Wave Propagation in Anisotropic Elasticity

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INTRODUCTION

Anisotropic wave propagation has received much attention in the last few years. The work of Lighthill [1], Ludwig [2], and Duff [3] has contributed much in understanding the phenomena. The basis of their approach consists in evaluating Fourier Integrals by the method of steepest descent. The main asymptotic results are given in an elegant form. For the two-dimensional problem Weitzner [4] gave explicit results for the full field in terms of the roots of a quartic. But the complicated form of the roots of a general quartic conceals the main features of solution. Bazer and Fleishman [5] used ray-theory to study the linearized problem of Magneto-Gas Dynamics.

Thomas [6] inaugurated the study of wave-propagation by means of the theory of singular surfaces. The method is powerful enough to deal with the nonlinear problems as easily as with the linear ones. It enables us to study, as the wave front moves, the variation of the strength of the wave, defined in a suitable way in terms of the discontinuity across the front. It may be regarded as the first step in building up of the full solution behind the front [7]. When it is known that the disturbance is strongest across the front, this study by itself is of great physical interest.

The present work is essentially a combination of the ideas of the theory of singular surfaces as given by Thomas [6] and those of ray-theory as given by Courant and Hilbert [7]. It draws heavily from these works. We arrive at a quite elegant approach to discuss completely anisotropic wave-propagation into a uniform medium. We believe that a number of results obtained here are new. Applications to nonlinear problems will be discussed in subsequent papers [8-10]. Extension to nonuniform and dispersive media offer new fields of study.

In Section 2, we note certain relevant important ideas of the theory of singular surfaces and those of the ray-theory. Then in Section 3 we apply

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these ideas to obtain the complete integral of the growth equation. The last section concludes with a discussion of the transversely isotropic medium which brings out the comparison of our results with those of Lighthill in the case of nondispersive media.

2. THE RAY THEORY AND THE THEORY OF SINGULAR SURFACES

Referred to a Cartesian orthogonal system \((x_i)\), we consider a moving surface \(\Sigma(t)\). This surface is called a singular surface or a wave-front if any of the field variables or their derivatives suffer discontinuities across it. We adopt a Gaussian system of coordinates \((u^a)\), \((a = 1,2)\) on the surface. Latin suffixes denote Cartesian tensors and, after a comma, denote partial differentiation with respect to \((x_i)\). Greek indices, as suffixes and superfixes, denote covariant and contra-variant components of a tensor in the system \((u^a)\), and after a comma, denote covariant derivative. We take \((g_{\alpha\beta})\) and \((b_{\alpha\beta})\) as the first and second fundamental forms of the surface \(\Sigma(t)\). The surface may be described in either of the forms as

\[
x_i = x_i(u^1, u^2, t), \quad f(x_1, x_2, x_3, t) = 0. \quad (2.1a, b)
\]

We assume in the present that the surface is moving into a uniform unstrained medium. Let \((G)\) be the velocity of the front normal to itself. The unit normal to the surface \((n_i)\), points into the medium ahead. Quite generally the incremental change \((\Delta F)\) of any function \((F)\) between two consecutive points \((x_i, t)\) and \((x_i + \Delta x_i, t + \Delta t)\) in space-time is given by

\[
\Delta F = \frac{\partial F}{\partial t} \Delta t + F_{,j} \Delta x_j. \quad (2.2)
\]

Let the consecutive points be taken now on the successive positions of the wave-front \(\Sigma(t)\). The element \((\Delta x_i)\) will then be denoted by \((dx_i)\). This determines a direction in space (to be identified with the Ray direction, defined subsequently). Let \((V_i)\) be the velocity of \(\Sigma(t)\) corresponding to this direction. Then we can set \(\Delta x_i = dx_i = V_i \Delta t\) and obtain

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + F_{,j} V_j. \quad (2.3)
\]

We next take these consecutive points on a normal trajectory and denote \((\Delta x_i)\) by \((\delta x_i)\) and set \(\Delta x_i = \delta x_i = Gn_i \Delta t\) and get

\[
\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + F_{,j} Gn_j. \quad (2.4)
\]
Since $f(x_i, t) = 0$ must remain a wave-front its delta-time derivative must vanish.

So we obtain

$$\frac{\partial f}{\partial t} + f_i G n_i = 0.$$  \hspace{1cm} (2.5)

We assume that $G \neq 0$ identically on any portion of the surface. When $(\partial f/\partial t) \neq 0$, we can solve $f(x_i, t) = 0$ as $f = W(x_i) - t = 0$ [7]. Let $(p_i)$ denote the gradient of $f(x_i, t)$. From (2.5) we then obtain

$$\mathbf{S} \equiv Gp - 1 = 0, \quad p_i^2 = p_j p_j, \quad p_i = p n_i. \hspace{1cm} (2.6)$$

Now $(G)$ may depend on $(x_i)$ and on $(n_i)$ and hence on $(p_i)$. Thus the equation (2.6) is a first order partial differential equation for the wave-front. Such an equation is solved by Charpit's method or, more commonly known as, the method of rays [7] as

$$V_i = \frac{dx_i}{dt} - \frac{\partial \mathbf{S}}{\partial p_i}, \quad \frac{dp_i}{dt} - \frac{\partial \mathbf{S}}{\partial x_i}. \hspace{1cm} (2.7)$$

The analogy with Hamiltonian equations in mechanics is clear. This system of first order ordinary differential equations for $(x_i)$ and $(p_i)$, regarded as independent, completely solves the partial differential equation (2.6), when their initial values are known. We assume $(G)$ and so $(\mathbf{S})$ is independent of $(x_i)$. Thus, we get

$$\frac{dp_i}{dt} = 0. \hspace{1cm} (2.8)$$

Since $p_i = p n_i$, the same holds for $(n_i)$. The first system of (2.7) determines the curves known as rays. The variable $(t)$ is any curve-parameter which we fix as time. Then $(V_i)$ may be called the 'ray velocity.' We thus identify the operator $(d/dt)$ as differentiation following the rays; the $(\delta/\delta t)$ is, of course, the one following the normal trajectories, as introduced by Thomas [6]. The ray velocity $(V_i)$ is easily obtained as

$$V_i = G n_i + (\delta_{ij} - n_i n_j) \frac{\partial G}{\partial n_j}. \hspace{1cm} (2.9)$$

It is clear that the normal component of $(V_i)$ is just $(G)$; but it has also a tangential component, given by

$$V_\alpha = V_i x_{i,\alpha}$$

$$= x_{i,\alpha} \frac{\partial G}{\partial n_j}. \hspace{1cm} (2.10)$$

The rays coincide with the normal trajectories only if $(G)$ is independent of $(n_i)$. We designate wave-propagation as anisotropic only if the normal
velocity of propagation of the front depends on the normal to the surface. If the wave is propagating into a nonhomogeneous or nonuniform medium then this velocity depends on the spatial coordinates also. The key to the study of anisotropic wave-propagation into a uniform medium is the result (2.8), which asserts that the direction of the normal remains unchanged as we move along the rays. Instead of studying the variation of the wave-strength by following the normal trajectories, we study its strength as we move along the rays. The basic formula which enables us to do this is obtained from (2.3) and (2.4) as

$$\frac{dF}{dt} = \frac{\delta F}{\delta t} + V^\alpha F_{,\alpha}.$$  \hspace{1cm} (2.11)

The above relation connects the variation of any surface quantity along the rays with that along the normal trajectories. By Hadamard's Lemma the tangential derivative of the jump in any quantity across the singular surface is the same as the jump in the tangential derivatives of the quantity on the two sides of the front. Hence the relation (2.11) continues to hold for the discontinuities too.

We next obtain a few more results leading to the second basic formula needed in our study. By definition we have

$$\frac{\delta x_i}{\delta t} = G n_i.$$  \hspace{1cm} (2.12)

Since $\{x_{,\alpha}\}$ are tangent vectors, the delta-time derivative *commutes* with tangential differentiation. We note also the following formulas needed in our study.

$$n_{,\alpha} = - b^a b_{a,\alpha},$$  \hspace{1cm} (2.13)

$$b_{a,\beta} = b_{a,\beta},$$  \hspace{1cm} (2.14)

$$\frac{\delta n_i}{\delta t} = - g^{\alpha\beta} G_{,\alpha} x_i,\beta.$$  \hspace{1cm} (2.15)

The formulas (2.13) and (2.14) are known after Weingarten and Mainardi-Codazzi, respectively, while (2.15) is obtained by Thomas [6]. In the present case the last result can also be obtained by noting that the $(d/dt)$ derivative of $(n_i)$ vanishes. Differentiating (2.12) with respect to $(u^\alpha)$ we can obtain

$$\frac{\delta g_{\alpha\beta}}{\delta t} = - 2 G b_{\alpha\beta},$$  \hspace{1cm} (2.16)

$$\frac{\delta g^{\alpha\beta}}{\delta t} = 2 G b^{\alpha\beta}.$$  \hspace{1cm} (2.17)
Taking the delta-time derivative of the Weingarten formulas after re-writing them in the form

\[ b_{\alpha}^{\beta} = -\xi^{\alpha} \eta_{\alpha,\beta}, \]  

we can obtain, with some calculations,

\[ \frac{\delta b_{\alpha}^{\beta}}{\delta t} = g^{\eta \gamma} G_{\alpha,\eta \gamma} + G_{\alpha}^{\psi \gamma} b_{\psi}^{\gamma}. \]  

We also have

\[ G_{,\alpha} = -\frac{\partial G}{\partial n_{j}} b_{\alpha}^{\beta} x_{j,\beta} \]

Using this and (2.11), we rewrite (2.19) as

\[ \frac{d b_{\alpha}^{\beta}}{d t} = (G_{\alpha}^{\gamma} b_{\gamma}^{\beta} - g^{\delta \gamma} V_{\delta,\alpha} b_{\delta}^{\gamma} ) + (V^{\gamma \beta}_{\sigma,\alpha} - g^{\psi \gamma} V_{\psi,\delta} b_{\delta}^{\gamma}). \]

Use of (2.14) enables us to prove that the second group of terms in (2.21) vanish. From this, by actual evaluation, we can obtain the second basic result needed for our purpose as

\[ \frac{d \log b}{d t} = G_{,\alpha}^{\alpha} - V_{,\alpha}^{\alpha}. \]

Here \( b \), the determinant of \( (b_{\alpha}^{\beta}) \), is the Gaussian curvature of the wave-front \( \Sigma(t) \).

3. **Wave Propagation in an Anisotropic Homogeneous Linearly Elastic Medium**

We now consider the most general anisotropic medium. The medium is taken to be linearly elastic; specifically our attention is restricted to infinitesimal theory. The density and the tensor of elastic modulii are assumed to be constant. We take that the modulii are nondimensionalized with respect to some modulus, say \( (M) \). We further transform the time as

\[ \tau = ct \quad \text{with} \quad \rho_{o} c^{2} = M. \]  

The new 'time' \( (\tau) \) is now of the same dimension as length. Also the actual velocity is obtained from all the velocities we obtain after multiplication by \( (c) \). The tensor of elastic modulii \( (C_{ijkl}) \), (now dimensionless) is symmetric in
(i, j) and (k, l) due to the symmetry of the stress-tensor $\sigma_{ij}$ (now dimensionless), and the strain-tensor $(\varepsilon_{ij})$. We further assume that the medium is hyperelastic so that there is a further symmetry in the first two and the last two indices.

Consider now a singular surface $\Sigma(t)$ moving into such a uniform anisotropic medium. If $(u_i)$ is the displacement vector we assume that its components are continuous across the wave front or the singular surface, while their first derivatives and their second derivatives are discontinuous across it. Using a square bracket to denote the discontinuities, we denote the jumps as

$$[u_i] = 0, \quad [u_{i,j}] n_j = \xi_i, \quad [u_{i,jk}] n_j n_k = \xi_i^+. \quad (3.2)$$

The 'shock' relations and the equations of motion for small deformations of the linear elasticity are [11]

$$[\sigma_{ij}] n_j = -C \left[ \frac{\partial u_i}{\partial t} \right], \quad (3.3)$$

$$\sigma_{ij,i} = \frac{\partial^2 u_i}{\partial x^2}. \quad (3.4)$$

The most general constitutive law is taken as

$$\sigma_{ij} = C_{ijkl} u_{k,l}. \quad (3.5)$$

The compatibility conditions now lead to [11]

$$G^2 \xi_i = a_{ik} \xi_k, \quad (3.6)$$

$$G^2 \xi_i^* - a_{ik} \xi_k^* = 2G \frac{\delta \xi_i}{\delta t} + \xi_i \frac{\delta G}{\delta t}$$

$$+ C_{ijkl} \left( \delta^{kq} \xi_{k,a} (n_j x_{l,b} + n_l x_{j,b}) - \xi_k b^{pq} x_{j,a} x_{l,b} \right), \quad (3.7)$$

where

$$a_{ik} = a_{ki} = C_{ijkl} a_{j,l}. \quad (3.7, a)$$

We note that the second rank tensor $(a_{ik})$ is symmetric in $(i, k)$ in view of the assumption that the medium is hyperelastic. The tensor of elastic modulii $(C_{ijkl})$ is given in terms of a $(6 \times 6)$ matrix $(c_{\eta\eta})$ obtained by the usual substitution rule as

$$(11) \rightarrow (1), \quad (22) \rightarrow (2), \quad (33) \rightarrow (3), \quad (23) \rightarrow (4), \quad (31) \rightarrow (5), \quad (12) \rightarrow (6).$$
Using these two-index modulii, we can write \((a_{ik})\) as

\[
a_{11} = c_{11}n_1^2 + c_{55}n_2^2 + c_{55}n_3^2 + 2(c_{15}n_1n_2 + c_{15}n_3n_1 + c_{55}n_2n_3),
\]

\[
a_{12} = c_{16}n_1^2 + c_{25}n_2^2 + c_{44}n_3^2 + (c_{12} + c_{46})n_1n_2 + (c_{45} + c_{25})n_2n_3
\]

\[+ (c_{14} + c_{56})n_1n_3.\]  

(3.8, a)

The others can be written down similarly. For an orthotropic medium we have

\[
a_{11} = c_{11}n_1^2 + c_{55}n_2^2 + c_{55}n_3^2, \quad a_{22} = (c_{44} + c_{22})n_2n_3,
\]

\[
a_{22} = c_{66}n_1^2 + c_{56}n_2^2 + c_{44}n_3^2, \quad a_{33} = (c_{55} + c_{33})n_3n_1,
\]

\[
a_{33} = c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2, \quad a_{12} = (c_{66} + c_{12})n_1n_2.\]  

(3.8, b)

For an isotropic medium we take \((\lambda, \mu)\) as Lame' constants, identify \((\mu)\) with \((M)\) and take \((\sigma)\) as the ratio of \((\lambda)\) to \((\mu)\). Then we get

\[
a_{ij} = (\sigma + 1)n_in_j + \delta_{ij}.\]  

(3.8, c)

For \(\varSigma(t)\) to be a singular surface, the vector \((\xi_i)\) cannot vanish identically. So the determinant of \((\xi_i)\) in (3.5) must vanish. So we obtain

\[
| a_{ik} - G^2S_{ik} | = 0.\]  

(3.9)

This is the equation giving the velocities of propagation. Now the equation (3.5) states that \((\xi_i)\) is an eigenvector of the symmetric matrix \((a_{ik})\) with eigenvalues \((G^2)\). So if \((L_i)\) is a normalised eigenvector of \((a_{ik})\), it must have the same eigenvalues \((G^2)\) and must be parallel to \((\xi_i)\). So we have [7]

\[
\frac{\xi_i}{L_i} = \psi, \quad \psi = L_i\xi_i, \quad G^2L_i = a_{ik}L_k.\]  

(3.10, a)

Since \((\xi_i)\) is a discontinuity-vector and \((L_i)\) is a unit vector \((\psi)\) may be called the 'strength' of the discontinuity. Let \((I), (II), (III)\) be the invariants of \((a_{ik})\), and \((A_{ki} = A_{ik})\) be the cofactor of \((a_{ik})\) in \(| a_{ik} |\), the determinant of \((a_{ik})\). Then the cubic (3.9) for the determination of the velocities of propagation can be written as

\[
G^8 - I G^4 + II G^2 - III = 0,\]  

(3.11, a)

To each root of this cubic in the square of the velocity \((G)\) there corresponds one eigenvector \((L_i)\). These are called the 'modes of propagation' [7].
The eigen-vectors corresponding to distinct roots are orthogonal. This enables us to study the growth of each mode to be made independently of the others [7]. Since \((a_{ik})\) is symmetric all the roots of the cubic are real. Moreover we have

\[ G^2 \xi_i \xi_i = a_{ik} \xi_i \xi_k. \] (3.12)

The right-hand member represents the change in the strain energy of the medium across the wave-front. Since this must be positive the right-hand member must be a positive definite quadratic form in \((\xi_i)\). So the following quantities must all be positive:

\[ a_{11}, \quad a_{22}, \quad a_{33}, \quad A_{11}, \quad A_{22}, \quad A_{33}, \quad |a_{ij}|. \] (3.13)

Further this shows that the square of the velocity is positive; the velocities are all thus real. All these conclusions have followed from our assumption that the medium is hyperelastic. The invariants \((I, II, III)\) are independent of \((n_i)\) only in the case of isotropy. Otherwise they depend on \((n_i)\). So the roots of the cubic (3.11), \((G^3)\), also depend on \((n_i)\). This is the basic feature, which we accepted as the definition, of anisotropic wave-propagation. Among various forms of the roots of a cubic, the simplest one, when all the roots are real, is the trigonometric form. Hence, defining two quantities \((p)\) and \((q)\) in terms of the invariants as

\[ p = \frac{1}{9} (I^2 - 3 II) \]

\[ = \frac{1}{18} \left\{ (a_{11} - a_{22})^2 + (a_{22} - a_{33})^2 + (a_{33} - a_{11})^2 + 6(a_{12}^2 + a_{23}^2 + a_{31}^2) \right\}, \] (3.14, a)

\[ q = \frac{2}{27} I^3 + III - \frac{1}{3} I II, \] (3.14, b)

and an angle \((x)\) in terms of \((p)\) and \((q)\) as

\[ 2p^{3/2} \cos 3x = q, \] (3.14, c)

we can write the roots of the cubic explicitly as

\[^{(1)} G^2 = \frac{1}{3} I + 2 \sqrt{p} \cos x, \] (3.14, d)

\[^{(2)} G^2 = \frac{1}{3} I + 2 \sqrt{p} \cos \left( x + \frac{2\pi}{3} \right), \] (3.14, e)

\[^{(3)} G^2 = \frac{1}{3} I + 2 \sqrt{p} \cos \left( x - \frac{2\pi}{3} \right). \] (3.14, f)
Again, since we have

\[(G^2 - G^2)^2 (G^2 - G^2)^2 (G^2 - G^2)^2 = 27 (4p^3 - q^3),\] (3.15)

the equation (3.14, c) always determines a real value of (x).

Further we note Jacobi's Theorems on adjugate determinants as

\[A_{22}A_{33} - A_{23}^2 = a_{11} \mid a_{ij} \mid, \quad A_{31}A_{12} - A_{11}A_{23} = a_{23} \mid a_{ij} \mid.\] (3.16)

By using these we can write the eigenvectors in a symmetric form as

\[L_i(A_{23} + G^2 a_{23}) = L_i(A_{11} + a_{11}) = L_i(a_{12}).\] (3.17)

The above result fails to hold when two roots of the cubic (3.11) coincide. However we then have

\[G^2 = a_{11} + a_{23} a_{31} - \gamma \] (3.18, c)

The eigenvectors \((L_i, L_i)\) are then indeterminate. Any two two mutually orthogonal directions orthogonal to \((L_i)\) can then be taken as \((L_i)\) and \((L_i)\).

This is the case of isotropy. We then have

\[G^2 = \sigma + 2, \quad G^2 = G^2 = 1, \quad L_i = n_i.\] (3.19)

Here any two directions tangential to \(\Sigma(t)\) serve as \((L_i)\) and \((L_i)\). However it is only in the case of isotropy that the modes reduce to one purely dilatational and two purely rotational ones. The quantity \((\psi)\) can be seen, from (3.10), to reduce to the discontinuity in the Divergence and the Rotation of the displacement vector \((u_i)\). For the general case, except when additional symmetries hold (see last section), no such interpretation is possible. Each mode is seen to be associated both with dilatation and rotation.

We now proceed to the study of the growth equation. Since \((L_i)\) is an eigenvector of the symmetric matrix \((a_{ik})\), multiplying (3.6) by \((L_i)\) we see that its left member must reduce to zero. So the inner product of the right member with \((L_i)\) leads to the growth equation. Using (2.11) and (3.10) we can write this growth equation as

\[\frac{2}{\psi} \frac{d\psi}{dt} = \frac{A}{G} + \frac{1}{G\psi} B^\psi \phi,\] (3.12)
where

\[ A = V^\alpha G_\alpha + F - H, \]  

(3.12, a)

\[ B^\alpha = 2GV^\alpha - C_{ijkl}L_iL_kg^\alpha\beta(n_i\mathbf{x}_{j,\beta} + n_j\mathbf{x}_{i,\beta}), \]  

(3.12, b)

with

\[ F = C_{ijkl}L_iL_kb^\alpha\beta x_{i,\alpha}x_{j,\beta}, \]  

(3.12, c)

\[ H = C_{ijkl}L_iL_kg^\alpha\beta(n_i\mathbf{x}_{j,\beta} + n_j\mathbf{x}_{i,\beta}). \]  

(3.12, d)

We first assert that the coefficient of the tangential derivative of \( \psi \) vanishes. This reduces the growth equation (3.12) to an ordinary differential equation along the rays. This central idea is elaborated by Courant and Hilbert [7] at various points. We further stress that, in view of (2.8), all other quantities in (3.12), such as \( (n_i) \) etc. can be treated as constants. In order to prove our assertion we first rewrite (3.10, b) as

\[ G^2 = C_{ijkl}n_in_jL_iL_k. \]  

(3.13)

Differentiate this with respect to \( (\psi^\alpha) \), use (2.20) and note the identity \( L_iL_{i,\alpha} = 0 \), since \( L_i \) is a unit vector. We then obtain

\[ 2GV^\beta b^\alpha = C_{ijkl}L_iL_kb^\alpha\beta(n_i\mathbf{x}_{j,\beta} + n_j\mathbf{x}_{i,\beta}). \]  

(3.14)

Since our main interest lies in the study of wave-propagation when the front is nonplanar, we can assume that \( b^\alpha \neq 0 \), identically. Then we obtain

\[ 2GV^\beta = C_{ijkl}L_iL_kb^\alpha\beta(n_i\mathbf{x}_{j,\beta} + n_j\mathbf{x}_{i,\beta}). \]  

(3.15)

This is the same condition as the vanishing of \( (B^\alpha) \), if we recognise it as its covariant form. We further differentiate (3.15) with respect to \( (\psi^\alpha) \) and then multiply by \( (g^{\alpha\beta}) \). After some straightforward calculations, we obtain the interesting result

\[ V^\alpha G_\alpha + GV^\alpha = G^2 b^\alpha + H - F. \]  

(3.16)

Substituting from this for \( (F-H) \) in \( (A) \) of (3.12, a), we obtain

\[ A = G(Gb^\alpha - V^\alpha) \]

\[ = G \frac{d}{dt} (\log b). \]  

(3.17)

Using (3.17) in (3.12) we finally obtain

\[ \psi b^{-1} = \text{constant}, \]  

(3.18)

as the result of integration.

No elaboration is needed to stress the elegance, the generality, and the precision of this result.
4. **TRANVERSELY ISOTROPIC MEDIUM**

Here we consider an illustration which brings out our results in comparable form with those of others. We take the constitutive law as

$$
\sigma_{11} = c_{11} u_{1,1} + c_{12} u_{2,2} + c_{13} u_{3,3}, \quad \sigma_{22} = c_{12} u_{1,1} + c_{11} u_{2,2} + c_{13} u_{3,3}, \quad \sigma_{33} = c_{13} u_{1,1} + c_{12} u_{2,2} + c_{33} u_{3,3},
$$

$$
\sigma_{31} = c_{44} u_{3,1} + c_{44} u_{1,3}, \quad \sigma_{12} = \frac{c_{11} - c_{12}}{2} (u_{1,2} + u_{2,1}). \quad (4.1)
$$

We use \((c_{44})\) as the typical modulus used to nondimensionalize the components of the stress tensor and to transform the time. Let the nondimensional elastic modulus be

$$
a = \frac{c_{11}}{c_{44}}, \quad b = \frac{c_{12}}{c_{44}}, \quad c = \frac{c_{13}}{c_{44}}, \quad d = \frac{c_{33}}{c_{44}}. \quad (4.2)
$$

Take cylindrical coordinates \((\rho, \phi, x_3)\) with \(\rho^2 = x_1^2 + x_2^2\). Let the normal make an angle \((\theta)\) with the \((x_3)\) axis. We assume that the wavefront is a surface of revolution. Since the velocity of propagation is found to be a function of \((\theta)\) only, it will remain a surface of revolution, once it is so. Let \((t_i)\) and \((\zeta_i)\) be unit tangents to \((\theta)\) and \((\phi)\) curves such that \((n_i, t_i, \zeta_i)\) form a right-handed system. Differentiation with respect to \((\theta)\) and \((\phi)\) are denoted by a dot and a dash, respectively. The \((\theta)\) and the \((\phi)\) curves are now the lines of curvatures and let their curvatures be \((K_1)\) and \((K_2)\), respectively. We thus have

$$
n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad t_i = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta),
$$

$$
\zeta_i = (-\sin \phi, \cos \phi, 0), \quad l_i = (-\cos \phi, -\sin \phi, 0), \quad (4.3)
$$

$$
n_i = t_i, \quad i_i = -n_i, \quad n_i' = \sin \theta \zeta_i. \quad (4.4)
$$

Lastly we introduce symmetric matrices \(a_{ij}(n, n), a_{ij}(t, t), a_{ij}(s, s), b_{ij}(n, t), b_{ij}(n, s), b_{ij}(n, 1),\) defined (typically) as (for \(a_{ij}(n, n)\), \(b_{ij}(n, t)\))

$$
a_{11} = an_1 n_1 + \frac{a - b}{2} n_2 n_2 + n_3 n_3,
$$

$$
a_{22} = \frac{a - b}{2} n_1^2 + an_2^2 + n_3^2, \quad a_{23} = (c + 1) n_2 n_3,
$$

$$
a_{33} = n_1^2 + n_2^2 + d n_3^2, \quad a_{31} = (c + 1) n_3 n_1,
$$

$$
b_{11} = 2an_1 t_1 + (a - b) n_2 t_2 + 2n_3 t_3, \quad b_{12} = \frac{a + b}{2} n_1 n_2,
$$

$$
b_{22} = (a - b) n_1 t_1 + 2an_2 t_2 + 2n_3 t_3, \quad b_{23} = (c + 1) (n_2 t_3 + n_3 t_2),
$$

$$
b_{33} = 2n_1 t_1 + 2n_2 t_2 + 2n_3 t_3, \quad b_{31} = (c + 1) (n_3 t_1 + n_1 t_3),
$$

$$
b_{32} = 2n_1 t_1 + 2n_2 t_2 + 2d n_3 t_3, \quad b_{13} = \frac{a + b}{2} (n_1 t_2 + n_2 t_1). \quad (4.5)
$$
Then the compatibility conditions lead us to
\[ a_{ij}(n, n) \xi_j = G^2 \xi_i, \quad (4.7) \]
\[ 2G \frac{d\phi}{dt} = A\psi + B\dot{\psi} + C\psi', \quad (4.8) \]
where
\[
A = K_1G^2 - K_1b_{ij}(n, t)L_jL_j - K_2\csc \theta L_iL_j' - a_{ij}(t, t)K_iL_j - b_{ij}(\xi, \zeta)K_2L_iL_j, \quad (4.8, a)
\]
\[ B = 2GGK_1 - K_1b_{ij}(n, t)L_jL_j, \quad (4.8, b) \]
\[ C = -K_2\csc \theta b_{ij}(n, \zeta)L_iL_j. \quad (4.8, c) \]

**Velocities and Eigenvectors**

This is a case when additional symmetries hold and the roots of the cubic can be obtained more easily. To start with, we first note that \((\xi_i)\) can be seen to be an eigenvector giving a mode corresponding to the rotation about the \((x_3)\) axis. This is described by
\[
^{(3)} L_i = \xi_i, \\
2G^2 = a - b + (2 - a + b) n_3^2, \\
\psi = 2[\omega_3], \\
2\omega_3 = e_{ij,k}u_{k,i}. \quad (4.9)
\]

Knowing one mode it is much easier to construct the others. Take any vector \((\cos \phi, \sin \phi, N)\) orthogonal to \((L_i)\). Multiply (4.7) by this and sum the equations; then equate the coefficients of \((\xi_i)\). This leads to two equations for \((N)\) and \((G^2)\). Thus we get the other two modes as
\[
^{(1)} 2G^2 = (a + 1) \sin^2 \theta + (d + 1) \cos^2 \theta + \sqrt{A}, \quad (4.10) \\
^{(1)} L_i = (R \cos \phi, R \sin \phi, S), \quad (4.10, a) \\
^{(2)} 2G^2 = (a + 1) \sin^2 \theta + (d + 1) \cos^2 \theta - \sqrt{A}, \quad (4.11) \\
^{(2)} L_i = (S \cos \phi, S \sin \phi, -R), \quad (4.11, a) 
\]
where
\[ A = P^2 + Q^2, \quad R = \left( \frac{\sqrt{A} - P}{2 \sqrt{A}} \right)^{1/3}, \quad S = \left( \frac{\sqrt{A} + P}{2 \sqrt{A}} \right)^{1/3}, \]
\[ P = (d - 1) \cos^2 \theta - (a - 1) \sin^2 \theta, \quad Q = 2(c + 1) \sin \theta \cos \theta. \]  

(4.12)

For isotropy, \( a = d = b + 2 = c + 2; \) \((L_i)\) reduces to \((n_i)\) giving the dilatational mode and \((L_i)\) to \((t_i)\) giving the shear mode. However, now each of these modes is accompanied by both.

**Growth Equation**

We now return to (4.5). We must first prove that it does not involve terms with tangential derivatives. Differentiating the relation \( G^2 = a_{ij}(n, n) L_i L_j \) with respect \((\theta)\) and \((\phi)\) we verify that \( B = C = 0. \) Repeated differentiations lead to
\[ b_{ij}(n, t) L_i L_j = G^2 + \dot{G} + G G - a_{ij}(t, t) L_i L_j, \]  
\[ b_{ij}(n, \zeta) L_i L_j = - a_{ij}(\zeta, \zeta) L_i L_j - \frac{1}{2} b_{ij}(n, t) L_i L_j. \]  

(4.13, 4.14)

By further direct evaluation we obtain
\[ A = - K_1 G(G + \ddot{G}) - G K_2 \cot \theta (\dot{G} \sin \theta + \ddot{G} \cos \theta). \]  

(4.15)

Let the initial manifold be \((x_i^0),\) given by
\[ x_i^0 = (g(\theta) \cos \phi, g(\theta) \sin \phi, f(\theta)), \]  
\[ f' = - \dot{g} \tan \theta. \]

(4.16, a)

The latter condition assures us that the normal makes an angle \( (\theta)\) with the \((x_3)\) axis. Integrating the ray equations
\[ V_i = \frac{dx_i}{dt} = G n_i + \dot{G} t_i, \]  
we obtain
\[ x_i = x_i^0 + V_i t. \]  

(4.17, 4.18)

From these we get the curvatures as
\[ K_1 = \frac{1}{\rho_1} = \frac{1}{\dot{g} \sec \theta + (G + \ddot{G}) t}, \]  
\[ K_2 = \frac{1}{\rho_2} = \frac{1}{g \cosec \theta + (\dot{G} + \ddot{G} \cot \theta) t}. \]  

(4.19, a, b)
It is well-known that the reciprocal wave-speed locus \((\gamma)\), a surface of revolution obtained by revolving the curve, \((1/G(\theta))\) versus \((\theta)\), about \((x_a)\) axis, plays a crucial role in deciding the singularity of the strength of the wave. Let us take that \(\Sigma(t)\), the wave-front and the reciprocal wave-speed locus are surfaces of revolution obtained from the revolution of curves \(\Sigma_0(t)\) and \((\gamma_0)\). Then a point of inflexion on \((\gamma_0)\) corresponds to a cusp on \(\Sigma_0(t)\) \([7]\). But the curvature at a point of inflexion of \((\gamma_0)\) vanishes. This, as we see below, determines the nature of the strength of the wave asymptotically. The curvatures of the reciprocal wave-speed locus are given by

\[
k_1' = (G + \ddot{G}) \frac{G^3}{V^8}, \tag{4.20, a}
\]

\[
k_2' = (G + \dot{G} \cot \theta) \frac{G}{V}, \tag{4.20, b}
\]

with

\[
V^2 = V_1 V_i.
\]

Integrating (4.8) we now obtain

\[
\psi^2 \rho_1 \rho_2 = \text{constant}, \tag{4.21}
\]

where

\[
\rho_1 = \dot{g} \sec \theta + \frac{V^8}{G^8} k_1', \tag{4.21, a}
\]

\[
\rho_2 = g \cosec \theta + \frac{V}{G} k_2'. \tag{4.21, b}
\]

The result (4.21), a particular case of (3.18), gives the strength of the wave \((\psi)\) for all times. There may exist points where any of the curvatures may become infinite, in a finite time. Such a point is called a focus. The strength, of course, tends to infinity as we approach such a point. Let \((P)\) be a point of \(\Sigma(t)\) corresponding to \((P^*)\) on \((\gamma)\). If none of the curvatures vanishes at \((P^*)\), then \((\psi)\) varies, asymptotically as \((t \to \infty)\), as \((t^{-1})\) at \((P)\). If one of the curvatures vanishes at \((P^*)\), then the propagation at \((P)\) is cylindrical; the strength there varies inversely as \((t^{1/2})\) for large times. If both the curvatures at \((P^*)\) vanish, the propagation at \((P)\) is planar; the strength remains constant in time. These asymptotic results correspond to those of Lighthill \([1]\), Ludwig \([2]\) and Duff \([3]\). However Lighthill’s results for a dispersive medium do not follow from our discussion. We hope to initiate the study subsequently.
CONCLUSION

The equation governing the strength of the wave along the rays is completely integrated. The product of the square of the strength and the Gaussian curvature remains constant. The asymptotic variation depends crucially on the nature of the reciprocal wave-speed locus, a result comparable with that of earlier work.

REFERENCES