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Inversion of Certain Extensions of Toeplitz Matrices

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It is shown how an algorithm for inverting Toeplitz matrices using $O(n^2)$ operations can be modified to deal with a certain type of extension, named conjugate-Toeplitz matrices. The block partitioned case is also included.

1. INTRODUCTION

It is well known that inversion of an $n \times n$ Toeplitz matrix can be achieved using $O(n^2)$ operations. In a previous paper [3] we defined some extensions of Toeplitz matrices, and showed how a number of properties of the standard case still applied. In this present paper we concentrate on what we termed *conjugate-Toeplitz* (*CT*) matrices, and obtain direct analogues of well-known algorithms for inverting Toeplitz and block Toeplitz matrices. The CT case is given in Section 2, with an illustrative numerical example in Section 3, and the block CT case is covered in Section 4. The somewhat tedious details of the verification of the recursion formulae are relegated to an Appendix. Finally, in Section 5, we demonstrate the elementary fact that block (conjugate) Toeplitz matrices are similar to partitioned matrices in which each block has (conjugate) Toeplitz form.

The motivation for this work is two fold. Firstly, it is of interest mathematically to extend the class of matrices whose inversion requires $O(n^2)$ operations. Secondly, there are a number of problems in signal processing where inversion of nonstandard Toeplitz-type matrices arise (e.g. [7]) and it is hoped that the matrices discussed below may find applications in that field.

It is necessary to establish some preliminary notation: For any complex number x, let $c(x)$ denote the complex conjugate $\bar{x}$, so that in particular $c^{2m}(x) = x$, $c^{2m-1}(x) = \bar{x}$ for all positive integers m, and $c^0(x) \equiv x$. An $n \times n$ matrix $A = [a_{ij}]$ is *conjugate-Toeplitz* (*CT*) if

$$a_{ij} = c^{i-1}(a_{i-j}), \qquad i, j = 1, 2, \dots, n. \tag{1.1}$$

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Notice that (1.1) implies the alternative definition

$$a_{i+1,j+1} = c(a_{ij}) \tag{1.2}$$

and if the a_{ij} are all real, then (1.1) and (1.2) reduce to the usual characterizations of a Toeplitz matrix. We shall use $c^r(A)$ to stand for the matrix having i,j element $c^r(a_{ij})$, and J to denote the "reverse unit matrix," having ones along the secondary diagonal and zeros elsewhere.

We note in passing, that if A is CT then JA has "conjugate-Hankel" form so there is no need to consider such cases separately.

2. The Inversion of Conjugate-Toeplitz Matrices

In [3] we showed how the algorithm to invert a Toeplitz matrix due to Trench [8, 9] could be extended to persymmetric and perhermitian CT matrices. We now give a corresponding algorithm for any strongly nonsingular CT matrix (i.e., all its leading principal minors are nonzero). If A_n is an $n \times n$ CT matrix as defined in (1.1), we need to introduce a related matrix $B_n = [b_{ij}]$ as follows.

Definition 2.1. $B_n = c^{n+1}(JA_n^T J)$.

It follows immediately that $B_n^{-1} = c^{n+1}(JA_n^{-T}J)$, and hence the first row and column of B_n^{-1} are respectively the last column and row of $c^{n+1}(A_n^{-1})$ with the elements reversed. Note also that $a_{11} = b_{11} = a_0$. The key idea is to apply the bordering technique of [8] to A_{n+1} and B_{n+1} simultaneously.

For $n \geqslant 1$, define

$$A_{n+1} = \begin{bmatrix} a_0 & r_n^T \\ s_n & \bar{A}_n \end{bmatrix}, \qquad A_{n+1}^{-1} = \begin{bmatrix} \alpha_n & e_n^T \\ f_n & M_n \end{bmatrix},$$
$$B_{n+1} = \begin{bmatrix} a_0 & u_n^T \\ v_n & \bar{B}_n \end{bmatrix}, \qquad B_{n+1}^{-1} = \begin{bmatrix} \beta_n & g_n^T \\ h_n & P_n \end{bmatrix}, \tag{2.2}$$

where $\bar{A}_1 = \bar{a}_0 = \bar{B}_1$, and

$$r_n = (a_{-1}, a_{-2}, \ldots, a_{-n})^T, \qquad s_n = (c(a_1), c^2(a_2), \ldots, c^n(a_n))^T,$$
$$u_n = (c(a_{-1}), c^2(a_{-2}), \ldots, c^n(a_{-n}))^T, \qquad v_n = (a_1, a_2, \ldots, a_n)^T. \tag{2.3}$$

It follows (see Remark 6 in the Appendix), that the following recursive steps generate the inverse of A_{n+1}.

Step 1. If $A_2^{-1} = [x_{ij}]$ then

$$\alpha_1 = x_{11}, \qquad \beta_1 = \bar{x}_{22}, \qquad e_1 = \bar{g}_1 = x_{12}, \qquad f_1 = \bar{h}_1 = x_{21}. \tag{2.4}$$

These are the initial values for the recursion.

Step 2. For each integer i, $1 \leqslant i \leqslant n-1$,

$$\gamma_i = c^{i+1}(\beta_i)\, a_{-(i+1)} + c^{i+1}(g_i^T)\, Jr_i, \tag{2.5}$$

$$\delta_i = c^{i+1}[\beta_i a_{i+1} + h_i^T Jc^{i+1}(s_i)], \tag{2.6}$$

$$\varepsilon_i = c^{i+1}[\alpha_i a_{-(i+1)} + e_i^T Jc^{i+1}(u_i)], \tag{2.7}$$

$$\eta_i = c^{i+1}(\alpha_i)\, a_{i+1} + c^{i+1}(f_i^T)\, Jv_i, \tag{2.8}$$

$$\alpha_{i+1} = \frac{\alpha_i + c^{i+1}(\beta_i \varepsilon_i \eta_i)}{1 - \gamma_i \delta_i c^{i+1}(\varepsilon_i \eta_i)}, \qquad \beta_{i+1} = \frac{\beta_i + c^{i+1}(\alpha_i \gamma_i \delta_i)}{1 - \varepsilon_i \eta_i c^{i+1}(\gamma_i \delta_i)}, \tag{2.9}$$

$$\begin{aligned} e_{i+1} &= \alpha_{i+1} \begin{bmatrix} \dfrac{1}{\alpha_i} e_i - \gamma_i c^{i+1}\left(\dfrac{1}{\beta_i} Jh_i\right) \\ -\gamma_i \end{bmatrix}, \\ f_{i+1} &= \alpha_{i+1} \begin{bmatrix} \dfrac{1}{\alpha_i} f_i - \delta_i c^{i+1}\left(\dfrac{1}{\beta_i} Jg_i\right) \\ -\delta_i \end{bmatrix}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} g_{i+1} &= \beta_{i+1} \begin{bmatrix} \dfrac{1}{\beta_i} g_i - \varepsilon_i c^{i+1}\left(\dfrac{1}{\alpha_i} Jf_i\right) \\ -\varepsilon_i \end{bmatrix}, \\ h_{i+1} &= \beta_{i+1} \begin{bmatrix} \dfrac{1}{\beta_i} h_i - \eta_i c^{i+1}\left(\dfrac{1}{\alpha_i} Je_i\right) \\ -\eta_i \end{bmatrix}. \end{aligned} \tag{2.11}$$

Step 3. If we now let $A_{n+1}^{-1} = [y_{ij}]$, then

$$y_{11} = \alpha_n, \quad y_{1j} = [e_n]_{j-1,1}, \quad y_{i1} = [f_n]_{i-1,1}, \quad i,j = 2, 3,..., n+1, \tag{2.12}$$

$$\begin{aligned} &y_{n+1,n+1} = \beta_n, \qquad y_{i,n+1} = c^{n+2}[h_n]_{n-i+1,1}, \\ &y_{n+1,j} = c^{n+2}[g_n]_{n-j+1,1}, \qquad i,j = 2, 3,..., n, \end{aligned} \tag{2.13}$$

$$y_{i+1,j+1} = \bar{y}_{ij} + \frac{y_{i+1,1}\, y_{1,j+1}}{y_{11}} - \frac{\bar{y}_{n+1,j}\, \bar{y}_{i,n+1}}{\bar{y}_{n+1,n+1}}, \qquad i,j = 2, 3,..., n-1. \tag{2.14}$$

The number of multiplications required using this method to determine A_{n+1}^{-1}

is of the order $8n^2$, compared with $O(n^3)$ using a standard method such as gaussian elimination.

Remark 1. If all a_{i-j} are real then A_{n+1} reduces to ordinary Toeplitz form, and $B_{n+1} = A_{n+1}$. Thus the amount of work is halved, ε_i, η_i, β_{i+1}, g_{i+1} and h_{i+1} being redundant, and we have the algorithm of [8], requiring $4n^2$ multiplications.

Remark 2. In the Toeplitz case, Zohar [9] removes a factor a_0 from A_{n+1} in (2.2) so that the element in the top left hand corner becomes unity. This cannot be done in the CT case as it destroys the CT pattern. The recursion given above however, shows that this step is unnecessary.

3. A Numerical Example of the Algorithm

Consider the fourth order CT matrix

$$A_4 = \begin{bmatrix} \frac{1-i}{2} & i & \frac{-1-i}{2} & \frac{-2+i}{5} \\ \frac{2-i}{5} & \frac{1+i}{2} & -i & \frac{-1+i}{2} \\ \frac{3-i}{10} & \frac{2+i}{5} & \frac{1-i}{2} & i \\ \frac{4-i}{17} & \frac{3+i}{10} & \frac{2-i}{5} & \frac{1+i}{2} \end{bmatrix}. \tag{3.1}$$

Step 1. We have

$$A_2 = \begin{bmatrix} \frac{1-i}{2} & i \\ \frac{2-i}{5} & \frac{1+i}{2} \end{bmatrix}, \qquad A_2^{-1} = \begin{bmatrix} \frac{-1+7i}{5} & \frac{8-6i}{5} \\ \frac{-4-2i}{5} & \frac{7+i}{5} \end{bmatrix}, \tag{3.2}$$

and from (2.4)

$$\alpha_1 = \frac{-1+7i}{5}, \qquad \beta_1 = \frac{7-i}{5},$$
$$e_1 = \bar{g}_i = \frac{8-6i}{5}, \qquad f_1 = \bar{h}_1 = \frac{-4-2i}{5}. \tag{3.3}$$

Step 2. Now putting $i = 1$ in (2.5)–(2.8) gives

$$\gamma_1 = \beta_1 a_{-2} + g_1 r_1 = -2 + i, \qquad \delta_1 = \beta_1 a_2 + h_1 s_1 = \frac{4+3i}{25}, \tag{3.4}$$

$$\varepsilon_1 = \alpha_1 a_{-2} + e_1 u_1 = \frac{-2-11i}{5}, \qquad \eta_1 = \alpha_1 a_2 + f_1 v_1 = \frac{-4+3i}{25}. \tag{3.5}$$

Hence from (2.9) and (3.3)–(3.5)

$$\alpha_2 = \frac{1+3i}{2}, \qquad \beta_2 = \frac{13-9i}{10}. \tag{3.6}$$

Lastly in this stage (2.10) and (2.11) give

$$e_2 = \begin{bmatrix} -1-3i \\ \dfrac{5+5i}{2} \end{bmatrix}, \qquad f_2 = \begin{bmatrix} \dfrac{-3-i}{5} \\ \dfrac{1-3i}{10} \end{bmatrix},$$

$$g_2 = \begin{bmatrix} \dfrac{1+7i}{5} \\ \dfrac{5+5i}{2} \end{bmatrix}, \qquad h_2 = \begin{bmatrix} -1+i \\ \dfrac{1-3i}{10} \end{bmatrix}. \tag{3.7}$$

These steps are now repeated for $i = 2$, giving

$$\gamma_2 = \bar{\beta}_2 a_{-3} + \bar{g}_2^T J r_2 = 1 + 3i, \qquad \delta_2 = \bar{\beta}_2 \bar{a}_3 + \bar{h}_2^T J s_2 = \frac{5+3i}{85},$$

$$\varepsilon_2 = \bar{\alpha}_2 \bar{a}_{-3} + \bar{e}_2^T J u_2 = -1 - 3i, \qquad \eta_2 = \bar{\alpha}_2 a_3 + \bar{f}_2^T J v_2 = \frac{11-27i}{425},$$

$$\alpha_3 = \frac{21+103i}{78}, \qquad \beta_3 = \frac{393-349i}{390}, \tag{3.8}$$

$$e_3 = \begin{bmatrix} \dfrac{4-33i}{13} \\ \dfrac{-45+95i}{26} \\ \dfrac{144-83i}{39} \end{bmatrix}, \qquad f_3 = \begin{bmatrix} \dfrac{-2-i}{3} \\ \dfrac{1-7i}{30} \\ \dfrac{6-17i}{195} \end{bmatrix},$$

$$g_3 = \begin{bmatrix} \dfrac{-8+19i}{15} \\[2ex] \dfrac{13+i}{6} \\[2ex] \dfrac{144+83i}{39} \end{bmatrix}, \qquad h_3 = \begin{bmatrix} \dfrac{-10+15i}{13} \\[2ex] \dfrac{-1-57i}{130} \\[2ex] \dfrac{6+17i}{195} \end{bmatrix}. \tag{3.9}$$

We now have the first and last rows and columns of A_4^{-1}. In fact if $A_4^{-1} = [y_{ij}]$, then y_{14} occurs as both $[e_3]_{31}$ and $[\bar{g}_3]_{31}$, and similarly y_{41} is duplicated. The rest of the elements of A_4^{-1} are obtained from (2.14), a single example of which is given below:

$$y_{23} = \frac{4-33i}{13} + \frac{\left(\dfrac{-2-i}{3}\right)\left(\dfrac{-45+95i}{26}\right)}{\left(\dfrac{21+103i}{78}\right)} - \frac{\left(\dfrac{-1-57i}{130}\right)\left(\dfrac{144+83i}{39}\right)}{\left(\dfrac{393-349i}{390}\right)}$$

$$= -2+i.$$

The complete inverse matrix is

$$A_4^{-1} = \begin{bmatrix} \dfrac{21+103i}{78} & \dfrac{4-33i}{13} & \dfrac{-45+95i}{26} & \dfrac{144-83i}{39} \\[2ex] \dfrac{-2-i}{3} & \dfrac{3-i}{2} & -2+i & \dfrac{13-i}{6} \\[2ex] \dfrac{1-7i}{30} & \dfrac{-4+3i}{5} & \dfrac{3+i}{2} & \dfrac{-8-19i}{15} \\[2ex] \dfrac{6-17i}{195} & \dfrac{-1+57i}{130} & \dfrac{-10-15i}{13} & \dfrac{393+349i}{390} \end{bmatrix}.$$

4. The Inversion of Block CT Matrices

A scheme for the inversion of block Toeplitz matrices using an extension of the Trench algorithm has been devised by Akaike [1]. We now generalize this result to the case of block CT matrices which satisfy the conditions in Remark 7. We first make some further definitions which will be needed in this section.

DEFINITION 4.1. The $mn \times mn$ reverse unit block matrix J_β is defined as

$$J_\beta = \begin{bmatrix} 0_m & \cdots & 0_m & I_m \\ \vdots & & & 0_m \\ 0_m & & & \vdots \\ I_m & 0_m & \cdots & 0_m \end{bmatrix}, \tag{4.1}$$

where I_m and 0_m are the unit and zero matrices of order m, respectively. Each row and column has n blocks.

DEFINITION 4.2. An $mn \times mn$ matrix $A_{m,n}$ is *block CT* if

$$A_{m,n} = \begin{bmatrix} A_0 & A_{-1} & \cdots & A_{-n+1} \\ c(A_1) & c(A_0) & \cdots & c(A_{-n+2}) \\ \vdots & & \cdots & \\ c^{n-1}(A_{n-1}) & c^{n-1}(A_{n-2}) & \cdots & c^{n-1}(A_0) \end{bmatrix}, \tag{4.2}$$

where $A_r = [a_{ij}^{(r)}]$ is an arbitrary $m \times m$ matrix and $c(A_r) = [\bar{a}_{ij}^{(r)}]$.

DEFINITION 4.3.

$$B_{m,n} = J_\beta c^{n+1}(A_{m,n}^T) J_\beta. \tag{4.3}$$

It can easily be seen that

$$B_{m,n} = \begin{bmatrix} A_0^T & c(A_{-1}^T) & \cdots & c^{n-1}(A_{-n+1}^T) \\ \vdots & \vdots & & \vdots \\ A_{n-1}^T & c(A_{n-2}^T) & \cdots & c^{n-1}(A_0^T) \end{bmatrix}. \tag{4.4}$$

In order to determine the inverse of $A_{m,(n+1)}$, we need to border $A_{m,(n+1)}$, $B_{m,(n+1)}$ and their inverses as follows:

$$\begin{aligned} A_{m,(n+1)} &= \begin{bmatrix} A_0 & R_n^T \\ S_n & \bar{A}_{m,n} \end{bmatrix}, & A_{m,(n+1)}^{-1} &= \begin{bmatrix} Y_n & E_n^T \\ F_n & M_{m,n} \end{bmatrix}, \\ B_{m,(n+1)} &= \begin{bmatrix} A_0^T & U_n^T \\ V_n & \bar{B}_{m,n} \end{bmatrix}, & B_{m,(n+1)}^{-1} &= \begin{bmatrix} Z_n & G_n^T \\ H_n & P_{m,n} \end{bmatrix}, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} R_n^T &= (A_{-1},..., A_{-n}), & S_n^T &= (c(A_1^T),..., c^n(A_n^T)), \\ U_n^T &= (c(A_{-1}^T),..., c^n(A_{-n}^T)), & V_n^T &= (A_1,..., A_n), \end{aligned} \tag{4.6}$$

and Y_n and Z_n are $m \times m$ matrices.

We now give the recursive steps required to obtain the inverse of $A_{m,(n+1)}$, corresponding to those in Section 2. The proof is given in the Appendix.

Step 1. Invert $A_{m,2}^{-1}$ and $B_{m,2}^{-1}$ by some standard method, and hence obtain Y_1, Z_1, E_1, F_1, G_1 and H_1. These are the starting values of the recursion.

Step 2. For each integer i, $1 \leqslant i \leqslant n-1$, determine

$$\gamma_i = A_{-(i+1)} c^{i+1}(Z_i^T) + R_i^T J_\beta c^{i+1}(G_i), \tag{4.7}$$

$$\delta_i = c^{i+1}[Z_i^T A_{i+1} + H_i^T J_\beta c^{i+1}(S_i)], \tag{4.8}$$

$$\varepsilon_i = c^{i+1}[A_{-(i+1)}^T Y_i^T + c^{i+1}(U_i^T) J_\beta E_i], \tag{4.9}$$

$$\eta_i = c^{i+1}(Y_i^T) A_{i+1}^T + c^{i+1}(F_i^T) J_\beta V_i. \tag{4.10}$$

Solve the following two bilinear equations for Y_{i+1} and Z_{i+1} using, for example, [5]:

$$Y_{i+1} - c^{i+1}(\varepsilon_i^T)\,\delta_i Y_{i+1} \gamma_i c^{i+1}(\eta_i^T) = Y_i + c^{i+1}(\eta_i Z_i \varepsilon_i)^T, \tag{4.11}$$

$$Z_{i+1} - c^{i+1}(\gamma_i^T)\,\eta_i Z_{i+1} \varepsilon_i c^{i+1}(\delta_i^T) = Z_i + c^{i+1}(\delta_i Y_i \gamma_i)^T. \tag{4.12}$$

Finally, obtain

$$\left.\begin{aligned} E_{i+1} &= \begin{bmatrix} E_i Y_i^{-T} - J_\beta c^{i+1}(H_i Z_i^{-1})\,\gamma_i^T \\ -\gamma_i^T \end{bmatrix} Y_{i+1}^T, \\ F_{i+1} &= \begin{bmatrix} F_i Y_i^{-1} - J_\beta c^{i+1}(G_i Z_i^{-T})\,\delta_i \\ -\delta_i \end{bmatrix} Y_{i+1}, \end{aligned}\right\} \tag{4.13}$$

$$\left.\begin{aligned} G_{i+1} &= \begin{bmatrix} G_i Z_i^{-T} - J_\beta c^{i+1}(F_i Y_i^{-1})\,\varepsilon_i^T \\ -\varepsilon_i^T \end{bmatrix} Z_{i+1}^T, \\ H_{i+1} &= \begin{bmatrix} H_i Z_i^{-1} - J_\beta c^{i+1}(E_i Y_i^{-T})\,\eta_i \\ -\eta_i \end{bmatrix} Z_{i+1}. \end{aligned}\right\} \tag{4.14}$$

Step 3. Determine the blocks inside the border of $A_{m,(n+1)}^{-1}$ using

$$\begin{aligned} [A_{m,(n+1)}^{-1}]_{i+1,j+1} = {} & [\bar{A}_{m,(n+1)}^{-1}]_{ij} + [F_n]_i\, Y_n^{-1}[E_n^T]_j \\ & - c^{n+1}\{[G_n]_{n-i+1}\, Z_n^{-T}[H_n^T]_{n-j+1}\}, \end{aligned} \tag{4.15}$$

where $[F_n]_i$ is the ith block in the block vector F_n, $[E_n^T]_j$, $[G_n]_{n-i+1}$, $[H_n^T]_{n-j+1}$ being defined similarly and $[A_{m,(n+1)}^{-1}]_{ij}$ is the ijth block in $A_{m,(n+1)}^{-1}$.

5. Block Pattern and Pattern Block Matrices

In the previous section we presented an algorithm for inverting a block CT matrix, extending similar results for block Toeplitz matrices [1] and block circulant matrices [4]. It does not seem to have been realised, however, that "pattern block" matrices are similar to "block pattern" matrices, as we will now show.

The result is first given for the Toeplitz case, followed by a statement of the necessary modifications for other patterns.

Definition 5.1. The *shift matrix* of order n is

$$Z = \begin{vmatrix} 0 & I_{n-1} \\ 0 & 0 \end{vmatrix}.$$

It is easy to see that

$$Z^r = \begin{vmatrix} 0 & I_{n-r} \\ 0_r & 0 \end{vmatrix}, \qquad r = 1, 2, \ldots, n-1, \tag{5.1}$$

and it can also easily be shown that $(Z^r)^T$ is the Moore–Penrose inverse of Z^r. This inverse, for convenience in the following analysis, will be denoted by Z^{-r}. Note that Z^0 denotes the unit matrix I_n. We can now give a series definition for a Toeplitz matrix.

If $A = [a_{ij}]$ is a Toeplitz matrix of order n, and $a_{1j} = a_{1-j}$ and $a_{i1} = a_{i-1}$, then we can write

$$A = \sum_{r=-(n-1)}^{n-1} a_{-r} Z^r. \tag{5.2}$$

In a similar fashion, if $\otimes$ denotes Kronecker product, then

$$\sum_{r=-(n-1)}^{n-1} Z^r \otimes A_{-r} = \begin{bmatrix} A_0 & A_{-1} & \cdots & A_{-n+1} \\ A_1 & A_0 & & \vdots \\ & & \cdots & \\ \vdots & & & A_{-1} \\ A_{n-1} & \cdots & A_1 & A_0 \end{bmatrix}, \tag{5.3}$$

where the A_i are arbitrary matrices of order m, is a *block Toeplitz* matrix of order mn. Furthermore, it is easy to see that

$$\sum_{r=-(n-1)}^{n-1} A_{-r} \otimes Z^r = \begin{bmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & \cdots & \vdots \\ B_{m1} & \cdots & B_{mm} \end{bmatrix} \tag{5.4}$$

is a *Toeplitz block* matrix, i.e., each matrix B_{ij} in (5.4) is itself Toeplitz. If, as before, we denote the r, s element of a matrix X by $[X]_{rs}$, then comparing (5.3) and (5.4) reveals that

$$[B_{rs}]_{1j} = [A_{1-j}]_{rs}, \qquad [B_{rs}]_{i1} = [A_{i-1}]_{rs}.$$

LEMMA 5.1. *The block Toeplitz matrix* (5.3) *is similar to the Toeplitz block matrix* (5.4).

Proof. If we introduce the vec-permutation matrix $I_{m,n}$, it is well known [6] that for arbitrary matrices $X(m \times m)$ and $Y(n \times n)$, then

$$I_{m,n}(X \otimes Y)\, I_{n,m} = Y \otimes X, \tag{5.5}$$

where

$$I_{m,n} I_{n,m} = I_{mn}; \tag{5.6}$$

I_{mn} denoting the unit matrix of order mn. Applying (5.5) to (5.3) and (5.4) term by term shows that

$$I_{m,n}\left(\sum_{r=-(n-1)}^{(n-1)} A_{-r} \otimes Z^r\right) I_{n,m} = \sum_{r=-(n-1)}^{n-1} Z^r \otimes A_{-r}. \quad ∎$$

Remark 3. In order to replace "Toeplitz" by "conjugate Toeplitz" in (5.2) we need to introduce an operator matrix:

DEFINITION 5.2. The operator matrix Z_c is defined by

$$Z_c = \begin{bmatrix} 0 & c^0 & 0 & \cdots & 0 \\ 0 & 0 & c^1 & & \cdot \\ \vdots & \vdots & & & \cdot \\ 0 & 0 & & & c^{n-2} \\ 0 & 0 & \cdots & & 0 \end{bmatrix},$$

where $xc^r \equiv c^r(x)$.

If we replace Z^r by $Z_c Z^{r-1}$ and Z^{-r} by $Z_c^T Z^{-r+1}$, $r \neq 0$, in (5.3) then it becomes a block CT matrix, and similarly (5.4) becomes a CT block matrix. The Lemma then applies to the CT case.

Remark 4. Toeplitz matrices can also be replaced by circulant matrices in the above argument, by setting

$$a_r = a_{-n+r}, \qquad r = 1, 2, \ldots, n-1. \tag{5.7}$$

In this case (5.7) allows the series (5.2) to be written as

$$A = a_0 I + \sum_{r=1}^{n-1} a_{-r}(Z^r + Z^{-(n-r)}) = a_0 I + \sum_{r=1}^{n-1} a_{-r}\pi^r,$$

where $\pi = \text{circ}(0, 1, 0, \ldots, 0)$, as defined by Davis [4]. This is the usual series representation of a circulant matrix. It then easily follows that a block circulant matrix is similar to a circulant block matrix, a fact not quoted in [4].

Remark 5. Lemma 5.1 and its corollaries are an expanded version of [2].

APPENDIX

The proof of the recursion steps for inverting a block CT matrix is given below. The inversion of a CT matrix follows directly as a corollary.

We begin from Eqs. (4.5). If we consider $A^{-1}_{m,(n+1)} A_{m,(n+1)} = I_{m(n+1)}$ we have

$$Y_n R_n^T + E_n^T \bar{A}_{m,n} = 0, \qquad F_n R_n^T + M_{m,n} \bar{A}_{m,n} = I_{mn}, \tag{A.1}$$

and so

$$Y_n^{-1} E_n^T = -R_n^T \bar{A}^{-1}_{m,n}, \qquad M_{m,n} = \bar{A}^{-1}_{m,n} - F_n R_n^T \bar{A}^{-1}_{m,n}. \tag{A.2}$$

Hence

$$M_{m,n} = \bar{A}^{-1}_{m,n} + F_n Y_n^{-1} E_n^T \tag{A.3}$$

and substituting (A.3) into (4.5) produces

$$A^{-1}_{m,(n+1)} = \begin{vmatrix} Y_n & E_n^T \\ F_n & \bar{A}^{-1}_{m,n} + F_n Y_n^{-1} E_n^T \end{vmatrix}. \tag{A.4}$$

Similarly, by considering $B^{-1}_{m,(n+1)} B_{m,(n+1)} = I$, we obtain

$$B^{-1}_{m,(n+1)} = \begin{vmatrix} Z_n & G_n^T \\ H_n & \bar{B}^{-1}_{m,n} + H_n Z_n^{-1} G_n^T \end{vmatrix}. \tag{A.5}$$

From Definition 4.3 we have

$$A^{-1}_{m,(n+1)} = J_\beta c^{n+2} (B^{-T}_{m,(n+1)}) J_\beta \tag{A.6}$$

and therefore from (A.5)

$$A^{-1}_{m,(n+1)} = c^{n+2} \begin{vmatrix} J_\beta \bar{B}^{-T}_{m,n} J_\beta + J_\beta G_n Z_n^{-T} H_n^T J_\beta & J_\beta G_n \\ H_n^T J_\beta & Z_n^T \end{vmatrix}. \tag{A.7}$$

We now take the $(i+1, j+1)$th block in (A.4) and the (i,j)th block in (A.7) and obtain respectively

$$\begin{aligned} [A^{-1}_{m,(n+1)}]_{i+1,j+1} &= [\bar{A}^{-1}_{m,n}]_{ij} + [F_n Y_n^{-1} E_n^T]_{ij} \\ &= [\bar{A}^{-1}_{m,n}]_{ij} + [F_n]_i Y_n^{-1} [E_n^T]_j, \end{aligned} \tag{A.8}$$

$$\begin{aligned} [A^{-1}_{m,(n+1)}]_{ij} &= c^{n+2} [J_\beta \bar{B}^{-T}_{m,n} J_\beta + J_\beta G_n Z_n^{-T} H_n^T J_\beta]_{ij}, \\ &= [A^{-1}_{m,n}]_{ij} + c^{n+2} \{ [G_n]_{n-i+1} Z_n^{-T} [H_n^T]_{n-j+1} \}. \end{aligned} \tag{A.9}$$

The notation used in (A.8) and (A.9) was explained in (4.15).

If we now take the conjugate of (A.9) and eliminate $[\bar{A}^{-1}_{m,n}]_{ij}$ in (A.8) and (A.9) we have

$$\begin{aligned} [A^{-1}_{m,(n+1)}]_{i+1,j+1} = [\bar{A}^{-1}_{m,(n+1)}]_{ij} &+ [F_n]_i Y_n^{-1} [E_n^T]_j \\ &- c^{n+1} \{ [G_n]_{n-i+1} Z_n^{\ T} [H_n^T]_{n-j+1} \} \end{aligned} \tag{A.10}$$

which shows that the $(i+1, j+1)$th block in $A^{-1}_{m,(n+1)}$ is obtained from the (i,j)th block and the border blocks only. This establishes (4.15).

We next obtain the recursive procedure for E_i, F_i, G_i and H_i. It can be seen that we can replace n by $i+1$ for $1 \leqslant i \leqslant n-1$ in Eqs. (A.1)–(A.10). Hence from (A.2), (A.6), (A.5) and (4.6) we have

$$\begin{aligned} E_{i+1} &= -\bar{A}^{-T}_{m,(i+1)} R_{i+1} Y_{i+1}^T = -c^{i+2} \{ J_\beta \bar{B}^{-1}_{m,(i+1)} J_\beta \} R_{i+1} Y_{i+1}^T \qquad \text{(A.11)} \\ &= -J_\beta c^{i+1} \begin{vmatrix} Z_i & G_i^T \\ H_i & \bar{B}^{-1}_{m,i} + H_i Z_i^{-1} G_i^T \end{vmatrix} \begin{pmatrix} A^T_{-(i+1)} \\ J_\beta R_i \end{pmatrix} Y_{i+1}^T \\ &= -J_\beta \begin{bmatrix} |c^{i+1}(Z_i) A^T_{-(i+1)} + c^{i+1}(G_i^T) J_\beta R_i| \\ c^{i+1}(H_i Z_i^{-1}) |c^{i+1}(Z_i) A^T_{-(i+1)} + c^{i+1}(G_i^T) J_\beta R_i| \\ + c^{i+1}(\bar{B}^{-1}_{m,1}) J_\beta R_i \end{bmatrix} Y_{i+1}^T \\ &= - \begin{vmatrix} J_\beta c^{i+1}(H_i Z_i^{-1}) \\ I_m \end{vmatrix} |c^{i+1}(Z_i) A^T_{-(i+1)} + c^{i+1}(G_i^T) J_\beta R_i| \, Y_{i+1}^T \\ &\quad - \begin{vmatrix} J_\beta c^{i+1}(\bar{B}^{-1}_{m,i}) J_\beta R_i \\ 0_m \end{vmatrix} Y_{i+1}^T \end{aligned}$$

which from (A.11) and (4.7) becomes

$$E_{i+1} = - \begin{vmatrix} J_\beta c^{i+1}(H_i Z_i^{-1}) \\ I_m \end{vmatrix} \gamma_i^T Y_{i+1}^T + \begin{vmatrix} E_i \\ 0_m \end{vmatrix} Y_i^{-T} Y_{i+1}^T. \qquad \text{(A.12)}$$

Similarly, we obtain from $B_{m,(n+1)}^{-1} B_{m,(n+1)} = I_{m(n+1)}$, and (4.9)

$$G_{i+1} = - \begin{vmatrix} J_\beta c^{i+1}(F_i Y_i^{-1}) \\ I_m \end{vmatrix} \varepsilon_i^T Z_{i+1}^T + \begin{vmatrix} G_i \\ 0_m \end{vmatrix} Z_i^{-T} Z_{i+1}^T. \qquad \text{(A.13)}$$

If we now consider $A_{m,(n+1)} A_{m,(n+1)}^{-1} = I_{m(n+1)}$, we obtain in a similar way to (A.1) and (A.11)

$$F_{i+1} = -\bar{A}_{m,(i+1)}^{-1} S_{i+1} Y_{i+1}$$

which gives, using (4.8),

$$F_{i+1} = - \begin{vmatrix} J_\beta c^{i+1}(G_i Z_i^{-T}) \\ I_m \end{vmatrix} \delta_i Y_{i+1} + \begin{vmatrix} F_i \\ 0_m \end{vmatrix} Y_i^{-1} Y_{i+1}. \qquad \text{(A.14)}$$

Lastly, from $B_{m,(n+1)} B_{m,(n+1)}^{-1} = I_{m(n+1)}$ and (4.10)

$$H_{i+1} = - \begin{vmatrix} J_\beta c^{i+1}(E_i Y_i^{-T}) \\ I_m \end{vmatrix} \eta_i Z_{i+1} + \begin{vmatrix} H_i \\ 0_m \end{vmatrix} Z_i^{-1} Z_{i+1}. \qquad \text{(4.15)}$$

(A.12)–(A.15) are the required formulae (4.13) and (4.14).

It remains to determine Y_{i+1} and Z_{i+1} to complete the recursion. From (A.4) and (A.6) the bottom right hand block, with n replaced by i, is

$$|A_{m,(i+1)}^{-1}|_{i+1,i+1} = |\bar{A}_{m,i}^{-1}|_{ii} + |F_i|_i Y_i^{-1} |E_i^T|_i = c^{i+2}(Z_i^T). \qquad \text{(A.16)}$$

If we omit the left hand term of (A.16) and replace i by $i+1$ we obtain

$$|\bar{A}_{m,(i+1)}^{-1}|_{i+1,i+1} + |F_{i+1}|_{i+1} Y_{i+1}^{-1} |E_{i+1}^T|_{i+1} = c^{i+3}(Z_{i+1}^T). \qquad \text{(A.17)}$$

Conjugating the first and last terms in (A.16) and substituting in (A.17) gives

$$c^{i+3}(Z_i^T) + |F_{i+1}|_{i+1} Y_{i+1}^{-1} |E_{i+1}^T|_{i+1} = c^{i+3}(Z_{i+1}^T). \qquad \text{(A.18)}$$

From (A.12) and (A.14) we have

$$|E_{i+1}^T|_{i+1} = -Y_{i+1}\gamma_i \qquad \text{and} \qquad |F_{i+1}|_{i+1} = -\delta_i Y_{i+1}$$

and hence

$$c^{i+3}(Z_i^T) + \delta_i Y_{i+1} \gamma_i = c^{i+3}(Z_{i+1}^T). \qquad \text{(A.19)}$$

Similarly, using $B^{-1}_{m,(n+1)}$ we obtain

$$c^{i+3}(Y_i^T) + \eta_i Z_{i+1} \varepsilon_i = c^{i+3}(Y_{i+1}^T). \tag{A.20}$$

Finally, eliminating Z_{i+1} from (A.19) and (A.20) produces (4.11), and eliminating Y_{i+1} from the same equations gives (4.12), which completes the recursion.

Remark 6. The analysis is similar to that of Akaike [1], with the crucial difference that both $A_{m,(n+1)}$ and $B_{m(n+1)}$ are considered simultaneously (see also Remark 1 in Section 2).

Remark 7. Necessary conditions for the application of the algorithm can be given either in terms of the original matrix $A_{m,(n+1)}$ or its inverse. Firstly, if all leading principal submatrices of $A_{m,(n+1)}$ of order rm, $r = 1, 2, \ldots, n$, are nonsingular then the algorithm is valid. Alternatively, applying Schur's determinantal expansion to (A.4) with n replaced by i, gives

$$|A^{-1}_{m,(i+1)}| = |Y_i| \, |\bar{A}^{-1}_{m,i} + F_i Y_i^{-1} E_i^T - F_i Y_i^{-1} E_i^T| = |Y_i| \, |\bar{A}^{-1}_{m,i}|. \tag{A.21}$$

Thus if Y_i is nonsingular for $i = 1, 2, \ldots, n$ then $A^{-1}_{m,(i+1)}$ exists, which is equivalent to the first condition above.

Remark 8. Equation (A.21) shows that it is possible to calculate the determinant of $A_{m,(n+1)}$ with only $0(n)$ extra operations. As part of the algorithm, Y_i is obtained from (4.11) and hence we have

$$|A_{m,(n+1)}| = c^n \, |A_{m,1}| \prod_{i=1}^{n} c^{n-i} \left(\frac{1}{|Y_i|} \right). \tag{A.22}$$

When $m = 1$ and all the elements are real in (A.22), then this reduces to a formula in [9].

Remark 9. If $m = 1$ each block reduces to a single element, and we immediately obtain the recursion steps for inversion of a CT matrix, described in Section 2. The condition of Remark 7 reduces to strong nonsingularity as defined in Section 2.

Remark 10. The number of multiplications required for inverting an $mn \times mn$ block CT matrix using the given algorithm is of the order of $8m^3n^2$ or $8m(mn)^2$. When $m = 1$ this reduces to $8n^2$ as indicated in Section 2.

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