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Divisorially Graded Rings, Related Groups, and Sequences

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INTRODUCTION

Generalized crossed products, sometimes called strongly graded rings, appear quite naturally in the theory of splitting rings for Azumaya algebras over rings with nontrivial Picard groups. This theorey and its relation to the existence of the seven term exact sequence of Chase–Harrison–Rosenberg has been studied by Kanzaki in [8], Caenepeel, Van den Bergh, and Van Oystaeyen in [3]. The aformentioned application of graded techniques is basically commutative in nature and the grading groups are usually finite, but also in the noncommutative case some techniques, of graded rings theory lead to fruitful applications, e.g., the use of generalized Rees rings in the study of maximal orders and their class groups, cf. [11, 10]. In the setting of orders, the class group is a more intrinsic invariant than the Picard group, and according to this the concept of a divisorially graded ring over an order is more natural than that of a generalized crossed product. Divisorially graded rings have been introduced by Van Oystaeyen (cf. [23] and in [11]), Le Bruyn and Van Oystaeyen studied divisorially graded P.I. rings; the more general definition, dealing with graded rings over a (semi-)prime Goldie ring has been introduced by Marubayashi in [12] and has also been used by Nastasescu, Nauwelaerts, and Van Oystaeyen in [16].

In this paper we study a ring A and a subring B and we derive some properties of divisorial B-bimodules contained in A. This leads to the construction of several groups and exact sequences relating them, much in the

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vein of some results of Miyashita [13] who treated the generalized crossed product case in much detail. The step from the generalized crossed product case to the case of divisorially graded rings present some typical problems concerning reflexive and divisorial bimodules. On the other hand, for orders and lattices over orders the notion of divisoriality is well established and it is known to be related to the center and properties of the lattice considered over the center by restrictions of scalars. Consequently, the fact that $Z(A)$ and $Z(B)$ need not be related in any sense causes difficulties when we are considering orders, say tame orders A and B over $Z(A)$ and $Z(B)$, and the usual concepts of divisoriality. We have solved this problem by restricting to the so-called *arithmetical situation* where the extensions $Z(A)$ over $Z(A) \cap Z(B)$ and $Z(B)$ over $Z(A) \cap Z(B)$ satisfy certain conditions (of PDE type). The arithmetical situation occurs frequently, actually in all concrete situations (e.g., gradation by finite groups) this will be the case. We pay particular attention to the case where B is commutative but not necessarily central in A.

1. Preliminaries

Let R be a ring graded by a group G such that R_e is a prime Goldie ring with classical ring of fractions $Q_{\mathrm{cl}}(R_e) = Q_e$. Let E_e be an injective envelope of the left R_e-module Q_e/R_e and consider the idempotent kernel functor κ on R_e-mod with filter $L(\kappa) = \{H \text{ left ideal of } R_e,\ \mathrm{Hom}_{R_e}(R_e/H, E_e) = 0\}$, in other words $H \in L(\kappa)$ if and only if for $r \in R_e$ and $q \in Q_e$ such that $(H : r)\, q \subset R_e$, $q \in R_e$ follows, where $(H : r) = \{x \in R_e,\ xr \in H\}$. In exactly the same way one defines the kernel functor κ' on right R_e-modules associated to the injective envelope of Q_e/R_e as a right R_e-module. Since R_e is a prime Goldie ring we have $\kappa(R_e) = 0$ and $Q_\kappa(R_e) = R_e$. For a left ideal L of R_e we define the κ-closure of L as $\mathrm{cl}(L) = \{x \in R_e,\ Hx \subset L \text{ for some } H \in L(\kappa)\}$. We say that L is κ-closed if $L = \mathrm{cl}(L)$. Since $Q_\kappa(R_e) = R_e$ it is clear that $\mathrm{cl}(L) = Q_\kappa(L)$ in R_e-mod. We say that R is *divisorially graded* if the following properties hold:

(1) R is κ and κ'-torsion-free.

(2) For all $\sigma, \tau \in G$, $Q_\kappa(R_\sigma R_\tau) = R_{\sigma\tau} = Q_{\kappa'}(R_\sigma R_\tau)$.

The second condition impies that $Q_\kappa(R_\sigma) = R_\sigma = Q_{\kappa'}(R_\sigma)$ for all $\sigma \in G$. From [16] we recall the following properties of divisorially graded rings:

(i) For all $\sigma \in G$, $Q_\kappa(R_\sigma R_{\sigma^{-1}}) = R_e$, i.e., $R_\sigma R_{\sigma^{-1}} \in L(\kappa)$.

(ii) If for some $\sigma, \tau \in G$, $R_\tau r_\sigma = 0$ with $r_\sigma \in R$ then $r_\sigma = 0$ and similarly $r_\sigma R_\tau = 0$ yields $r_\sigma = 0$.

(iii) The set $S_e = \{S \in R_e,\ s \text{ regular in } R_e\}$ is a regular left and right

Ore set of R. The left ring of fractions with respect to S_e is isomorphic to the right ring of fractions and it is denoted by Q^g. Then Q^g is strongly graded by G and $(Q^g)_e = Q_{cl}(R_e) = Q_e$ where e is the neutral element of G.

(iv) A strongly graded ring (i.e., $R_\sigma R_\tau = R_{\sigma\tau}$, $\sigma, \tau \in G$) is automatically divisorially graded.

We now consider a subring B of A and we assume that B is a prime Goldie ring; let $Z(B)$, $Z(A)$ be the centre of B (resp. A). The theory we are about to expound may be developed for relative maximal orders in the sense of Le Bruyn, cf. [9], but here we restrict attention to a more concrete situation where B is a maximal order over a Krull domain $Z(B)$ in a central simple algebra $Q(B)$. By a result of Chamarie [4], see also D.1.18 in [26], it follows that $\kappa = \kappa'$ is a central kernel functor in this case and it corresponds to the prime ideals $p \in X^{(1)}(Z(B))$ in the sense: $\kappa = \inf\{\kappa_p, p \in X^{(1)}(Z(B))\}$. From now on we do not distinguish between κ and κ', and we write κ_1 instead, thus indicating that it corresponds to $X^{(1)}(Z(B))$. Any $Z(B)$-module M is said to be *divisorial* if it is torsion-free and if $M = \bigcap \{M_p, p \in X^{(1)}(Z(B))\}$. Let K be the field of fractions of $Z(B)$, then we define

$$Z(B): M = \{f \in \mathrm{Hom}_{Z(B)}(K \otimes M, K), f(M) \subset Z(B)\}$$

and we write M^* for the $Z(B)$-module $\mathrm{Hom}_{Z(B)}(M, Z(B))$. We may view $Z(B): (Z(B): M)$ as a $Z(B)$-submodule of $K \otimes M$, as $Z(B)$ modules, in natural way. If M is a $Z(B)$-lattice (cf. Fossum [7]) then $Z(B): M$ is isomorphic to M^* and M is divisorial if and only if $M = M^{**}$, i.e., if and only if M is a reflexive $Z(B)$-module or if and only if $M = \bigcap \{M_p, p \in X^{(1)}(Z(B))\}$. By the assumption that B is a maximal $Z(B)$-order it follows in particular that B is divisorial $Z(B)$-lattice and from Lemma 2.5(2), p. 12 in [26, II], we maintain that a finitely generated reflexive B-module will also be reflexive as a $Z(B)$-module. Actually we may strenghten this result as follows: if ${}_BM$ is torsion-free of finite rank (in particular if M is κ_1-finitely generated, i.e., M contains a finitely generated M' such that M/M' is κ_1-torsion, then it is reflexive as a left B-module if and only if it is reflexive (necessarily of finite rank) as a $Z(B)$-module and reflexivity is then equivalent to $M = \bigcap \{M_p / p \in X^{(1)}(Z(B))\}$, i.e., to divisoriality (see Proposition 1.8 in [26, II] or Corollary 1.9(4) on p. 240 [26]).

A two-sided B-submodule P of A is said to be *divisorial in A* if and only if $P = Q_{\kappa_1}(P)$ and there exists a two-sided B-submodule Q of A, also satisfying $Q_{\kappa_1}(Q) = Q$, such that $(*)$ $Q_{\kappa_1}(PQ) = Q_{\kappa_1}(QP) = B$. Clearly P (also Q) are torsionfree left B-modules of finite rank, so we may apply the foregoing remark to them. From $(*)$ it follows that P (also Q) has "rank one" because for all $p \in X^{(1)}(Z(B))$, $P_p = B_p$ in B_p-mod. Now a maximal order B

over a Krull domain is κ_1-noetherian in the sense of [26] hence every κ_1-finitely generated left B-module is also κ_1-finitely presented (Lemma 4.2 of [26, II, p. 76]) and we may apply the criterion given in Proposition 4.15, p. 83, of [26, II] to deduce that a P which is divisorial in A is κ_1-invertible in the sense of [26, II] p. 50, i.e., the isomorphism class of the B-bimodule P represents an element of the relative Picard group $\mathrm{Pic}(B, \kappa_1)$. By checking the definition it follows that the class of Q, $[Q]$, is the inverse of $[P]$ and $Q_{\kappa_1}(Q \otimes P) = Q_{\kappa_1}(QP) = B = Q_{\kappa_1}(PQ) = Q_{\kappa_1}(P \otimes Q)$. In case B is commutative, $\mathrm{Pic}(B, \kappa_1) = \mathrm{Cl}(B)$ the class group of the Krull domain $B = Z(B)$. If B is a reflexive Azumaya algebra (cf. [25, 26, 27]) then $\mathrm{Pic}(B, \kappa_1)$ equals the central class group $\mathrm{CCl}(B)$ of the maximal order B as defined in [9] or [26]. From the remarks and results summarized above it is clear that the two-sided B-submodules which are divisorial in A form a group $D_B(A)$. We write $\mathrm{Aut}_B(A)$ for the group of *all* B-automorphism of A. Clearly $\mathrm{Aut}_B(A)$ acts on $D_B(A)$ by letting $\sigma \in \mathrm{Aut}_B(A)$ act on $P \in D_B(A)$ by $P \mapsto P^\sigma$.

With assumption on B as above, if A is divisorially graded by G, such that $A_e = B$, then each A_σ, $\sigma \in G$, is an element of $D_B(A)$ because $Q_{\kappa_1}(A_\sigma A_\sigma - 1) = Q_{\kappa_1}(A_\sigma - 1 A_\sigma) = B$ hence $\sigma \to A_\sigma$ determines a group morphism $G \to D_B(A) \to^j \mathrm{Pic}(B, \kappa_1)$ where $j(A_\sigma) = [A_\sigma]$ is the isomorphism class of the B-bimodule A_σ. First, we will derive some results assuming only that $Q_{\kappa_1}(A) = A$.

2. Actions and Exact Sequences

As in Section 1 we consider a maximal order B over a Krull domain $R = Z(B)$ and a ring A containing B. We assume moreover that A is divisorial as an R-module, i.e., $A = \bigcap \{A_p, p \in X^{(1)}(R)\}$ as R-modules (also as B-modules).

Consider $P \in D_B(A)$ and $a \in C_B(A) = \{x \in A,\ xb = bx$ for every $b \in B\}$. Since p is central in B, A_p is a B_p-module and P_p is an invertible B_p-module. Therefore, to P_p we may associate an automorphism $\sigma_{A_p} \in \mathrm{Aut}_{Z(A)_p}(C_{B_p}(A_p))$ which may be derived from a decomposition of 1 in $P_p(P_p)^{-1} = B_p$, cf. [15], such that for all $y \in C_{B_p}(A_p)$, $P_p y = \sigma_{A_p}(y) P_p$ elementwise. Now from $A = \bigcap \{A_p, p \in X^{(1)}(R)\}$ it follows that $A \subset A_p$ for all $p \in X^{(1)}(R)$, hence $C_B(A) \subset C_{B_p}(A_p)$ (note that A_p need not be a ring here).

Consequently for all $a \in C_B(A)$, $x \in P$ we obtain $xa = \sigma_{A_p}(a)\, x$, because $P \subset P_p$ as P is κ_1-invertible (see Sect. 1). Again by centrality of p it follows easily that $C_{B_p}(A_p) = (C_B(A))_p$ (also using the fact that A and $C_B(A)$ are κ_1-torsion-free). Now for all $x \in P$ we obtain $(\sigma_{A_p}(a) - \sigma_{A_q}(a))\, x = 0$, p, q in $X^{(1)}(R)$, consequently $(\sigma_{A_p}(a) - \sigma_{A_q}(a))\, P = 0$ and $(\sigma_{A_p}(a) - \sigma_{A_q}(a))\, PQ = 0$

for Q such that $Q_{\kappa_1}(PQ)=B$. Hence $\sigma_{A_p}(a)-\sigma_{A_q}(a)\in\kappa_1(K\otimes_R A)=0$ (note A is divisorial hence torsion-free as an R-module). Consequently $\sigma_{A_p}(a)\in\bigcap\{(C_B(A))_q,\ q\in X^{(1)}(R)\}$, which is contained in $\bigcap\{A_q,\ q\in X^{(1)}(R)\}=A$ and also, $\sigma_{A_p}(a)\in C_B(A)$. So we have obtained a uniquely determined automorphism $\sigma\in\mathrm{Aut}_{Z(A)}(C_B(A))$ corresponding to P such that $xa=\sigma(a)\,x$ holds for all $a\in C_B(A)$, $x\in P$. It is obvious that we may replace A by any A-bimodule M, which is $Z(B)$-compatible, so we have

2.1. Proposition. *Let $P\in D_B(A)$ and let M be an A-bimodule which is $Z(B)$-compatible and divisorial as a $Z(B)$-module. To P we may associate an automorphism $\sigma(P)\in\mathrm{Aut}_{Z(A)}(C_B(A))$ which may be induced by taking a decomposition $1=\sum_i x_ix_i'$ in $P_pQ_p=B_p$ for some p in $X^1(R)$ in the sense that $\sigma(P)(a)=\sum x_iax_i'$, and such that for all $a\in C_B(A)$, $Pa=\sigma(P)(a)\,P$ elementwise. In a similar way one obtains a map $\sigma_M(P)\colon C_B(M)\to C_B(M)$ which is semilinear (left and right) $C_B(A)$-automorphism.*

Proof. For $\sigma(P)$ we take σ_{A_p} constructed above and we already established that this is independent of the chosen $p\in X^1(R)$. Since $[P_p]\in\mathrm{Pic}(B_p)$ it is well known, cf. [2], that σ_{A_p} may be defined by picking a decomposition $1=\sum x_ix_i'$ as in the proposition (and σ_{A_p} is of course no depending on the chosen decomposition). The statement concerning $\sigma_M(P)$ may be proved in the same way, actually here the proof is very similar to the lines of proof used in [13].

2.2. Corollary. Suppose we are given a group morphism $\Phi\colon G\to D_B(A)$, $g\to P_g$ for some group G. Then there is a canonical action of G on $C_B(A)$ given by the composition of:

$$\begin{aligned}\Psi_A\colon\quad & G\to D_B(A)\to\mathrm{Aut}(C_B(A))\\ & g\to P_g\to\sigma(P_g)=\sigma_g\end{aligned}$$

This action of G is compatible with the canonical action of G on $C_B(M)$ defined by the group morphism $\Psi_M\colon G\to D_B(A)\to\mathrm{Aut}_{\Psi_A}(C_B(M))$, where the latter group is the group of left and right Ψ_A-semilinear $C_B(A)$-bimodule automorphism.

In many concrete situations the inclusion $B\subset A$ will be an extension of rings in the sense of C. Procesi [19] (e.g., $A=BC_A(B)$) or more general, $Z(B)\subset Z(A)$ will hold. But this is not always the case and in particular when A is some graded ring over B the extension $Z(B)\cap Z(A)\to Z(B)$ plays an important part. In the latter case $Z(A)\cap Z(B)$ will be the fixed ring of some action of some quotient group $\bar{G}$ of G a finite group on most occasions. We say that $M\in D_B(A)$ is *centrally controlled* if for every $P\in X^1(Z(B))$ localization at $p=P\cap Z(A)$ requires that $M_p\in\mathrm{Pic}(B_p)$. For a group morphism $\Phi\colon G\to D_B(A)$ we say that G is centrally controlled if every $P_g=\Phi(g)$ is centrally controlled.

2.3. PROPOSITION. *Let $\delta_B(A)$ be the subset of centrally controlled elements of $D_B(A)$, then $\delta_B(A)$ is a subgroup of $D_B(A)$.*

Proof. Let $P \in \delta_B(A)$ and let P' be the inverse of P in $D_B(A)$. By assumption, for every $p = P \cap Z(A)$, $P \in X^1(Z(B))$, $P_p \in \mathrm{Pic}(B_p)$. Since P is κ_1-invertible it is κ_1-finitely presented, i.e., $P = Q_{\kappa_1}(M)$ for some finitely presented $Z(B)$-module M, but using $P = \mathrm{Hom}_B(\mathrm{Hom}_B(P, B) = P^{**}$ one easily derives that P is finitely presented itself. Since $P' = P^{-1}$ in $\mathrm{Pic}(B, \kappa_1)$ and $P^{-1} = \mathrm{Hom}_B(P, B)$ holds in $\mathrm{Pic}(B, \kappa_1)$ (cf. [26, II]) it follows that $(P')_p = (\mathrm{Hom}_B(P, B))_p = \mathrm{Hom}_{B_p}\ (P_p, B_p) \in \mathrm{Pic}(B_p)$. It is clear that $\kappa_1(Z(B)) \leqslant \kappa_1(Z(B)_p) = (\kappa_1)_p$, hence $Q_{(\kappa_1)_p}(P_p P'_p(Q_{\kappa_1}(PP')) = B_p$ and thus $P_p \in D_{B_p}(A_p)$, $P'_p \in D_{B_p}(A_p)$ while $Q_{(\kappa_1)_p}(P_p \otimes P'_p) = Q_{(\kappa_1)p}(P_p P'_p)$ follows from the fact that B_p is a maximal $Z(B)_p$-order just like for B and $D_B(A)$ in Section 1. Since P_p and P'_p are in $\mathrm{Pic}(B_p)$ it follows that $P_p P'_p = B_p$, so $P' \in \delta_B(A)$ is the inverse of P. If P and Q are in $D_B(A)$ then so is $Q_{\kappa_1}(PQ)$ so if $P, Q \in \delta_B(A)$ then $(Q_{\kappa_1}(PQ))_p \in D_{B_p}(A_p)$. Since P_p, $Q_p \in D_{B_p}(A_p)$ it follows on one hand that $PQ \subset Q_{(\kappa_1)_p}(P_p Q_p) = P_p Q_p \subset (PQ)_p$ (note P_p, Q_p are invertible B_p-modules by assumption). Since $(\kappa_1)_p \geqslant \kappa_P$ (note: $I \subset p$ in $Z(A) \cap Z(B)$ yields $Z(B)_p I$ is not contained in P_p for all $P_p \in X^1(Z(B)_p)$ such that $P_p \cap Z(A) \subset p$ and thus also $(\kappa_1)_p = \inf\{\kappa_{P_p}, P_p \in X^1(Z(B)_p),$ $P \cap Z(A) \subset p\}$ as one easily checks) it follows from the foregoing that $P_p Q_p$ is equal to (PQ_p), hence $P_p Q_p = Q_{(\kappa_1)_p}(P_p Q_p) = (Q_{\kappa_1}(PQ)_p) \in D_{B_p}(A_p)$ or $PQ \in \delta_B(A)$.

2.4. *Remark.* The main technical difficulty in the above proof is that the localization induced by p in $Z(B)$-mod need not be larger than κ_1 in general. If we assume that κ_1 on $Z(B)$-mod is induced by $\kappa_1(Z(A) \cap Z(B))$ or, more generally, by $\kappa_1 = \inf\{\kappa_p, p = P \cap Z(A)$ for some $P \in X^1(Z(B))\}$ then is very easy to check that $\delta_B(A) = D_B(A)$. Indeed, for every p prime in $Z(A) \cap Z(B)$ as above we then have that $\bar{\kappa}_p \geqslant \kappa_1$ ($\bar{\kappa}_p$ the induced kernel functor on $Z(B)$-mod) and consequently, from $P \in D_B(A)$ and $Q_{\kappa_1}(P \otimes_B P') = B$ it follows that $P_p \otimes_{B_p} P'_p = B_p$, hence $P_p \in \mathrm{Pic}(B_p)$ or $P \varepsilon \delta_B(A)$.

Let us point out that we have $\kappa_1(\overline{Z(A) \cap Z(B)}) = \kappa_1(Z(B))$ in particular when $Z(B)$ is integral over $Z(A) \cap Z(B)$.

2.5. PROPOSITION. *If $P \in \delta_B(A)$ then $A = Q_{\kappa_1}(P \otimes_B A) = Q_{\kappa_1}(A \otimes_B P^{-1})$.*

Proof. Let us establish $A = Q_{\kappa_1}(A \otimes P^{-1})$, the other equality may be proved in exactly the same way. Localizing $A \otimes P^{-1} \to AP^{-1}$ at κ_1 yields a B-bimodule map: γ: $Q_{\kappa_1}(A \otimes P^{-1}) \to Q_{\kappa_1}(AP^{-1}) = A$. Let K be the kernel of $A \otimes P^{-1} \to A$. Localizing at $p = Q \cap Z(A)$ for some $Q \in X^1(Z(B))$ yields $P_p^{-1} \in \mathrm{Pic}(B_p)$ because $P \in \delta_B(A)$ (and p is central in A and in B), hence

$K_p = 0$ for every such p, because $A_p \otimes_{B_p} P_p^{-1} = A_p$ follows from that fact that P_p^{-1} is invertible in A_p. In particular K has to be κ_1-torsion but then it follows from $\kappa_1(A) = 0$ that $K = \kappa_1(A \otimes P^{-1})$. Since Im γ contains AP^{-1} we finally obtain that $Q_{\kappa_1}(A \otimes P^{-1}) = A$.

The foregoing may be used to derive a reflexive analogous of certain results of [13] but in doing so we will restrict attention to the so-called "arithmetical situation" (A.S.) given by:

(AS1) The extension $Z(B)$ of $C = Z(A) \cap Z(B)$ satisfies PDE and $\bar{\kappa} = \kappa_1(Z(B))$ ($\bar{\kappa}$ induced by κ as in Remark 2.4).

(AS2) The extension $C \to Z(A)$ satisfies PDE.

(AS3) The ring A is a tame order over $Z(A)$.

It is clear that the PDE condition for $C \subset Z(A)$ yields that the kernel functor $\overline{\kappa_1(C)}$ induced by $\kappa(X^1(C))$ on $Z(A)$-mod satisfied $\overline{\kappa_1(C)} \leqslant \kappa(X^1(Z(A))$. In the case where $Z(B)$ is integral over C then (AS1) holds and moreover $\kappa_1(C)$ induces $\kappa(X^1(Z(B))$ on $Z(B)$-mod.

2.6. Proposition. (1) *Let B be a tame order over a Krull domain $Z(B)$ and let the prime ring A be divisorially graded over B by a finite group G such that $/G/^{-1} \in B$, then A is a tame $Z(A)$-order, in particular $Z(A)$ is a Krull domain. Moreover, both $Z(B)$ and $Z(A)$ are integral over $C = Z(A) \cap Z(B)$, and we are in the arithmetical situation.*

(2) *If A is divisorially graded by the torsionfree abelian group G over B such that B is a maximal $Z(B)$-order in $Q_{\mathrm{cl}}(B)$ then A is a maximal order in $Q_{\mathrm{cl}}(A)$. (The same statement holds for poly-infinite-cyclic groups if A is strongly graded).*

Proof. (1) Write $A = \bigoplus_{\sigma \in G} A_\sigma$. Since A_σ is a divisorial B-bimodule and B is a tame order over $Z(B)$, it follows that A is finitely generated (and not only relatively finitely generated) as a B-module. Since $B = A_e$ is a P.I. ring which is integral over $Z(B)$ while $Z(B)$ is integral over $Z(B)^G$ (the fix ring of the canonical action of G) it follows that B is integral over $Z(B)^G$. If $r_\sigma \in A_\sigma$, for any $\sigma \in G$, then $r_\sigma^n \in B$ for some $n \mid /G/$. Hence r_σ is integral over $Z(B)^G$. If $0 \neq r \in A$, say $r = r_{\sigma_1} + \cdots + r_{\sigma_k}$, let $T = Z(B)^G\{r_{\sigma_1}, ..., r_{\sigma_k}\}$ be the $Z(B)^G$-subalgebra of A generated by $\{r_{\sigma_1}, ..., r_{\sigma_k}\}$. Since T is a P.I. ring and monomials in $r_{\sigma_1}, ..., r_{\sigma_k}$ are integral over $Z(B)^G$ it follows from [21, p. 152] that T is finitely generated as a $Z(B)^G$-module, hence r is integral over $Z(B)^G$ and in particular $Z(B)$ and $Z(A)$ are integral over $Z(A) \cap Z(B) = Z(B)^G$. This proves that we are in the arithmetical situation. The rest of the proof is identical to the proof of Theorem 3.1 in [17] if one takes into account that for every $q \in X^1(Z(B)^G)$, A_q is strongly graded over B_q so that the properties used "locally" in the proof of Theorem 3.1 in [17] are still valid in this case.

(2) (cf. [16]). Note that in this case $Z(A)$ is G-graded over $Z(A)_0 = Z(A) \cap Z(B)$ and it is possible to prove that $Z(A)$ is integral over a subring D which is divisorially graded over $Z(A)_0$; at $q \in X^1(Z(B))$, A_q is strongly graded and $Z(A_q)$ is a scaled Rees ring in the sense of [23]. Consequently $Z(B)$ and $Z(A)$ will have PDE over $Z(A)_0 = Z(B)^G$. If the A_σ are centrally controled for all $\sigma \in G$ then we are again in the arithmetical situation.

Now we consider B-bimodules P, P' and A-bimodules Q, Q' together with left and right B-linear maps $\phi: P \to Q$ and $\phi': P' \to Q'$. An isomorphism between $\phi: P \to Q$ and $\phi': P' \to Q'$ is then given by a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\phi} & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ P' & \xrightarrow{\phi'} & Q' \end{array}$$

where f is a B-bimodule isomorphism, g an A-bimodule isomorphism and all (iso)morphisms are defined over C. A substantial part of the theory may now be developed in the absence of (AS2) but since we are focussing on the divisorial techniques in this paper it is not very restrictive to impose condition (AS2). The situation of Proposition 2.6(1) is the most interesting for us.

Let $R_B(A)$ be the set of isomorphism classes $[\phi]$ of $\phi: P \to Q$ as introduced above, where $P \in \mathrm{Pic}_{Z(B)}(B, \kappa_1(Z(B))$ and Q is an A-bimodule aver $Z(A)$ such that the morphism $(A \otimes_B P)^{**} \to Q$ induced by $a \otimes p \to a\phi(p)$ is an isomorphism. In the AS situation $\kappa_1 = \kappa_1(Z(B))$ equals to the kernel functor induced on $Z(B)$-mod by $\kappa_1(C)$. We may therefore induce an operation in $R_B(A)$ by taking $(\_\_ \otimes \_\_)^{**}$ where $(P \otimes_B P')^{**} = Q_{\kappa_1}(P \otimes_B P') = P \perp_B P'$, $(Q \otimes_A Q')^{**} = Q_{\kappa_1(Z(A))}(Q \otimes_A Q')$ since A is tame, and $(\phi \otimes \phi')^{**}$ is defined by taking the composition,

$$\begin{aligned}(P \otimes_B P')^{**} &\to Q_{\kappa_1}(Q \otimes_A Q') = Q_\kappa(Q \otimes_A Q') \\ &\to Q_{\kappa(Z(A))}(Q \otimes_A Q') = (Q \otimes_A Q')^{**}.\end{aligned}$$

We simply write $[\phi][\phi'] = [\phi \perp \phi']$ for this operation. Since $(A \otimes_B (P \otimes_B P')^{**})^{**} = (A \otimes_B P \otimes_B P')^{**} = ((A \otimes_B P) \otimes_A (A \otimes_B P'))^{**} = (Q \otimes Q')^{**}$, where $**$ at the end of the formula refers to the $**$ in A-mod, it follows that $[\phi \perp \phi']$ is indeed an element of $R_B(A)$. The inclusion $B \to A$ is the identity for this operation; for $\phi: P \to Q$ such that $[\phi] \in R_B(A)$ one easily verifies that $[\phi^*]$ with $\phi^*: P^* \to Q^*$ (where

$P^* = \mathrm{Hom}_B({}_BP, {}_BB)$, $Q^* = \mathrm{Hom}_A({}_AQ, {}_AA)$ and $\phi^*(p^*)$ defined by $a\phi(p) \to ap^*(p)$ for all $a \in A$, $p \in P$) is an inverse for $[\phi]$ in $R_B(A)$. If we only assume that A and B are tame and $Z(A)$ and $Z(B)$ satisfy PDE over C while $\kappa_1(Z(A))$ is larger than the kernel functor induced by $\kappa_1(Z(B))$ on A-mod we say that we are in the *weak* AS.

The definition of centrally controlled elements of $D_B(A)$ may be extended to elements of $\mathrm{Pic}_C(B, \kappa_1)$ in the obvious way, so that we obtain a subgroup $\underline{\mathrm{Pic}}_C(B, \kappa_1)$ which coincides with $\mathrm{Pic}_C(B, \kappa_1)$ in the AS situation. The subset $\rho_B(A)$ of $R_B(A)$ consisting those $[\phi] \in R_B(A)$ such that $\phi: P \to Q$ is such that $P \in \underline{\mathrm{Pic}}_C(B, \kappa_1)$ is easily seen to be a subgroup of $R_B(A)$ in the weak AS situation (check that $(\phi \perp \phi')^{**}$ as defined above still makes sense in the weak AS situation), denoted by $\rho_B(A)$, which equals $R_B(A)$ in the AS situation.

2.7. THEOREM. *In the weak* AS *situation we obtain the following exact sequences*:

(a) $1 \to U(Z(A)) \to U(C_A(B)) \to^{\alpha} \delta_B(A) \to \underline{\mathrm{Pic}}_C(B, \kappa_1)$,

(b) $1 \to U(Z(A)) \to U(C_A(B)) \to^{\beta} \mathrm{Aut}_B(A) \to \mathrm{Pic}_C(A, \kappa_1(Z(A))$,

(c) $1 \to U(Z(A))\,U(Z(B)) \to U(Z(A)) \to^{\alpha} \delta_B(A) \to^{\varepsilon} \rho_B(A) \to^{\pi_A} \mathrm{Pic}_C(A, \kappa_1(Z(A)))$,

(d) $1 \to U(Z(A))\,U(Z(B)) \to U(Z(B) \to^{\beta} \mathrm{Aut}_B(A) \to_{\eta} \rho_B(A) \to^{\pi_B} \underline{\mathrm{Pic}}_C(B, \kappa_1)$.

In the AS *situation we may replace $\delta_B(A)$ by $D_B(A)$, $\rho_B(A)$ by $R_B(A)$ and $\underline{\mathrm{Pic}}_C(B, \kappa_1)$ by* $\mathrm{Pic}_C(B, \kappa_1)$.

Proof. Once all maps are defined, verification of exactness of the sequence is straightforward and left to the reader, so we just point out the definition of the nonobvious maps here.

(a) For $d \in U(C_A(B))$ define $\alpha(d) = Bd$.

(b) For $d \in U(C_A(B))$ define $\beta(d)(a) = dad^{-1}$ and we note that the inner automorphism of A given by d maps to the trivial element in $\mathrm{Pic}_C(A, \kappa_1(Z(A))$.

(c) If $P \in \delta_B(A)$ then $\varepsilon(P) = [i]$ where i is the inclusion $i: P \to A$. If $\phi: P \to Q$ represents $[\phi] \in \rho_B(A)$ then $\pi_A([\phi]) = [Q]$.

(d) To $f \in \mathrm{Aut}_B(A)$ we associate $[\psi]$ where $\psi: B \to Au_f$, $b \to bu_f$, where $Au_f = {}_1A_f$, i.e., $Au_f \cdot a = Af(a)\,u_f$ for all $a \in A$. (Note that $[\psi] \in \rho_B(A)$ since B is centrally controlled). If $[\phi] \in \rho_B(A)$ then $\pi_B(\phi) = [P]$, where $\phi: P \to Q$ represents $[\phi]$.

In the arithmetical situation we have already seen that $\delta_B(A) = D_B(A)$, $\rho_B(A) = R_B(A)$, and $\underline{\mathrm{Pic}}_C(B, \kappa_1) = \mathrm{Pic}(B, \kappa_1)$.

When B is commutative we also have $\mathrm{Pic}(B, \kappa_1) = \mathrm{Cl}(B)$ and this yields interesting results, e.g., when A is a reflexive Azumaya algebra over B or just any maximal order.

We include another particular result in case B is commutative.

2.8. THEOREM. *Suppose that $P \in I_B(A)$, i.e., P is invertible in the absolute sense, and B is commutative. Let the automorphism induced by P on $Z(B) = B$ be denoted by σ, then for every $p \in P$, $p' \in P^{-1}$ we have $\sigma(p'p) = pp'$; moreover if $\sum a_i a_i' = 1$ is a decomposition of 1 in $PP^{-1} = B$ then also $\sum a_i' a_i = 1$. A similar statement is still true if we only assume that $P \in \delta_B(A)$ in the arithmetical situation.*

Proof. First, note that it suffices to establish the result in the case where $P \in I_B(A)$ because by the usual localization argument at $p \in X^1(Z(A) \cap Z(B))$, combined with the remarks at the beginning of Section 2 the divisorial case will follow. So fix $\sum a_i a_i' = 1$ with $a_i \in P$, $a_i' \in P^{-1}$. Since $B \subset C_B(A)$ we obtain for every $b \in B$ that $\sigma(b) = \sum a_i b a_i'$. Consider $p' \in P^{-1}$, $p \in P$, then with $\lambda = \sum a_i' a_i$: $(*)$ $\sigma(p'p) = \sum a_i p' p a_i' = \sum p a_i' a_i p = p\lambda p' = pp'\sigma(\lambda)$. On the other hand, $pp' = \sum pp' a_i a_i' = \sigma(p' a_i) p a_i'$ and also $a_k' \sigma(p' a_i) = p' a_i a_k'$ for each $a_k' \in P^{-1}$; thus $a_k' pp' = \sum a_k' \sigma(p' a_i) a_i' = \sum p' a_i a_k' p a_i' = \sum p' p a_i a_i' a_k' = p' p \lambda a_k$. For any $a_k \in P$ we obtain: $a_k a_k' pp' = a_k p' p \lambda a_k$. By taking $a_k = a_i$, $a_k' = a_i'$ and summing over i: $(**)$ $pp' = \sum p' p \lambda a' = \sigma(p'p)\sigma(\lambda)$. From $(*)$ and $(**)$ we obtain $pp' = pp'\sigma(\lambda)^2$ and as elements of the form pp' generate B additively the foregoing relation leads to $\sigma^2(\lambda) = 1$. Now $pp' = pp' = \sigma(pp')pp' = pp'\sigma(\lambda)pp' = \sigma(\lambda)(pp')^2$. Consequently $1 - \sigma(\lambda)$ annihilates $(pp')^2$ for all $p \in P$, $p' \in P^{-1}$. Write $\rho_i = a_i a_i'$. Then $1 = \rho_1 + \cdots + \rho_r$ and $1 = 1^{r+1} = \sum \rho_1^{v_1} \cdots \rho_r^{v_r}$ where each v_i is at least 2. Therefore $(1 - \sigma(\lambda)).\ 1 = 0$ (since $\rho_1, \ldots, \rho_r$ commute). So $\sigma(\lambda) = 1$, hence $\lambda = 1$ and $(*)$ yields $\sigma(p'p) = pp'$.

3. GROUPS ASSOCIATED TO DIVISORALLY GRADED RINGS

In this section we consider a maximal order over a Krull domain $Z(B)$ and a given homomorphism $\Phi: G \to \mathrm{Pic}(B, \kappa_1)$ such that the composition $G \to \mathrm{Pic}(B, \kappa_1) \to \mathrm{Aut}(Z(B))$ defines an action of G on $Z(B)$ with the property that $Z(B)$ satisfies (AS1) over $Z(B)^G$ (e.g., if $Z(B)$ is integral over $Z(B)^G$, so in particular when G maps to a finite subgroup of $\mathrm{Aut}(Z(B))$). We define the divisorially graded ring $\check{B}(\phi)$ as $\oplus_{\sigma \in G} J_\sigma$, fixing representatives J_σ for $\Phi(\sigma)$ and B-bimodule isomorphism $f_{\sigma,\tau}: Q_{\kappa_1}(J_\sigma \otimes_B J_\tau) \to J_{\sigma\tau}$ which may be used to determine the multiplication in $B(\Phi)$ as follows: for

$x_\sigma \in J_\sigma$, $y_\tau \in J_\tau$ put $x_\sigma y_\tau = f_{\sigma,\tau}(x_\sigma \otimes y_\tau)$ extended B-bilinearly and extended to the localization in the canonical way. Now we consider the inclusion $B \to \check{B}(\Phi)$.

3.1. LEMMA. *The group morphism Φ: $G \to \mathrm{Pic}(B, \kappa_1)$ (as above) gives rise to a commutative diagram of group morphisms*

$$\begin{tikzcd} & \delta_B(\check{B}(\Phi)) \arrow[d, "{[\ ]}"] \\ G \arrow[ur, "\Phi'"] \arrow[r, "\Phi"'] & \underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1) \end{tikzcd}$$

Proof. Define $\phi': G \to \delta_B(\Phi))$ by $\sigma \to J_\sigma$; note that $J_\sigma \in \delta_B(B(\Phi))$ because of the (AS1) condition for the extension $Z(B)^G \to Z(B)$. Hence it is also clear that $\Phi(\sigma) = [J_\sigma]$ is in $\underline{\mathrm{Pic}}(B, \kappa_1)$ and even in $\underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1)$ because each J_s commutes with $Z(B)^G$.

In the sequel we will write $\Delta = \check{B}(\Phi)$ and Φ for the morphism $G \to \delta_B(\Delta)$, for notational convenience. Let us point out that any condition on G (and) or on B that makes Δ into a tame order will entail the weak-AS situation for $B \to \Delta$ whereas for a finite group with $/G/^{-1} \in B$ we are in the AS situation by Proposition 2.6. For B-bimodules N and M we say that $N \mid M$ if N is a direct summand of a finite number of copies of M. We say that N is similar to M, written $N \sim M$ if $N \mid M$ and $M \mid N$. Extending the definition we will say that $N \mid_{\kappa_1} M$ if there exists a B-bilinear surjection $M \oplus \cdots \oplus M \to N \to 0$ which splits locally at every $p = P \cap Z(B)^G$ for $P \in X^1(Z(B))$. We say that M is divisorially similar to N if $N \mid_{\kappa_1} M$ and $M \mid_{\kappa_1} N$.

3.2. LEMMA. *Let M be divisorially similar to N, then*

(1) *If $[N] \in \mathrm{Pic}(B, \kappa_1)$ then $[M] \in \mathrm{Pic}(B, \kappa_1)$.*

(2) *If $[N] \in \mathrm{Pic}_{Z(B)^G}(B, \kappa_1)$ then also $[M]$.*

Proof. If we are in the situation of (1) or (2) then N is κ_1-finitely (presented) generated and from the existence of a morphism $N \oplus \cdots \oplus N \to M \to 0$ it follows that M is κ_1-finitely (presented) generated too (note: since B is a tame order the foregoing statement holds over B as well as over $Z(B)$). By Proposition 4.15. [26, II] it suffices to check whether $M_P \in \mathrm{Pic}(B_P)$ for all $P \in X^1(Z(B))$ to prove (1) and $M_p \in \mathrm{Pic}(B_p)$ for all p of the form $p = P \cap Z(B)^G$, $P \in X^1(Z(B))$ in order to establish (2). In each of the cases considered these properties follow from the corresponding properties for N_p and M_p and the suitable exact localization of the sequences defining the divisorial similarity of M and N.

We define $\gamma_B(\Delta)$ to be the set of graded isomorphism classes of divisorially graded rings $\oplus_{\sigma \in G} H_\sigma$ over $B = H_e$ such that, for all $\sigma \in G$ we have $H_\sigma \sim_{\kappa_1} J_\sigma$.

The multiplication of $\oplus_{\sigma \in G} H_\sigma$ is given by a "factor set" $h_{\sigma,\tau}: Q_{\kappa_1}(H_\sigma \otimes H_\tau) \to H_{\sigma\tau}$ and we will write (H, h) for this divisorially graded ring and $[H, h]$ for its graded isomorphism class, so in particular $\Delta = (J, f)$ will be denoted by (J, j) for symmetry reasons. We now define an operation in $\gamma_B(A)$ as follows. Consider $[V, v]$ and $[W, w]$ in $\gamma_B(\Delta)$ and define their product $[U, u]$ by putting $U_\sigma = Q_{\kappa_1}(V_\sigma \otimes_B J_{\sigma^{-1}} \otimes_B W_\sigma)$ and $u_{\sigma\tau} = Q_{\kappa_1}(v_{\sigma\tau} \otimes j_{\sigma^{-1}\tau^{-1}} \otimes w_{\sigma\tau})$; let us check that we have done well, by providing some easy lemmas.

3.3. Lemma. (1) *If M is a divisorial $Z(B)$-module such that $M \mid_{\kappa_1} Z(B)$, then $Q_{\kappa_1}(\mathrm{End}_B({}_B(B \otimes_{Z(B)} M))) = Q_{\kappa_1}(B \otimes_{Z(B)} \mathrm{End}_{Z(B)}(M))$, and $Q_{\kappa_1}(B \otimes_{Z(B)} M) \mid_{\kappa_1} B$. Also $C_B(B_{Z(B)} M) = M$ and if $M \in \mathrm{Pic}(Z(B), \kappa_1) = \mathrm{Cl}(Z(B))$ then $B \otimes_{Z(B)} M \in \mathrm{Pic}(B, \kappa_1)$.*

(2) *If $M \mid_{\kappa_1} B$ then $M = Q_{\kappa_1}(B \cdot C_B(M)) \cong Q_{\kappa_1}(B \otimes C_B(M)$ and $C_B(M) \mid_{\kappa_1} Z(B)$. Also $\mathrm{End}_{Z(B)}(C_B(M)) \cong \mathrm{End}_B({}_B M_B)$, $\mathrm{End}_B({}_B M) \cong Q_{\kappa_1}(B_{Z(B)} \mathrm{End}_B(M))$. If $M \mid_{\kappa_1} B$ and $M' \mid_{\kappa_1} B$ then the B-bimodules M and M' are isomorphic if and only if $C_B(M) \cong C_B(M')$ are $Z(B)$-isomorphic.*

(3) *If $M \mid_{\kappa_1} B$, and $M' \mid_{\kappa_1} B$ then $Q_{\kappa_1}(C_B(M \otimes_B M') \cong Q_{\kappa_1}(C_B(M) \otimes_{Z(B)} C(M'))$ and there exists an isomorphism $t: Q_{\kappa_1}(M \otimes_B M') \to Q_{\kappa_1}(M' \otimes_B M)$, $m \otimes m' \to m' \otimes m$, where $m \in M$, $m' \in C_B(M')$ (extended linearly to the whole $Q_{\kappa_1}(M \otimes M')$ after localizing).*

Proof. If $M \mid_{\kappa_1} B$ then from the B-bilinear $B \oplus \cdots \oplus B \to M$ it follows that M is generated as a left (or right) B-module by $C_B(M)$, i.e., $M = BC_B(M)$ (even if we write $M = Q_{\kappa_1}(BC_B(M))$ sometimes). All the statements of the lemma may be derived from of Lemmas 2.3 and 2.4 and Corollaries 1, 2, 3, in [13] (note: it is essential here that κ_1 is a central localization).

For the definition of (U, u) we now note:

$$Q_{\kappa_1}(Q_{\kappa_1}(V_\sigma \otimes_B J_{\sigma^{-1}} \otimes_B W_\sigma) \otimes_B Q_{\kappa_1}(V_\tau \otimes_B J_{\tau^{-1}} \otimes_B W_\tau)$$
$$= Q_{\kappa_1}(V_\sigma \otimes B\, J_{\sigma^{-1}} \otimes_B W_\sigma \otimes_B V_\tau \otimes_B J_{\tau^{-1}} \otimes_B W_\tau).$$

Furthermore $Q_{\kappa_1}(V_\tau \otimes_B J_\sigma^{-1}) \mid_{\kappa_1} B$ and $Q_{\kappa_1}(J_{\sigma^{-1}} \otimes_B W_\sigma) \mid_{\kappa_1} B$ and

$$Q_{\kappa_1}(J_{\sigma^{-1}} \otimes_B W_\sigma) \otimes_B Q_{\kappa_1}(V_\tau \otimes_B J_{\tau^{-1}})$$
$$\xrightarrow[t]{\cong} Q_{\kappa_1}(V_\tau \otimes_B J_{\tau^{-1}}) \otimes_B Q_{\kappa_1}(J_{\sigma^{-1}} \otimes_B W_\sigma).$$

By localizing $v_{\sigma,\tau} \otimes j_{\tau^{-1},\sigma^{-1}} \otimes w_{\sigma,\tau}$ we obtain a B-linear

$$Q_{\kappa_1}(v_{\sigma,\tau} \otimes_B j_{\tau^{-1},\sigma^{-1}} \otimes_B w_{\sigma,\tau})\colon\ Q_{\kappa_1}(V_\sigma \otimes_B V_\tau \otimes_B J_{\tau^{-1}} \otimes_B J_{\sigma^{-1}} \otimes_B W_\sigma \otimes_B W_\tau)$$
$$\to Q_{\kappa_1}(V_{\sigma\tau} \otimes_B J_{(\sigma\tau)^{-1}} \otimes_B W_{\sigma\tau}$$

which defines

$$u_{\sigma\tau}\colon\ Q_{\kappa_1}(Q_{\kappa_1}(V_\sigma \otimes_B J_{\sigma^{-1}} \otimes_B W_\sigma) \otimes_B Q_{\kappa_1}(V_\tau \otimes_B J_{\tau^{-1}} \otimes_B W_\tau)$$
$$\to Q_{\kappa_1}(V_{\sigma\tau} \otimes_B J_{(\sigma\tau)^{-1}} \otimes_B W_{\sigma\tau}).$$

One easily checks that $u_{\sigma,\tau}$ is a factor set and using the transposition map t one may establish that we have obtained an associative operation on $\gamma_B(A)$ such that $[\Delta] = [J, j]$ is the unit element. The inverse of $[V, v]$ is defined by putting $W_\sigma = Q_{k_1}(J_\sigma \otimes V_\sigma^* \otimes J_\sigma)$ where $V_\sigma^* = \mathrm{Hom}_B({}_B V_\sigma, {}_B B) \in \mathrm{Pic}(B_{,\kappa 1})$, and

$$v_{\sigma\tau}^*\colon\ Q_{\kappa_1}(J_\sigma \otimes_B V_\sigma^* \otimes_B J_\sigma \otimes_B J_\tau \otimes_B V_\tau^* \otimes_B J_\tau) \to Q_{\kappa_1}(J_{\sigma\tau} \otimes_B V_{\sigma\tau}^* \otimes_B J_{\sigma\tau})$$

is obtained via the transposition map, because the $\underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1) \to \gamma_B(\Delta)$, $[P] \to [\oplus_{\sigma \in G} Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P), j_{\sigma\tau}^P]$, where $j_{\sigma\tau}\colon Q_\kappa^{\ 1}(P \otimes J_\sigma \otimes^* P \otimes P \otimes J_\tau \otimes^* P) \to Q_{\kappa_1}(P \otimes J_\sigma \otimes J_\tau \otimes^* P) \to Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P)$ defines the multiplication of the divisorially graded ring $\oplus_{\sigma \in G} Q_{\kappa_1}(P \otimes J \otimes^* P)$. We have obtained

3.4. PROPOSITION. *There is a commutative diagram of abelian groups*

$$\begin{tikzcd}
\underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1)^G \arrow[r] \arrow[d] & \gamma_B^g(\Delta) \arrow[d] \\
\underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1)^{[G]} \arrow[r] & \gamma_B(\Delta)
\end{tikzcd}$$

In $\rho_B(\Delta)$ we define $\rho_B(\Delta)^{(G)} = \{[\phi] \in \rho_B(\Delta), \phi\colon P \to M \ \ Q_{\kappa_1}(J_\sigma \phi(P)) = \psi_{\kappa_1}(\phi(P)\, J_\sigma)$ for all $\sigma \in G\}$. Note that the κ_1-flatness of Δ over B entails that the kernel of $P \to \Delta \otimes P$ is κ_1-torsion, i.e., zero since P is κ_1-closed, so $P \to Q_{\kappa_1}(\Delta \otimes P) = M$ and therefore we may identify P and $\phi(P)$ in M.

It is the clear that $[\phi] \in \rho_B(\Delta)^{(G)}$ exactly when there exists a B-bimodule isomorphism $f_\sigma\colon P \to Q_{\kappa_1}(J_\sigma \otimes P \otimes J_{\sigma^{-1}})$ such that the following diagram is commutative:

$$\begin{tikzcd}
P \arrow[rr, "\phi"] \arrow[d, "f_\sigma"'] & & M \\
Q_{\kappa_1}(J_\sigma \otimes P \otimes J_{\sigma^{-1}}) \arrow[urr, "Q_{\kappa_1}(\sigma_\phi)"'] & &
\end{tikzcd} \qquad (*)$$

where ${}^{\sigma}\phi(x_\sigma \otimes p \otimes x'_{\sigma^{-1}}) = x_\sigma \phi(p)\, x'_{\sigma^{-1}}$ for all $x_\sigma \in J_\sigma$, $p \in P$, $x'_{\sigma^{-1}} \in J_{\sigma^{-1}}$. Indeed if for all $\sigma \in G$ we have $Q_{\kappa_1}(J_\sigma \phi(P)) = Q_{\kappa_1}(\phi(P)\, J_\sigma)$ then from $Q_{\kappa_1}(J_\sigma \otimes P) \to Q_{\kappa_1}(\Delta \otimes P) = M$ we obtain that $Q_{\kappa_1}(J_\sigma \otimes P) = Q_{\kappa_1}(J_\sigma \phi(P))$; on the other hand, if K is the kernel of the map $\phi(P) \otimes J_\sigma \to M$, $p \otimes x \to px$ then by localization at $p = P \cap Z(B)^G$, $P \in X^1(Z(B))$, if follows that K is κ_1-torsion and then $Q_{\kappa_1}(\phi(P) \otimes J_\sigma) = Q_{\kappa_1}(\phi(P)\, J_\sigma)$, i.e., we obtain

$$Q_{\kappa_1}(\phi(P) \otimes J_\sigma) = Q_{\kappa_1}(\phi(P)\, J_\sigma) = Q_{\kappa_1}(J_\sigma \phi(P)) = Q_{\kappa_1}(J_\sigma \otimes \phi(P)).$$

Identifying P and $\phi(P)$ in M we thus obtain

$$\begin{aligned} P &= Q_{\kappa_1}(P \otimes J_\sigma \otimes J_{\sigma^{-1}}) = Q_{\kappa_1}(Q_{\kappa_1}(P \otimes J_\sigma) \otimes J_{\sigma^{-1}}) = Q_{\kappa_1}(Q_{\kappa_1}(PJ_\sigma) \otimes J_{\sigma^{-1}}) \\ &= Q_{\kappa_1}(J_\sigma \otimes P \otimes J_{\sigma^{-1}}), \end{aligned}$$

which defines the B-bimodule isomorphism f_σ. Again by localizing at all $p = P \cap Z(B)^G$ for all $P \in X^1(Z(B))$ it follows easily that $(*)$ leads to $Q_{\kappa_1}(PJ_\sigma) = Q_{\kappa_1}(J_\sigma P)$ for all $\sigma \in G$. Considering $Q_{\kappa_1}((*) \otimes J_\sigma)$ yields

$$\begin{tikzcd}
Q_{\kappa_1}(P \otimes J_\sigma) \arrow[r, "Q_{\kappa_1}(\phi \otimes J_\sigma)"'] \arrow[d] & Q_{\kappa_1}(M \otimes J_\sigma) \arrow[r, "\mu"] & Q_{\kappa_1}(M) \\
Q_{\kappa_1}(J \otimes P \otimes J_{\sigma^{-1}} \otimes J_\sigma) \arrow[urr, "Q_{\kappa_1}({}^{\sigma}\phi \otimes J_\sigma)"] \arrow[d] & & \\
Q_{\kappa_1}(J_\sigma \otimes P) \arrow[uurr, "v"'] & &
\end{tikzcd}$$

where μ is the localization of $m \otimes x_\sigma \to mx_\sigma$; $m \in M$, $x \in J_\sigma$, and u is the composition $\mu Q_{\kappa_1}(\phi \otimes J_\sigma)$. We have $\operatorname{Im} u = \operatorname{Im} v$. Clearly $Q_{\kappa_1}(Im\, u) = Q_{\kappa_1}(PJ_\sigma)$. On the other hand, the definition of ${}^{\sigma}\phi$ is such that one can easily verify that $Q_{\kappa_1}(\operatorname{Im} v) = Q_{\kappa_1}(J_\sigma P)$ and hence $\operatorname{Im} u = \operatorname{Im} v$ entails: $Q_{\kappa_1}(J_\sigma P) = Q_{\kappa_1}(PJ_\sigma)$. (Note: it is possible to derive these equivalences from Miyashita's results by the usual "local" argumentation). The foregoing argumentation also applies to P^* representing the inverse of $[P]$ and it is not hard to check that ϕ^*: $P^* \to M^*$, where $M^* = \operatorname{Hom}_\Delta({}_\Delta M, {}_\Delta \Delta)$ is again representing an element of $\rho_B(\Delta)^{(G)}$. Therefore $\rho_B(\Delta)^{(G)}$ is closed under taking inverses and so it is a subgroup of $\rho_B(\Delta)$ (this statement can also be checked "locally" in the trivial way). Writing $\operatorname{Aut}_B(\Delta)^{(G)} = \{\alpha \in \operatorname{Aut}_B(\Delta),\ \alpha(J_\sigma) = J_\sigma \text{ for all } \sigma \in G\}$ we obtain

3.5. Proposition. *There is an exact sequence*:

$$1 \to U(Z(B)^{(G)}) \to U(Z(B)) \xrightarrow{\beta} \operatorname{Aut}_B(\Delta) \xrightarrow{\eta} \rho_B(\Delta)^{(G)} \xrightarrow[\pi_B]{} \underline{\operatorname{Pic}}\, Z(B)^G(B, \kappa_1)^{(G)}$$

Proof. The above sequence is a subsequence of the one in Theorem 2.6(d) and it will suffice to establish that an $f \in \mathrm{Aut}_B(\Delta)$ maps in $\rho_B(\Delta)$. Now $\eta(f)$: $B \to {}_1\Delta_f$ is in $\rho_B(\Delta)^{(G)}$ if and only if $f(J_\sigma) = J_\sigma$ for all $\sigma \in G$.

3.6. LEMMA. *The maps in Proposition* 3.4 *yield an exact sequence*

$$\rho_B(\Delta)^{(\Delta)} \to \underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1)^{(G)} \xrightarrow{\varepsilon} \gamma_B^g(\Delta).$$

Proof. Let $\psi: P \to M$ represent some $[\psi] \in \rho_B^g(\Delta)^{(G)}$, then $[P] \in \mathrm{Pic}_{Z(B)^G}(B, \kappa_1)$ and it maps to $[Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P), {}^P j]$. Now $Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P) = Q_{\kappa_1}(Q_{\kappa_1}(J_\sigma \otimes P \otimes J_{\sigma^{-1}}) \otimes J_\sigma \otimes^* P)$ since $[\psi] \in \rho_B(\Delta)^{(G)}$, and the latter is further isomorphic to $Q_{\kappa_1}(J_\sigma \otimes P \otimes J_{\sigma^{-1}} \otimes J_\sigma \otimes^* P) = Q_{\kappa_1}(J_\sigma \otimes P \otimes Q_{\kappa_1}(J_{\sigma^{-1}} \otimes J_\sigma) \otimes^* P) = Q_{\kappa_1}(J_\sigma \otimes Q_{\kappa_1}(P \otimes^* P)) = J_\sigma$, as B-bimodules.

Let $h_\sigma: Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P) \to J_\sigma$ be the above isomorphism, then we obtain a commutative diagram

$$\begin{tikzcd}
Q_{\kappa_1}(Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P) \otimes Q_{\kappa_1}(P \otimes J_\tau \otimes^* P)) \arrow[r, "{}^P j_{\sigma\tau}"'] \arrow[d] & Q_{\kappa_1}(P \otimes J_{\sigma\tau} \otimes^* P) \arrow[d] \\
Q_{\kappa_1}(J_\sigma \otimes J_\tau) \arrow[r, "j_{\sigma\tau}"'] & J_{\sigma\tau}
\end{tikzcd}$$

If $[P] \in \underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1)$ is in the kernel of ε then $\Delta^1 = (\oplus_{\sigma \in G} Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P), P_j) = (\oplus_{\sigma \in G} J_\sigma, j)$. First, we establish that $\Delta^1 = \mathrm{End}_\Delta(Q_{\kappa_1}(P \otimes \Delta)_\Delta)$, then it is evident that the fact that $\Delta^1 = \Delta$ entails $Q_{\kappa_1}(P \otimes \Delta) \in \mathrm{Pic}_{Z(B)^G}(\Delta, \kappa_1(Z(\Delta)))$ and so the canonical ϕ: $P \to Q_{\kappa_1}(P \otimes \Delta)$ determines an element $[\phi] \in \rho_B(\Delta)^{(G)}$ mapping to $[P]$.

For $p \otimes x \otimes p' \in P \otimes J_\sigma \otimes^* P$ we define, for all $\tau \in G$, $p \otimes x \otimes p'$: $P \otimes J_\tau \to P \otimes J_{\sigma\tau}$, $q \otimes y \to p \otimes xp'(q)\, y$, thus $Q_{\kappa_1}(p \otimes x \otimes p')$: $Q_{\kappa_1}(P \otimes J_\tau) \to Q_{\kappa_1}(P \otimes J_{\sigma\tau})$ determines an element of degree σ in $\mathrm{HOM}_\Delta(Q_{\kappa_1}(P \otimes \Delta), Q_{\kappa_1}(P \otimes \Delta))$ and $\mathrm{Hom}_\Delta(Q_{\kappa_1}(P \otimes \Delta), Q_{\kappa_1}(P \otimes \Delta)) = \mathrm{HOM}_\Delta(Q_{\kappa_1}(P \otimes \Delta), Q_{\kappa_1}(P \otimes \Delta))$ follows from the fact that $P \otimes \Delta$ is κ_1-finitely generated and the divisorial version of the well-known graded result concerning Hom and HOM, Corollary I.2.11 in [15]. Note that $\mathrm{HOM}_\Delta(Q_{\kappa_1}(P \otimes \Delta), Q_{\kappa_1}(P \otimes \Delta))$ is a κ_1-closed B-module by Corollary 1.14 in [26, II]. It is obvious to check that we have actually defined a graded ring morphism (of degree zero):

$$\Omega: \quad \bigoplus_{\sigma \in G} Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P) \to \mathrm{HOM}_\Delta(Q_{\kappa_1}(P \otimes \Delta)_\Delta, Q_{\kappa_1}(P \otimes \Delta)_\Delta).$$

Both rings are divisorially graded by G over B. First, Ω is monomorphism; pick $x, y \in Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P)$ such that $\Omega(x) = \Omega(y)$, then $\Omega(Q_{\kappa_1}(P \otimes J_{\sigma^{-1}} \otimes^* P)\, x) = \Omega(Q_{\kappa_1}(P \otimes J_{\sigma^{-1}} \otimes^* P)\, y)$. Since $\Omega\,|_B$ is an isomorphism it follows that $Q_{\kappa_1}(P \otimes J_{\sigma^{-1}} \otimes^* P)(x - y) = 0$ and further $Q_{\kappa_1}(P \otimes J \otimes^* P \otimes P \otimes J_{\sigma^{-1}} \otimes^* P)(x - y) = 0$, i.e., $x - y = 0$. Now divisorially graded rings over the same ring in degree zero such that one constains the other are necessarily equal, so Ω is an isomorphism.

Define $\beta_B(\Delta)$ by the exact sequence

$$\underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1)^{[G]} \to \gamma_B(\Delta) \xrightarrow{\xi} \beta_B(\Delta)$$

3.7. Proposition. *There is an exact sequence*:

$$\underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1)^{(G)} \to \gamma_B^g(\Delta) \to \beta_B(\Delta).$$

Proof. Semi-exactness is obvious. If $[\oplus_{\sigma \in G} J_\sigma, f_{\sigma,\tau}]$ is in the kernel of ξ then there exists a $[P]$ in $\mathrm{Pic}_{Z(B)^G}(B, \kappa_1)^{[G]}$ such that $[P] \to [\oplus_{\sigma \in G} J_\sigma, f_{\sigma,\tau}]$ under the map $Pic_{Z(B)^G}(B, \kappa_1)^{[G]} \to \gamma_B(\Delta)$. So we have that $Q_{\kappa_1}(P \otimes J_\sigma \otimes^* P) = J_\sigma$ and $Q_{\kappa_1}(J_{\sigma^{-1}} \otimes P \otimes J_\sigma \otimes^* P) = B$, so $Q_{\kappa_1}(J_{\sigma^{-1}} \otimes P \otimes J_\sigma \otimes^* P) \otimes P = P$ and hence $Q_{\kappa_1}(J_{\sigma^{-1}} \otimes P \otimes J_\sigma) = P$ or $[P] \in \mathrm{Pic}_{Z(B)^G}(B, \kappa_1)^{(G)}$.

We now extend further some definitions and results of Miyashita [13].

Put $\underline{\mathrm{Pic}}_0(B, \kappa_1) = \{[P] \in \underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1),\ P \sim_\kappa B\}$. Since this group consist of classes of B-bimodules M which are of the form $M = Q_{\kappa_1}(B \otimes C_B(M)) = BC_B(M)$, it is obvious that $\underline{\mathrm{Pic}}_0(B, \kappa_1) = \mathrm{Pic}_{Z(B)^G}(Z(B), \kappa_1)$.

We may now define a group homomorphism:

$$\gamma_B(\Delta) \to Z^1(B, \underline{\mathrm{Pic}}_0(B, \kappa_1)),$$

$$\left[\bigoplus_{\sigma \in G} V_\sigma, v\right] \to (\sigma \to Q_{\kappa_1}(V_\sigma \otimes J_{\sigma^{-1}}),$$

which obviously gives rise to the exact sequence:

$$1 \to \gamma_B^g(\Delta) \to \gamma_B(\Delta) \to Z^1(G, \underline{\mathrm{Pic}}_0(B, \kappa_1).$$

If we define $\bar{H}^1(G, \underline{\mathrm{Pic}}_0(B, \kappa_1)$ by the exact sequence

$$\begin{tikzcd}
\mathrm{Pic}_{Z(B)^G}(B, \kappa_1) \arrow[r] \arrow[dr] & Z^1(G, \underline{\mathrm{Pic}}_0(B, \kappa_1) \arrow[r] & \bar{H}^1(G, \underline{\mathrm{Pic}}_0(B, \kappa_1). \\
 & \gamma_B(\Delta) \arrow[u] &
\end{tikzcd}$$

This will lead to the exactness of the sequence:

$$\gamma_B(\Delta) \to \beta_B(\Delta) \to \bar{H}^1(G, \mathrm{Pic}_0(B, \kappa_1) \to H^3(G, U(Z(B))).$$

Combining these sequences we obtain a seven terms exact sequence generalizing the Chase–Rosenberg sequence of the reflexive Brauer group of a Krull domain:

3.8. THEOREM. *Let B be a maximal order over a Krull domain $Z(B)$; let $\Phi: G \to \mathrm{Pic}(B, \kappa_1)$ be a given such that $G \to \mathrm{Pic}(B, \kappa_1) \to \mathrm{Aut}(Z(B)$ defines an action of G on $Z(B)$ such that $Z(B)$ satisfies* (AS1) *over $Z(B)^G$ then for $\Delta = B(\Phi)$, the following sequence is exact*:

$$1 \to U(Z(B)^G) \to U(Z(B)) \to \mathrm{Aut}_B(\Delta)^G \to \rho_B(\Delta)^{(G)}$$
$$\to \underline{\mathrm{Pic}}_{Z(B)^G}(B, \kappa_1) \to \gamma_B^g(\Delta) \to \beta_B(\Delta) \to \bar{H}^1(G, \underline{\mathrm{Pic}}_0(B, \kappa_1)$$
$$\to H^3(G, U(Z(B)).$$

Further research of some terms in this sequense may prove to be interesting, in particular the groups $\gamma_B(\Delta)$, $\beta_B(\Delta)$ may be of interest even if B is commutative (hence a Krull domain in our situation) or when Δ is a reflexive Azumaya algebra. If B is a Galois extension of B^G and a maximal commutative subring of the reflexive Azumaya algebra Δ then the sequence in Theorem 3.8 reduces to the reflexive version of the Chase–Rosenberg sequence (and the related generalized crossed product results) expounded in [24].

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