppose that the diameters of the new T_i's at the 1st stage are less than $\varepsilon = \varepsilon/2^0$ where $2^0 = 1$. We remark that, unlike in the proof of Theorem 1, the two first stage tori T_i in this proof are not rotated, simply so that the 2^mth stages rather than the (2^m-1)st stages are the most interesting ones in the subsequent discussion.

Centerlines $S_{ij...k}$ of each new $T_{ij...k}$ will zig-zag along S and be so close and parallel to S that we ignore the lengths at the turns and estimate the length of $S_{ij...k}$ by the sum of the lengths of the projections of its pieces on S. The bends in the zig-zagging of $S_{ij...k}$ are determined by two kinds of turns—those that occur at "linking elbows" and those forced by the fact that the centerlines lie in zig-zagging tubes. These interior turns in the zig-zagging tubes were created by "hooking elbow" turns at previous stages. A "hooking elbow" turn at one stage (as well as an interior turn at that stage) causes an interior turn at the next.

Our goal is to describe new $T_{ij\ldots k}$'s (which for convenience we continue to call $T_{ij\ldots k}$'s) so that for each integer m with $0 \leq m \leq n$, the diameters of the $T_{ij\ldots k}$'s at the 2^mth stage are less than $\varepsilon/2^m$.

Each $S_{ij\ldots k}$ will be the union of two arcs joined at their ends and which otherwise run parallel to each other and to the same zig-zagging arc $P_{ij\ldots k}$. We regard the length of $S_{ij\ldots k}$ to be about twice the length of $P_{ij\ldots k}$ and the diameter of $T_{ij\ldots k}$ to be about the same as the diameter of $P_{ij\ldots k}$. If $P_{ij\ldots k}$ is of length less than ε and it bends precisely at all the points that divide it into 2^m equal pieces, we suppose that the diameter of $P_{ij\ldots k}$ (and hence that the diameter of $T_{ij\ldots k}$) is less than $\varepsilon/2^m$.

We use Fig. 9 to show how S_{i1} and S_{i2} lie in T_i. Note that we have used the "standard mistake" but we are only at the 2^1th stage and only require that diameters be less than $\varepsilon/2^1$.

We draw $P_{ij\ldots k}$ as straight or curved depending on which is convenient. We place circles on $P_{ij\ldots k}$ to show where it bends. Figure 10a shows P_i straight (with no circle on it and P_{ij} with one bend (and one circle) and P_{ijk} with two bends (and two circles). Figure 10b shows P_{ijk} with two circles and P_{ijks} with three. Note that the bend points of P_{ijk} are at the quarter and three-quarter marks but there is none at its center. The diameter of P_{ijk} is about the same as that of P_{ij} but that of P_{ijks} is only about half as much.

All of the $P_{ij\ldots k}$'s at any stage have their bends at similar places. We denote a typical $P_{ij\ldots k}$ at the mth stage by Q_m and show it straight but with circles at the bends. Figure 11 shows Q_1, $Q_2, \ldots, Q_8$. It is to be noticed that if the length of Q_1 is less than ε, the length of the pieces between turns of Q_1, Q_2, Q_4, Q_8 are less than ε, $\varepsilon/2$, $\varepsilon/4$, $\varepsilon/8$ respectively so the diameter of the T's at the 1st, 2nd, 4th, and 8th stages have diameter less than $\varepsilon/2^0$, $\varepsilon/2^1$ $\varepsilon/2^2$, $\varepsilon/2^3$.

We now make certain observations about the Q's that will help us define other Q's and make similar observations about them. Each of the Q's is symmetric about its center, as is each of the halves of a Q. (For considering a half, we ignore any circle at an end of the half.) In general, we will define Q's so that if they are broken into 2^i equal pieces, each of these pieces is alike and is a shrunken version of a previously defined Q. If m is odd, Q_m does not have a circle

Fig. 9.

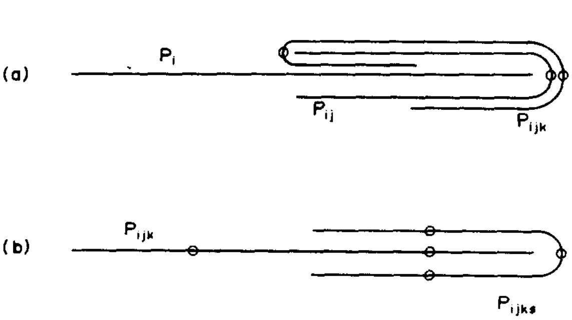

Fig. 10.

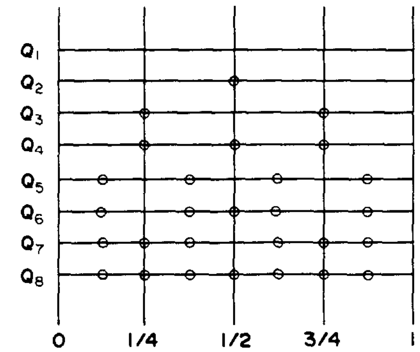

Fig. 11.

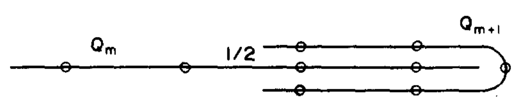

Fig. 12.

at its center and Q_{m+1} is formed from Q_m by placing a circle on Q_m at 1/2. If m is even, Q_{m+1} does not have a circle at its center and each half of it looks like a one-half multiple of Q_r where $r=(m+2)/2$. Once we have obtained Q_i's $(i=1, 2, \ldots, m)$ satisfying these observations, these observations produce a unique Q_{m+1} satisfying them.

Finally, we show how we can slide the T_i's at the $(m+1)$st stage about in those at the mth to get Q's as we have described. Figure 12 shows how to do this if m is odd and Q_m has no circle at its center. The extra circle at the center of Q_{m+1} is where Q_{m+1} goes around the end. We considered each T_i at the mth stage and rotated the inner part of it containing the T_i's at the $(m+1)$st stage in it through $\pi/2$ radians. We note that each half of Q_{m+1} (as does each half of Q_m) looks like a reduced copy of Q_r for $r=(m+1)/2$.

Figure 13 shows how the slide should be done if m is even. The "hooking elbows" occur at $1/2^t$ and $1-1/2^t$ where $1/2^t$ is the largest of the numbers 1/2, 1/4, 1/8, ... denoting a point where Q_m is not circled. Rotation of an $(m+1)$st stage T_i in an mth stage T_i is through $\pi/2^t$.

Actually, if 1/2 is uncircled, m is odd and we are at the preceding case. If $1/2^t$ is the first such uncircled point of Q_m, then all points of the forms $s/2^t$ are uncircled for s odd and circled for s even. The symmetry and equivalences of the pieces of Q_m of size $1/2^t$ shows Q_{m+1} has circles at the required points.

A few philosophical comments seem appropriate now that Theorem 2 is complete. Figure 11 appears simple but it was not easy to discover. It was actually found by working backwards. If in eight steps one had Q_8, what would Q_7 look like? There was only one possibility. The big gap would have to be in the middle. Once Q_7 were found, what would Q_6 look like? Where would the big gaps be? Actually, more complicated Q's like Q_{32}, Q_{128}, Q_{2^m} were being considered at the time, but Q_8 illustrates the principle just as well and it is easier to work back to Q_1 from Q_8 than from Q_{128}. This suggests several adages.

Learning to understand a complicated situation may be aided by considering an easier one. [*Q_8 is easier than Q_{2^m}.*]

In trying to find the steps of a proof it is sometimes useful to work backwards and get the last steps first. [*Count* 8, 7, 6, ... *rather than* 1, 2, 3, ...]

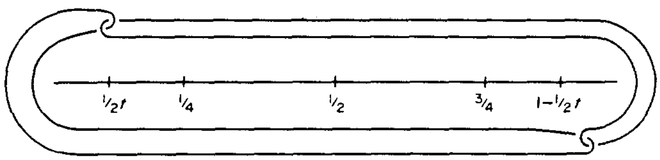

Fig. 13.

In trying to solve a difficult problem, do not overlook the possibility of rephrasing it and looking at the problem from a different point of view. [Decompositions were used to study unions.]

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Department of Mathematics
The University of Texas at Austin
Austin, TX 78712
U.S.A