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On joint universality of the Riemann zeta-function and Hurwitz zeta-functions ☆

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In honour of my teacher Professor Jonas Kubilius at the occasion of his 90th birthday

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ABSTRACT

We construct classes of composite functions of the Riemann zeta-function and Hurwitz zeta function with transcendental parameter which are universal in the sense that their shifts uniformly on compact subsets of some region approximate any analytic function. For example, the functions $c_1\zeta(s) + c_2\zeta(s,\alpha)$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $e^{\zeta(s)+\zeta(s,\alpha)}$ and $\sin(\zeta(s) + \zeta(s,\alpha))$ are universal.

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1. Introduction

Let α, $0 < \alpha \leqslant 1$, be a fixed parameter, and $s = \sigma + it$ be a complex variable. The Hurwitz zeta-function $\zeta(s,\alpha)$ is defined, for $\sigma > 1$, by Dirichlet series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

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and is meromorphically continued to the whole complex plane. The point $s = 1$ is its unique simple pole with residue 1. For $\alpha = 1$, the function $\zeta(s, \alpha)$ reduces to the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1.$$

It is well known that both the functions $\zeta(s)$ and $\zeta(s, \alpha)$ with transcendental or rational parameter α are universal in the sense that their shifts $\zeta(s+i\tau)$ and $\zeta(s+i\tau, \alpha)$ approximate uniformly on compact subsets of some region any analytic function. The universality property for $\zeta(s)$ was discovered by S.M. Voronin. The last version of the Voronin theorem can be found in [6] and [14], and is stated as follows. Let $D = \{s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1\}$, and $\operatorname{meas}\{A\}$ denote the Lebesgue measure of measurable set $A \subset \mathbb{R}$.

Theorem 1. *Let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T]\colon \sup_{s \in K} \big|\zeta(s+i\tau) - f(s)\big| < \varepsilon\Big\} > 0.$$

The universality of the function $\zeta(s, \alpha)$ is a more complicated problem, and is solved [1,4,10,13] only in the cases of transcendental or rational parameter α. The case of algebraic irrational α remains an open problem. So, we have the statement.

Theorem 2. *Suppose that α is a transcendental or rational number $\neq 1, \frac{1}{2}$. Let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T]\colon \sup_{s \in K} \big|\zeta(s+i\tau, \alpha) - f(s)\big| < \varepsilon\Big\} > 0.$$

The cases of rational $\alpha = 1$ or $\frac{1}{2}$ are excluded in Theorem 2 because $\zeta(s, 1) = \zeta(s)$ and

$$\zeta\left(s, \frac{1}{2}\right) = \left(2^s - 1\right)\zeta(s).$$

Note that in Theorem 2 we have so-called the strong universality of the function $\zeta(s, \alpha)$ because, differently from Theorem 1, the approximated function $f(s)$ can have zeros on K.

In [12], H. Mishou obtained a very interesting theorem on the joint universality of the functions $\zeta(s)$ and $\zeta(s, \alpha)$. We state this theorem.

Theorem 3. *(See [12].) Suppose that α is a transcendental number. Let K_1 and K_2 be compact subsets of the strip D with connected complements. Moreover, for each $j = 1, 2$, let the functions $f_j(s)$ be continuous on K_j and analytic in the interior of K_j, and additionally the function $f_1(s)$ is non-vanishing on K_1. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T]\colon \sup_{s \in K_1} \big|\zeta(s+i\tau) - f_1(s)\big| < \varepsilon,\ \sup_{s \in K_2} \big|\zeta(s+i\tau, \alpha) - f_2(s)\big| < \varepsilon\Big\} > 0.$$

Universality is a very interesting and useful property of zeta-functions. Therefore, it is an important problem to extend the class of universal functions. In [7], we began to study some classes of functions F such that the composite function $F(\zeta(s))$ is universal in the above sense. Let $H(G)$ denote the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Among other results, we proved the following theorem. Let $S = \{g \in H(D)\colon\ g^{-1}(s) \in H(D) \text{ or } g(s) \equiv 0\}$. For $F : H(D) \to H(D)$, and $a_1, \ldots, a_r \in \mathbb{C}$, denote

$$H_{F(0);a_1,\ldots,a_r}(D) = \left\{g \in H(D)\colon\ \left(g(s) - a_j\right)^{-1} \in H(D),\ j = 1, \ldots, r\right\} \cup \left\{F(0)\right\}.$$

Theorem 4. *Suppose that $F : H(D) \to H(D)$ is a continuous function such that $F(S) \supset H_{F(0);a_1,\ldots,a_r}(D)$. For $r = 1$, let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous and $\neq a_1$ function on K which is analytic in the interior of K. For $r \geqslant 2$, let $K \subset D$ be a compact subset, and $f(s) \in H_{F(0);a_1,\ldots,a_r}(D)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T]\colon \sup_{s\in K} \left|F\left(\zeta(s + i\tau)\right) - f(s)\right| < \varepsilon\right\} > 0.$$

Theorem 4 is a slightly modified, corrected and extended version of Theorem 7 of [7]. For $r = 2$, the hypotheses of Theorem 7 of [7] must be changed by those of Theorem 4 because in its proof the function $h_{a,b}(s)$ can take the value b (p. 2330 of [7]). In Theorem 4, the image $F(S)$ contains a comparatively simple set, and this allows to obtain the universality for some elementary functions F. For example, taking $r_1 = 1$ and $a_1 = 0$, we obtain the universality of the function $\zeta^n(s)$, $n \in \mathbb{N}$. If $r = 2$ and $a_1 = -1$, $a_2 = 1$, then we have the universality of $\sin\zeta(s)$ and $\cos\zeta(s)$.

In general, the following result (Theorem 5) is valid. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and let d denote the spherical metric, i.e., for $s_1, s_2, s \in \mathbb{C}$,

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1 + |s_1|^2}\sqrt{1 + |s_2|^2}}, \qquad d(s, \infty) = \frac{2}{\sqrt{1 + |s|^2}}, \quad d(\infty, \infty) = 0.$$

Let $M(D)$ stand for the space of meromorphic functions $g : D \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $\{g_n\colon\ n \in \mathbb{N}\} \in M(D)$ converges to $g \in M(D)$ if, for every compact subset $K \subset D$,

$$\sup_{s\in K} d\left(g_n(s), g(s)\right) \underset{n\to\infty}{\longrightarrow} 0.$$

$H(D)$ is a subspace of $M(D)$.

Theorem 5. *Suppose that the function $\hat{F} : H(D) \to M(D)$ is continuous, $f(s) \in \hat{F}(S) \cap H(D)$, and $K \subset D$ is a compact subset. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T]\colon \sup_{s\in K} \left|\hat{F}\left(\zeta(s + i\tau)\right) - f(s)\right| < \varepsilon\right\} > 0.$$

For example, if $f(s)$ is a non-vanishing analytic function on D, then $f(s)$ can be approximated uniformly on compact subsets $K \subset D$ by shifts $\zeta^{-1}(s + i\tau)$. The same is also true for $f(s) \in H(D)$ and $\frac{\zeta'(s+i\tau)}{\zeta(s+i\tau)}$.

The universality of composite functions $F(\zeta(s, \alpha))$ is considered in [9]. This paper, for example, contains the following statement.

Theorem 6. *(See [9].) Suppose that $F : H(D) \to H(D)$ is a continuous function such that, for each polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non-empty, and that α is a transcendental number. Let K and $f(s)$ be the same as in Theorem 2. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T]\colon \sup_{s\in K} \left|F\left(\zeta(s + i\tau, \alpha)\right) - f(s)\right| < \varepsilon\right\} > 0.$$

For example, it follows from Theorem 6 that the function

$$F\big(\zeta(s,\alpha)\big) = c_1\zeta'(s,\alpha) + \cdots + c_r\zeta^{(r)}(s,\alpha), \quad c_1,\dots,c_r \in \mathbb{C}\setminus\{0\},$$

with transcendental α is universal in the sense of Theorem 6.

The aim of this paper is to consider the universality of composite functions $F(\zeta(s),\zeta(s,\alpha))$. In what follows, we suppose that the parameter α is transcendental.

Theorem 7. *Suppose that $F: H^2(D) \to H(D)$ is a continuous function such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S \times H(D))$ is non-empty. Let K and $f(s)$ be the same as in Theorem 2. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\operatorname{meas}\Big\{\tau \in [0,T]: \sup_{s\in K}\big|F\big(\zeta(s+i\tau),\zeta(s+i\tau,\alpha)\big) - f(s)\big| < \varepsilon\Big\} > 0.$$

The hypothesis of Theorem 7 that the set $(F^{-1}G) \cap (S \times H(D))$ is non-empty for every open set $G \subset H(D)$ is rather complicated and general. The next theorem gives simpler sufficient conditions for the universality of the function $F(\zeta(s),\zeta(s,\alpha))$. Let V be an arbitrary positive number,

$$D_V = \Big\{s \in \mathbb{C}: \frac{1}{2} < \sigma < 1,\ |t| < V\Big\}$$

and

$$S_V = \big\{g \in H(D_V): g^{-1}(s) \in H(D_V) \text{ or } g(s) \equiv 0\big\}.$$

We will use the notation $H^2(D_V, D) = H(D_V) \times H(D)$.

Theorem 8. *Let K and $f(s)$ be the same as in Theorem 2, and $V > 0$ be such that $K \subset D_V$. Suppose that $F: H^2(D_V, D) \to H(D_V)$ is a continuous function such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S_V \times H(D))$ is non-empty. Then the assertion of Theorem 7 is true.*

For example, Theorem 8 implies the universality of the functions

$$F\big(\zeta(s),\zeta(s,\alpha)\big) = c_1\zeta(s) + c_2\zeta(s,\alpha)$$

and

$$F\big(\zeta(s),\zeta(s,\alpha)\big) = c_1\zeta'(s) + c_2\zeta'(s,\alpha), \quad c_1, c_2 \in \mathbb{C}\setminus\{0\}.$$

Now let $a_1,\dots,a_r$ be arbitrary distinct numbers, and

$$H_r(D) = \begin{cases} H(D) & \text{if } r = 0, \\ \{g \in H(D): (g(s)-a_j)^{-1} \in H(D),\ j = 1,\dots,r\}. \end{cases}$$

Theorem 9. *Suppose that $F: H^2(D) \to H(D)$ is a continuous function such that $F(S \times H(D)) \supset H_r(D)$ (is the set $H(D)$ if $r = 0$). In the case $r = 0$, let K and $f(s)$ be the same as in Theorem 2. If $r = 1$, let $K \subset D$ be the same as in Theorem 2, and let $f(s)$ be a continuous and $\neq a_1$ function on K, and analytic in the interior of K. For $r \geqslant 2$, let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_r(D)$. Then the assertion of Theorem 7 is true.*

For example, taking $r = 1$ and $a_1 = 0$, we obtain the universality of the function $e^{\zeta(s)+\zeta(s,\alpha)}$. If $r = 2$ and $a_1 = 1$, $a_2 = -1$, then we have that the shifts $\sinh(\zeta(s+i\tau)+\zeta(s+i\tau,\alpha))$ approximate analytic functions on D which do not take values 1 and -1.

The joint universality of functions $F = (F_1, F_2) : H^2(D) \to H^2(D)$ will be considered in a forthcoming paper.

2. Lemmas

For proofs of the theorems, we will apply the probabilistic approach. So, we need some definitions. As usual, denote by $\mathcal{B}(X)$ the class of Borel sets of the space X. Let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ be the unit circle on the complex plane. Define the infinite-dimensional tori

$$\Omega_1 = \prod_p \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes p, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By the Tikhonov theorem, Ω_1 and Ω_2 with the product topology and pointwise multiplication are compact topological Abelian groups. Similarly, the product $\Omega = \Omega_1 \times \Omega_2$ is also a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega_1(p)$ and $\omega_2(m)$ the projections of $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ to the coordinate spaces γ_p and γ_m, respectively, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the $H^2(D)$-valued random element $\underline{\zeta}(s, \omega)$, $\omega = (\omega_1, \omega_2) \in \Omega$, by the formula

$$\underline{\zeta}(s, \omega) = \big(\zeta(s, \omega_1), \zeta(s, \alpha, \omega_2)\big),$$

where

$$\zeta(s, \omega_1) = \prod_p \left(1 - \frac{\omega_1(p)}{p^s}\right)^{-1}$$

and

$$\zeta(s, \alpha, \omega_2) = \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^s}.$$

We note that, for almost all $\omega \in \Omega$, the product and the series both converge uniformly on compact subsets of the strip D, and thus they define an $H^2(D)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s, \omega)$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H\big(\omega \in \Omega\colon \underline{\zeta}(s, \omega) \in A\big), \quad A \in \mathcal{B}\big(H^2(D)\big).$$

For $A \in \mathcal{B}(H^2(D))$, and $\underline{\zeta}(s) = (\zeta(s), \zeta(s, \alpha))$, we set

$$P_T(A) = \frac{1}{T}\operatorname{meas}\big\{\tau \in [0, T]\colon \underline{\zeta}(s+i\tau) \in A\big\}.$$

Lemma 10. *Suppose that α is a transcendental number. Then P_T converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.*

The lemma is Theorem 1 from [12].

In the sequel, we will use the following well-known fact from the theory of the weak convergence of probability measures. Let X_1 and X_2 be two metric spaces, and let $h : X_1 \to X_2$ be a

$(\mathcal{B}(X_1), \mathcal{B}(X_2))$-measurable function, i.e.,

$$h^{-1}\mathcal{B}(X_2) \subset \mathcal{B}(X_1).$$

Then every probability measure P on $(X_1, \mathcal{B}(X_1))$ induces on the space $(X_2, \mathcal{B}(X_2))$ the unique probability measure Ph^{-1} defined by the formula

$$Ph^{-1}(A) = P\big(h^{-1}A\big), \quad A \in \mathcal{B}(X_2).$$

Clearly, if the function $h: X_1 \to X_2$ is a continuous, then it is $(\mathcal{B}(X_1), \mathcal{B}(X_2))$-measurable.

Lemma 11. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $(X_1, \mathcal{B}(X_1))$, the function $h: X_1 \to X_2$ is continuous, and P_n converges weakly to P as $n \to \infty$. Then $P_n h^{-1}$ also converges weakly to Ph^{-1} as $n \to \infty$.*

The lemma is a particular case of Theorem 5.1 from [2].

Lemma 12. *Suppose that the function $\hat{F}: H(D) \to M(D)$ is continuous. Then*

$$\frac{1}{T}\operatorname{meas}\big\{\tau \in [0, T]\colon\ \hat{F}\big(\zeta(s+i\tau)\big) \in A\big\}, \quad A \in \mathcal{B}\big(M(D)\big),$$

converges weakly to the distribution of the random element $\hat{F}(\zeta(s, \omega_1))$ as $T \to \infty$.

Proof. The lemma is a consequence of Lemma 9 from [7], Lemma 11, and of the continuity of the function $\hat{F}$. $\square$

As it was mentioned above, throughout the paper we suppose that α is a transcendental number.

Lemma 13. *Suppose that the function $F: H^2(D) \to H(D)$ is continuous. Then*

$$P_{T,F}(A) \stackrel{\text{def}}{=} \frac{1}{T}\operatorname{meas}\big\{\tau \in [0, T]\colon\ F\big(\underline{\zeta}(s+i\tau)\big) \in A\big\}, \quad A \in \mathcal{B}\big(H(D)\big),$$

converges weakly to the distribution of the random element $F(\underline{\zeta}(s, \omega))$ as $T \to \infty$.

Proof. Clearly, we have that $P_{T,F} = P_T F^{-1}$. The continuity of the function F together with Lemmas 10 and 11 yields the weak convergence of $P_{T,F}$ to $P_{\underline{\zeta}} F^{-1}$ as $T \to \infty$. However, for $A \in \mathcal{B}(H(D))$,

$$P_{\underline{\zeta}} F^{-1}(A) = P_{\underline{\zeta}}\big(F^{-1}A\big) = m_H\big(\omega \in \Omega\colon\ \underline{\zeta}(s, \omega) \in F^{-1}A\big) = m_H\big(\omega \in \Omega\colon\ F\big(\underline{\zeta}(s, \omega)\big) \in A\big).$$

This shows that $P_{\underline{\zeta}} F^{-1}$ is the distribution of the $H(D)$-valued random element $F(\underline{\zeta}(s, \omega))$. $\square$

For $V > 0$, denote by $P_{T,V}$ and $P_{\underline{\zeta},V}$ the restrictions to the space $(H^2(D_V, D), \mathcal{B}(H^2(D_V, D)))$ of the probability measures P_T and $P_{\underline{\zeta}}$, respectively. Let $\underline{\zeta}_V(s, \omega)$ be the $H^2(D_V, D)$-valued random element with the distribution $P_{\underline{\zeta},V}$.

Lemma 14. *$P_{T,V}$ converges weakly to $P_{\underline{\zeta},V}$ as $T \to \infty$.*

Proof. Obviously, the function $h_V : H^2(D) \to H^2(D_V, D)$ given by the formula $h_V(g_1(s), g_2(s)) = (g_1(s)|_{s\in D_V}, g_2(s))$, $g_1, g_2 \in H(D)$, is continuous. Therefore, the lemma is a corollary of Lemmas 10 and 11. □

Lemma 15. *Suppose that the function $F : H^2(D_V, D) \to H(D_V)$ is continuous. Then*

$$\frac{1}{T}\text{meas}\bigl\{\tau \in [0, T]:\ F\bigl(\underline{\zeta}(s+i\tau)\bigr) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H(D_V)\bigr),$$

converges weakly to the distribution of the random element $F(\underline{\zeta}_V(s,\omega))$ as $T \to \infty$.

Proof. The lemma follows from Lemmas 14 and 11 in the same way as Lemma 13 from Lemmas 10 and 11. □

For the proof of universality theorems for zeta-functions, we also need the explicit form for the supports of the limit measures in limit theorems in the space of analytic functions. The spaces $H(D)$, $H^2(D)$, $H(D_V)$ and $H^2(D_V, D)$ are separable. For brevity, denote them by H. Therefore, the support of a probability measure P on $(H, \mathcal{B}(H))$ is a minimal closed set $S_P \subset H$ such that $P(S_P) = 1$. The set S_P consists of all elements $g \in H$ such that, for every open neighbourhood G of g, the inequality $P(G) > 0$ is satisfied. If ξ is a random element with the distribution P, then the support of ξ is that of the measure P.

We recall that

$$S = \bigl\{g \in H(D):\ g^{-1}(s) \in H(D) \text{ or } g(s) \equiv 0\bigr\}.$$

Lemma 16. *The support of the measure $P_{\underline{\zeta}}$ is the set $S \times H(D)$.*

Proof. The lemma follows from [12, p. 46]. It also is a particular case of the corresponding statements for zeta-functions with periodic coefficients [5,8]. □

Lemma 17. *For arbitrary $V > 0$, the support of the measure $P_{\underline{\zeta},V}$ is the set $S_V \times H(D)$.*

Proof. The lemma is proved by using precisely the same arguments as in the case of the measure $P_{\underline{\zeta}}$. □

Denote by $P_{\underline{\zeta},F}$ the distribution of the random element $F(\underline{\zeta}(s,\omega))$.

Lemma 18. *Suppose that the function $F : H^2(D) \to H(D)$ satisfies the hypotheses of Theorem 7. Then the support of the measure $P_{\underline{\zeta},F}$ is the whole of $H(D)$.*

Proof. Let g be an arbitrary element of the space $H(D)$. We take an arbitrary open neighbourhood G of the element g. Since the function F is continuous, the set $F^{-1}G$ is also open. Moreover, the hypothesis

$$\bigl(F^{-1}G\bigr) \cap \bigl(S \times H(D)\bigr) \neq \emptyset$$

implies the existence of an element $\widehat{g}$ which simultaneously belongs to the sets $F^{-1}G$ and $S \times H(D)$. Thus, $F^{-1}G$ is an open neighbourhood of the element $\widehat{g}$. Therefore, in view of Lemma 16,

$$m_H\bigl(\omega \in \Omega:\ \underline{\zeta}(s,\omega) \in F^{-1}G\bigr) > 0.$$

Hence,

$$m_H\bigl(\omega \in \Omega:\ F\bigl(\underline{\zeta}(s,\omega)\bigr) \in G\bigr) = m_H\bigl(\omega \in \Omega:\ \underline{\zeta}(s,\omega) \in F^{-1}G\bigr) > 0.$$

Since g and G are arbitrary, this gives the assertion of the lemma. □

Now we recall the Mergelyan theorem on approximation of analytic functions by polynomials [11], see also [15].

Lemma 19. *Suppose that $K \subset \mathbb{C}$ is a compact subset with connected supplement, and $f(s)$ is a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K}\bigl|f(s) - p(s)\bigr| < \varepsilon.$$

Denote by $P_{\underline{\zeta},F,V}$ the distribution of the random element $F(\underline{\zeta}_V(s,\omega))$.

Lemma 20. *Suppose that the function $F : H^2(D_V, D) \to H(D_V)$ satisfies the hypotheses of Theorem 8. Then the support of the measure $P_{\underline{\zeta},F,V}$ is the whole of $H(D_V)$.*

Proof. We recall that the space $H(G)$ is metrisable. It is well known, see, for example, [3], that there exists a sequence $\{K_l:\ l \in \mathbb{N}\} \subset G$ of compact subsets such that $K_l \subset K_{l+1}$, $l \in \mathbb{N}$,

$$G = \bigcup_{l=1}^{\infty} K_l,$$

and if $K \subset G$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. For $g_1, g_2 \in H(G)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l}|g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l}|g_1(s) - g_2(s)|}.$$

Then, clearly, ρ is a metric on $H(G)$ which induces the topology of uniform convergence on compacta. It is easily seen that $\rho(g_1, g_2)$ is small if $\sup_{s\in K_l}|g_1(s) - g_2(s)|$ is sufficiently small for the set K_l with rather large $l \in \mathbb{N}$. Thus, the approximation in the space $H(G)$ reduces to that on the sets of the type K_l. Moreover, in the case $H(D)$ or $H(D_V)$ we can choose the sets K_l, $l \in \mathbb{N}$, to be with connected complement, for example, we can take closed rectangles.

Let g be an arbitrary element of $H(D_V)$, and G be its arbitrary open neighbourhood. Suppose that $K \subset D_V$ is a compact subset with connected complement. Then, in virtue of Lemma 19, for every $\varepsilon > 0$, we can find a polynomial $p = p(s)$ such that

$$\sup_{s\in K}\bigl|g(s) - p(s)\bigr| < \varepsilon.$$

Thus, if ε is small enough, we may assume that $p \in G$, too. Therefore, by the hypotheses of the lemma, the set $F^{-1}G$ is open and contains an element of the set $S_V \times H(D)$. So, Lemma 17 implies the inequality

$$m_H\bigl(\omega \in \Omega:\ \underline{\zeta}_V(s,\omega) \in F^{-1}G\bigr) > 0.$$

Hence,

$$P_{\underline{\zeta},F,V}(G) = m_H\big(\omega \in \Omega\colon F\big(\underline{\zeta}_V(s,\omega)\big) \in G\big) = m_H\big(\omega \in \Omega\colon \underline{\zeta}_V(s,\omega) \in F^{-1}G\big) > 0.$$

Since g and G are arbitrary, this proves the lemma. $\square$

Lemma 21. *Suppose that the function $F : H^2(D) \to H(D)$ satisfies the hypotheses of Theorem 9. Then the support of the measure $P_{\underline{\zeta},F}$ contains the closure of the set $H_r(D)$ (is set $H(D)$ if $r = 0$).*

Proof. Under hypotheses of the lemma, for each element $h \in H_r(D)$, there exists an element $(g_1, g_2) \in S \times H(D)$ such that $F(g_1, g_2) = h$. Thus, for every open neighbourhood G of h, the open set $F^{-1}G$ contains an element of $S \times H(D)$. Therefore, by Lemma 16,

$$m_H\big(\omega \in \Omega\colon \underline{\zeta}(s,\omega) \in F^{-1}G\big) > 0,$$

and this yields

$$P_{\underline{\zeta},F}(G) = m_H\big(\omega \in \Omega\colon F\big(\underline{\zeta}(s,\omega)\big) \in G\big) = m_H\big(\omega \in \Omega\colon \underline{\zeta}(s,\omega) \in F^{-1}G\big) > 0. \tag{1}$$

Therefore h is an element of the support of $P_{\underline{\zeta},F}$. Hence, the set $H_r(D)$ is a subset of the support $P_{\underline{\zeta},F}$. Since the support is a closed set, the support of $P_{\underline{\zeta},F}$ includes the closure of $H_r(D)$. $\square$

3. Proofs of theorems

First we recall an equivalent in terms of open sets of the weak convergence of probability measures.

Lemma 22. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(X, \mathcal{B}(X))$. Then P_n, as $n \to \infty$, converges weakly to P if and only if*

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G)$$

for every open set $G \subset X$.

The lemma is a part of Theorem 2.1 of [2].

Proof of Theorem 4. Denote by $P_{\zeta,F}$ the distribution of the random element $F(\zeta(s,\omega_1))$ defined on the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, where m_{1H} is the Haar measure on $(\Omega_1, \mathcal{B}(\Omega_1))$. First we observe that the support of $P_{\zeta,F}$ includes the closure of the set $H_{F(0);a_1,\dots,a_r}(D)$. Indeed, let g be an arbitrary element of $H_{F(0);a_1,\dots,a_r}(D)$. Then there exists $g_1 \in S$ such that $F(g_1) = g$. Since the function F is continuous, for every open neighbourhood G of g, the set $F^{-1}G$ is also open and contains the element g_1. Thus, by Lemma 13 of [7],

$$m_{1H}\big(\omega_1 \in \Omega_1\colon F\big(\zeta(s,\omega_1)\big) \in G\big) = m_{1H}\big(\omega_1 \in \Omega_1\colon \zeta(s,\omega_1) \in F^{-1}G\big) > 0.$$

This shows that g is an element of the support of $P_{\zeta,F}$. Therefore, the set $H_{F(0);a_1,\dots,a_r}(D)$ is a subset of the support of $P_{\zeta,F}$. Thus, the support of $P_{\zeta,F}$ contains the closure of $H_{F(0);a_1,\dots,a_r}(D)$.

Suppose that $r = 1$. By Lemma 19, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} \big|f(s) - p(s)\big| < \frac{\varepsilon}{4}. \tag{2}$$

Since $f(s) \neq a_1$ on K, we have that $p(s) \neq a_1$ on K as well if ε is small enough. Therefore, we can define a continuous branch of $\log(p(s) - a_1)$ which will be an analytic function in the interior of K. By Lemma 19 again, there exists a polynomial $q(s)$ such that

$$\sup_{s \in K} \left| p(s) - a_1 - \mathrm{e}^{q(s)} \right| < \frac{\varepsilon}{4}. \tag{3}$$

Let $h_{a_1}(s) = \mathrm{e}^{q(s)} + a_1$. Then, $h_{a_1}(s) \in H(D)$ and $h_{a_1}(s) \neq a_1$. Therefore, by the above remark, the function $h_{a_1}(s)$ is an element of the support of $P_{\zeta,F}$. Moreover, inequalities (2) and (3) imply

$$\sup_{s \in K} \left| f(s) - h_{a_1}(s) \right| < \frac{\varepsilon}{2}. \tag{4}$$

Define

$$\mathcal{G}_1 = \left\{ g \in H(D): \sup_{s \in K} \left| g(s) - h_{a_1}(s) \right| < \frac{\varepsilon}{2} \right\}.$$

Then we have that $P_{\zeta,F}(\mathcal{G}_1) > 0$, and Lemma 9 of [7] together with Lemma 22 shows that

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T]: \sup_{s \in K} \left| F\big(\zeta(s + i\tau)\big) - h_{a_1}(s) \right| < \frac{\varepsilon}{2} \right\} > 0.$$

This and (4) prove the theorem in the case $r = 1$.

Now let $r \geqslant 2$. Define

$$\mathcal{G}_2 = \left\{ g \in H(D): \sup_{s \in K} \left| g(s) - f(s) \right| < \varepsilon \right\}.$$

Since $f(s) \in H_{F(0);a_1,\ldots,a_r}(D)$, it is an element of the support of $P_{\zeta,F}$. Therefore, $P_{\zeta,F}(\mathcal{G}_2) > 0$, and we have by Lemma 9 of [7] and Lemma 22 that

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T]: \sup_{s \in K} \left| F\big(\zeta(s + i\tau)\big) - f(s) \right| < \varepsilon \right\} > 0.$$

The theorem is proved. □

Proof of Theorem 5. Denote by $P_{\zeta,\hat{F}}$ the distribution of the random element $\hat{F}(\zeta(s, \omega_1))$. It is easily seen that the support of the measure $P_{\zeta,\hat{F}}$ is the closure of $\hat{F}(S)$. If g is an arbitrary element of $\hat{F}(S)$, and G is its any open neighbourhood, then, by Lemma 13 of [7]

$$m_{1H}\big(\omega_1 \in \Omega_1: \zeta(s, \omega_1) \in \hat{F}^{-1} G\big) > 0.$$

Therefore,

$$m_{1H}\big(\omega_1 \in \Omega_1: \hat{F}\big(\zeta(s, \omega_1)\big) \in G\big) = m_{1H}\big(\omega_1 \in \Omega_1: \zeta(s, \omega_1) \in \hat{F}^{-1} G\big) > 0.$$

Moreover,

$$m_{1H}\big(\omega_1 \in \Omega_1: \hat{F}\big(\zeta(s, \omega_1)\big) \in \hat{F}(S)\big) = m_{1H}\big(\omega_1 \in \Omega_1: \zeta(s, \omega_1) \in S\big) = 1.$$

Thus, the support of $P_{\zeta,\hat{F}}$ is the closure of $\hat{F}(S)$.

Since the analytic function $f(s) \in \hat{F}(S)$, the end of the proof is the same as that of Theorem 4 in the case $r \geqslant 2$. $\square$

Proof of Theorem 7. By Lemma 19, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{5}$$

Define

$$G = \left\{ g \in H(D)\colon \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

In view of Lemma 18, the polynomial $p(s)$ is an element of the support of the measure $P_{\underline{\zeta},F}$. Since G is an open neighbourhood of $p(s)$, we have that $P_{\underline{\zeta},F}(G) > 0$. Taking into account Lemmas 13 and 22, we deduce from this that

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\{\tau \in [0,T]\colon F(\underline{\zeta}(s+i\tau)) \in G\} \geqslant P_{\underline{\zeta},F}(G) > 0.$$

Thus, the definition of the set G yields

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T]\colon \sup_{s\in K} \left|F\big(\zeta(s+i\tau), \zeta(s+i\tau,\alpha)\big) - p(s)\right| < \frac{\varepsilon}{2}\right\} > 0.$$

Combining this with (5) proves the theorem. $\square$

Proof of Theorem 8. We follow the proof of Theorem 7, and in place of Lemmas 18 and 13, we apply Lemmas 20 and 15, respectively. $\square$

Proof of Theorem 9. The case $r = 0$ uses Lemmas 21 and 13, and completely coincides with the proof of Theorem 7.

The case $r = 1$. As in the proof of Theorem 4, we find that there exists a function $h_{a_1} \in H(D)$ such that $h_{a_1}(s) \neq a_1$ and inequality (4) holds. Thus, by Lemma 21, $h_{a_1}(s)$ is an element of the support of the measure $P_{\underline{\zeta},F}$. Therefore, in notation of the proof of Theorem 4, $P_{\underline{\zeta},F}(\mathcal{G}_1) > 0$. Hence, in view of Lemmas 13 and 22,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T]\colon \sup_{s\in K} \left|F\big(\zeta(s+i\tau), \zeta(s+i\tau,\alpha)\big) - h_{a_1}(s)\right| < \frac{\varepsilon}{2}\right\} > 0.$$

Combining this with (4) gives the assertion of the theorem in the case $r = 1$.

The case $r \geqslant 2$. We preserve the notation used in the proof of Theorem 4. Since $f(s) \in H_r(D)$, by Lemma 21, $f(s)$ is an element of the support of $P_{\underline{\zeta},F}$. Therefore $P_{\underline{\zeta},F}(\mathcal{G}_2) > 0$. Now the definition of $\mathcal{G}_2$ and Lemmas 13 and 22 yield the inequality

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T]\colon \sup_{s\in K} \left|F\big(\zeta(s+i\tau), \zeta(s+i\tau,\alpha)\big) - f(s)\right| < \varepsilon\right\} > P_{\underline{\zeta},F}(\mathcal{G}_2) > 0.$$

The theorem is proved. $\square$

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