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The convergence of He's variational iteration method for solving integral equations

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ABSTRACT

In this paper, several integral equations are solved by He's variational iteration method in general case, then we consider the convergence of He's variational iteration method for solving integral equations.

1. Introduction

Various kinds of analytical methods and numerical methods [1,2] were used to solve integral equations. In this paper, we apply He's variational iteration method (shortly VIM) [3–8] to solve integral equations. The method can solve various different nonlinear equations [9–11]. The variational iteration method is used in [12] to solve some problems in calculus of variations. This technique is used in [13] to solve the Fokker–Planck equation. Authors of [14] applied the variational iteration method to solve the Lane–Emden differential equation. This method is employed in [15] to solve the Klein–Gordon partial differential equations. Authors of [16] used the variational iteration method to solve a model describing biological species living together. Also the approach is employed in [17] to solve a parabolic inverse problem. He's variational iteration method is proposed in [18] to solve the Cauchy reaction–diffusion problem. This method is used in [19] to solve a biological population model. For more applications of the method the interested reader is referred to [20–24].

To illustrate the basic idea of the method, we consider a general nonlinear system:

$$L[u(t)] + N[u(t)] = g(t),$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a given continuous function. The basic character of the method is to construct functional for the system, which reads

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda(s)[Lu_n(s) + N\tilde{u}_n(s) - g(s)]\mathrm{d}s,$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory, u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

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2. Fredholm integral equation of the second kind

Now we consider the Fredholm integral equation of the second kind in general case, which reads

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\mathrm{d}t, \tag{2.1}$$

where $K(x,t)$ is the kernel of the integral equation. There is a simple iteration formula for Eq. (2.1) in the form

$$u_{n+1}(x) = f(x) + \lambda \int_a^b K(x,t)u_n(t)\mathrm{d}t. \tag{2.2}$$

Next we consider the existence of solution for Eq. (2.1) by considering the convergence of (2.2) by the next theorem.

Theorem 2.1. *Consider the iteration scheme*

$$u_0(x) = f(x)$$
$$u_{n+1}(x) = f(x) + \lambda \int_a^b K(x,t)u_n(t)\mathrm{d}t,$$

for $n = 0, 1, 2, \ldots$, to construct a sequence of successive iterations $\{u_n(x)\}$ to the solution of Eq. (2.1). In addition, let

$$\int_a^b \int_a^b K^2(x,t)\mathrm{d}x\mathrm{d}t = B^2 < \infty, \tag{2.3}$$

and assume that $f(x) \in L^2(a,b)$. Then, if $|\lambda| < 1/B$, the above iteration converges in the norm of $L^2(a,b)$ to the solution of Eq. (2.1).

Example 2.2. Consider the integral equation

$$u(x) = \sqrt{x} + \lambda \int_0^1 xtu(t)\mathrm{d}t, \tag{2.4}$$

its iteration formula reads

$$u_{n+1}(x) = \sqrt{x} + \lambda \int_0^1 xtu_n(t)\mathrm{d}t, \tag{2.5}$$

and

$$u_0(x) = \sqrt{x}. \tag{2.6}$$

Substituting Eq. (2.6) into Eq. (2.5), we have the following results

$$u_1(x) = \sqrt{x} + \lambda \int_0^1 xt\sqrt{t}\mathrm{d}t = \sqrt{x} + \frac{2\lambda x}{5}.$$
$$u_2(x) = \sqrt{x} + \lambda \int_0^1 xt\left[\sqrt{t} + \frac{2\lambda t}{5}\right]\mathrm{d}t = \sqrt{x} + \left[\frac{2\lambda}{5} + \frac{2\lambda^2}{15}\right]x.$$
$$u_3(x) = \sqrt{x} + \lambda \int_0^1 xt\left[\sqrt{t} + \left(\frac{2\lambda}{5} + \frac{2\lambda^2}{15}\right)t\right]\mathrm{d}t = \sqrt{x} + \left[\frac{2\lambda}{5} + \frac{2\lambda^2}{15} + \frac{2\lambda^3}{45}\right]x.$$

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sqrt{x} + \left[\frac{2}{5.3^0}\lambda + \frac{2}{5.3^1}\lambda^2 + \frac{2}{5.3^2}\lambda^3 + \cdots\right]x = \sqrt{x} + \left[\frac{2}{5}\sum_{i=1}^{n}\frac{\lambda^i}{3^{i-1}}\right]x.$$

The above sequence is convergent if $|\lambda| < 3$ and the exact solution is

$$\lim_{n\longrightarrow\infty} u_n(x) = \sqrt{x} + \frac{6\lambda}{5(3-\lambda)}x = u(x).$$

Note that by Theorem 2.1 we have

$$\int_a^b \int_a^b K^2(x,t)\mathrm{d}x\mathrm{d}t = \int_0^1 \int_0^1 (xt)^2\mathrm{d}x\mathrm{d}t = \frac{1}{9} = B^2.$$

Then if $|\lambda| < 3$ Eq. (2.5) is convergent.

Example 2.3. Consider the integral equation

$$u(x) = x + \lambda \int_0^1 (1 - 3xt)u(t)\mathrm{d}t, \tag{2.7}$$

its iteration formula reads

$$u_{n+1}(x) = x + \lambda \int_0^1 (1 - 3xt)u_n(t)\mathrm{d}t, \tag{2.8}$$

and

$$u_0(x) = x. \tag{2.9}$$

Substituting Eq. (2.9) into Eq. (2.8), we have the following results

$$u_1(x) = x + \lambda \int_0^1 (1 - 3xt)t\mathrm{d}t = (1 - \lambda)x + \frac{1}{2}\lambda.$$

$$\begin{aligned} u_2(x) &= x + \lambda \int_0^1 (1 - 3xt)\left[(1 - \lambda)t + \frac{1}{2}\lambda\right]\mathrm{d}t \\ &= (1 - \lambda)x + \frac{1}{2}\lambda + \frac{\lambda^2}{4}x. \\ u_3(x) &= x + \lambda \int_0^1 (1 - 3xt)\left[(1 - \lambda)t + \frac{1}{2}\lambda + \frac{\lambda^2}{4}t\right]\mathrm{d}t \\ &= (1 - \lambda)x + \frac{\lambda^2}{4}(1 - \lambda)x + \frac{1}{2}\lambda + \frac{\lambda^3}{8}. \end{aligned}$$

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sum_{i=0}^{n}\left[\left(\frac{\lambda^2}{4}\right)^i \lambda\left(\frac{1}{2} - x\right) + \left(\frac{\lambda^2}{4}\right)^i\right] + (1 + (-1)^n)\frac{\lambda^{2n+2}}{2^{2n+3}}x.$$

The above sequence is convergent if $|\frac{\lambda^2}{4}| < 1$ i.e., $-2 < \lambda < 2$ and the exact solution is

$$\lim_{n\longrightarrow\infty} u_n(x) = \frac{2\lambda}{4 - \lambda^2} + \frac{4(1 - \lambda)}{4 - \lambda^2}x = u(x).$$

Note that by Theorem 2.1 we have

$$\int_a^b \int_a^b K^2(x, t)\mathrm{d}x\mathrm{d}t = \int_0^1 \int_0^1 (1 - 3xt)^2\mathrm{d}x\mathrm{d}t = \frac{1}{2} = B^2.$$

Then if $|\lambda| < \sqrt{2}$ i.e., $-\sqrt{2} < \lambda < \sqrt{2}$, Eq. (2.8) is convergent.

3. Volterra integral equations of the second kind

First, we consider the Volterra integral equations of the second kind, which reads

$$u(x) = f(x) + \lambda \int_a^x K(x, t)u(t)\mathrm{d}t, \tag{3.1}$$

where $K(x, t)$ is the kernel of the integral equation.

As in the case of the Fredholm integral equation we can use variational iteration method to solve Volterra integral equations of the second kind. However, there is one important difference: if $K(x, t)$ and $f(x)$ are real and continuous, then the series converges for all values of λ (see [25]).

Example 3.1. Consider the integral equation

$$u(x) = x + \lambda \int_0^x (x - t)u(t)\mathrm{d}t, \tag{3.2}$$

its iteration formula reads

$$u_{n+1}(x) = x + \lambda \int_0^x (x - t)u_n(t)\mathrm{d}t, \tag{3.3}$$

and

$$u_0(x) = x. \tag{3.4}$$

Substituting Eq. (3.4) into Eq. (3.3), we have the following results

$$u_1(x) = x + \lambda \int_0^x (x-t)t\,\mathrm{d}t = x + \lambda \frac{x^3}{3!}.$$

$$u_2(x) = x + \lambda \int_0^x (x-t)\left[t + \lambda \frac{t^3}{3!}\right]\mathrm{d}t = x + \lambda \frac{x^3}{3!} + \lambda^2 \frac{x^5}{5!}.$$

$$\begin{aligned} u_3(x) &= x + \lambda \int_0^x (x-t)\left[t + \lambda \frac{t^3}{3!} + \lambda^2 \frac{t^5}{5!}\right]\mathrm{d}t \\ &= x + \lambda \frac{x^3}{3!} + \lambda^2 \frac{x^5}{5!} + \lambda^3 \frac{x^7}{7!}. \end{aligned}$$

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sum_{i=0}^{n} \lambda^i \frac{x^{2i-1}}{(2i-1)!}.$$

The above sequence is convergent for all λ.

Example 3.2. Consider the following integro-differential equation

$$u''(x) = -1 + \lambda \int_0^x (x-t)u(t)\mathrm{d}t, \tag{3.5}$$

which is equivalent to

$$u(x) = 1 - \frac{x^2}{2!} + \frac{\lambda}{3!} \int_0^x (x-t)^3 u(t)\mathrm{d}t, \tag{3.6}$$

for more details see [26]. Eq. (3.6) is a Volterra integral equation of the second kind where its iteration formula reads

$$u_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{\lambda}{3!} \int_0^x (x-t)^3 u_n(t)\mathrm{d}t, \tag{3.7}$$

and

$$u_0(x) = 1 - \frac{x^2}{2!}. \tag{3.8}$$

Substituting Eq. (3.8) into Eq. (3.7), we have the following results

$$u_1(x) = 1 - \frac{x^2}{2!} + \frac{\lambda}{3!} \int_0^x (x-t)^3 \left[1 - \frac{t^2}{2!}\right]\mathrm{d}t = 1 - \frac{x^2}{2!} + \lambda \left[\frac{x^4}{4!} - \frac{x^6}{6!}\right].$$

$$\begin{aligned} u_2(x) &= 1 - \frac{x^2}{2!} + \frac{\lambda}{3!} \int_0^x (x-t)^3 \left[1 - \frac{t^2}{2!} + \lambda \left(\frac{t^4}{4!} - \frac{t^6}{6!}\right)\right]\mathrm{d}t \\ &= 1 - \frac{x^2}{2!} + \lambda \left[\frac{x^4}{4!} - \frac{x^6}{6!}\right] + \lambda^2 \left[\frac{x^8}{8!} - \frac{x^{10}}{10!}\right]. \end{aligned}$$

$$\begin{aligned} u_3(x) &= 1 - \frac{x^2}{2!} + \frac{\lambda}{3!} \int_0^x (x-t)^3 \left[1 - \frac{t^2}{2!} + \lambda \left(\frac{t^4}{4!} - \frac{t^6}{6!}\right) + \lambda^2 \left[\frac{t^8}{8!} - \frac{t^{10}}{10!}\right]\right]\mathrm{d}t \\ &= 1 - \frac{x^2}{2!} + \lambda \left[\frac{x^4}{4!} - \frac{x^6}{6!}\right] + \lambda^2 \left[\frac{x^8}{8!} - \frac{x^{10}}{10!}\right] + \lambda^3 \left[\frac{x^{12}}{12!} - \frac{x^{14}}{14!}\right]. \end{aligned}$$

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sum_{i=0}^{n} \lambda^i \frac{x^{4i}}{(4i)!} + \sum_{i=0}^{n} \lambda^i \frac{x^{4i+2}}{(4i+2)!}.$$

The above sequence is convergent for all λ. Note, for $\lambda = 1$ the above sequence converges to $\cos x$ which is the exact solution for

$$u''(x) = -1 + \int_0^x (x-t)u(t)\mathrm{d}t. \tag{3.9}$$

4. Conclusion

In this paper, we have applied He's variational iteration method in general case and considered its convergence.

References

[1] A.M. Wazwaz, Two methods for solving integral equation, Appl. Math. Comput. 77 (1996) 79–89.
[2] A.M. Wazwaz, A reliable treatment for mixed Volterra–Fredholm integral equations, Appl. Math. Comput. 127 (2002) 405–414.
[3] J.H. He, Variational iteration method—a kind of nonlinear analytical technique: Some examples, Int. J. Nonlinear Mech. 34 (1999) 699–708.
[4] J.H. He, A review on some new recently developed nonlinear analytical techniques, Int. J. Nonlinear Sci. Numer. Simul. 1 (2000) 51–70.
[5] J.H. He, Non-perturbative methods for strongly nonlinear problems, Dissertation, de-Verlag im Internet GmbH, Berlin, 2006.
[6] J.H. He, Variational iteration method—some recent results and new interpretations, J. Comput. Appl. Math. 207 (2007) 3–17.
[7] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2000) 115–123.
[8] J.H. He, X.H. Wu, Construction of solitary solutions and compacton-like solution by Variational iteration method, Chaos Solitons Fractals 29 (2006) 108–113.
[9] N. Bildik, A. Konuralp, The use of Variational iteration method, differential transform methods and Adomian decomposition method for solving different type of nonlinear partial differential equation, Int. J. Nonlinear Sci. Numer. Simul. 7 (2006) 65–70.
[10] M. Dehghan, M. Tatari, Identifying and unknown function in parabolic equation with overspecified data via He's VIM, Chaos Solitons Fractals 36 (2008) 157–166.
[11] N.H. Sweilam, Variational iteration method for solving cubic nonlinear Schrodinger equation, J. Comput. Appl. Math. 207 (2007) 155–163.
[12] M. Tatari, M. Dehghan, Solution of problems in calculus of variations via He's variational iteration method, Physics Letters A 362 (2007) 401–406.
[13] M. Dehghan, M. Tatari, The use of He's variational iteration method for solving the Fokker–Planck equation, Phys. Scripta 74 (2006) 310–316.
[14] M. Dehghan, F. Shakeri, Approximate solution of a differential equation arising in astrophysics using the variational iteration method, New Astron. 13 (2008) 53–59.
[15] F. Shakeri, M. Dehghan, Numerical solution of the Klein–Gordon equation via He's variational iteration method, Nonlinear Dynamics 51 (2008) 89–97.
[16] F. Shakeri, M. Dehghan, Solution of a model describing biological species living together using the variational iteration method, Math. Comput. Modelling 48 (2008) 2175–2188.
[17] M. Dehghan, M. Tatari, Identifying an unknown function in a parabolic equation with overspecified data via He's variational iteration method, Chaos Solitons Fractals 36 (2008) 157–166.
[18] M. Dehghan, F. Shakeri, Application of He's variational iteration method for solving the Cauchy reaction–diffusion problem, J. Comput. Appl. Math. 214 (2008) 435–446.
[19] M. Dehghan, F. Shakeri, Numerical solution of a biological population model using He's variational iteration method, Comput. Math. Appl. 54 (2007) 1197–1209.
[20] J.H. He, X.H. Wu, Variational iteration method: New development and applications, Comput. Math Appl. 54 (2007) 881–894.
[21] J.H. He, Variational iteration method — Some recent results and new interpretations, Comput. Appl. Math. 207 (2007) 3–17.
[22] H. Ozer, Application of the variational iteration method to the boundary value problems with jump discontinuities arising in solid mechanics, Int. J. Nonlinear Sci. Numer. Simul. 8 (2007) 513–518.
[23] J. Biazar, H. Ghazvini, He's variational iteration method for solving hyperbolic differential equations, Int. J. Nonlinear Sci. Numer. Simul. 8 (2007) 311–314.
[24] Z.M. Odibat, S. Momani, Application of variational iteration method to Nonlinear differential equations of fractional order, Int. J. Nonlinear Sci. Numer. Simul. 7 (2006) 27–34.
[25] C.E. Fröberg, Introduction to Numerical Analysis, Addison-Wesley Pub Company, 1968.
[26] A.M. Wazwaz, A First Course in Integral Equation, World Scientific, Singapore, 1997.