The Levitzki Radical in Associative and Jordan Rings

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1. INTRODUCTION

If \( R \) is an associative ring in which \( 2x = a \) has a unique solution for all \( a \in R \), then it is of interest to consider the attached ring \( R^+ \), where \( R^+ \) is the same additive group as \( R \) but multiplication in \( R^+ \) is given by \( a \cdot b = \frac{1}{2}(ab + ba) \) (Here \( ab \) represents the multiplication in \( R \) and \( \frac{1}{2}a \) is the element \( x \) for which \( 2x = a \)). \( R^+ \) is a linear Jordan ring by virtue of the fact that it satisfies the identity \( x^2 \cdot (x \cdot y) = x \cdot (x^2 \cdot y) \) and its linearizations. Similarly, if \( R \) is equipped with an involution \( * \), then we also have the attached subring \( S \) of \( R^+ \) where \( S \) is the set of \( * \)-symmetric elements of \( R \).

There are many known interrelationships between \( R \), \( R^+ \), and \( S \). For example, Herstein [6, 7] has shown that \( R \) is simple if and only if \( R^+ \) is simple if and only if \( S \) is simple. McCrimmon [9] extended this result to arbitrary rings not necessarily satisfying the condition that \( 2x = a \) has a solution. In addition \( \text{Rad } R = \text{Rad } R^+ \), where \( \text{Rad} \) denotes the Jacobson, nil, or prime radical [3, 9]. Finally, \( \text{Rad } S = S \cap \text{Rad } R \) if \( \text{Rad} \) denotes the Jacobson or prime radical [3, 9].

In this paper, we compare \( \mathcal{L}(R), \mathcal{L}(R^+), \) and \( \mathcal{L}(S) \) where \( \mathcal{L} \) denotes the Levitzki radical. Unless otherwise stated (Lemma 5 and Theorem 2) we will assume that \( R \) satisfies the characteristic condition mentioned above. In Section 2 we show that \( \mathcal{L}(R) = \mathcal{L}(R^+) \) and as a by-product we see that there exist finitely generated nil Jordan algebras that are not nilpotent. In Section 3, we study the relationship between \( \mathcal{L}(S) \) and \( S \cap \mathcal{L}(R) \) and show that \( \mathcal{L}(S) = S \cap \mathcal{L}(R) \). Along the way we show that if \( R \) is an algebra over a field \( F \) with "enough" elements, then if \( S \) is nil of index \( n \), then \( R \) is nil of index \( \leq 2n \).
In the following \([x, y]\) denotes \(xy - yx\).

**Lemma 1.** \(R \cdot R\) is an ideal of the associative ring \(R\).

*Proof.* It is easy to see that for every \(x, y, u, v, \) in \(R\)

\[xy = x \cdot y + \frac{1}{2}[x, y], \quad (1)\]

and

\[[u \cdot v, y] = [u, y] \cdot v + [v, y] \cdot u. \quad (2)\]

In (1) let \(v = u \cdot v\). Then \((u \cdot v)y \equiv \frac{1}{2}[u \cdot v, y] \mod R \cdot R\). On the other hand, (2) shows that \([u \cdot v, y] \in R \cdot R\). Therefore, \((u \cdot v)y \in R \cdot R\). Similarly, \(y(u \cdot v) \in R \cdot R\), so that \(R \cdot R\) is an ideal of \(R\).

In any ring \(R\) (not necessarily associative) define \(R^{(0)} = R\), \(R^{(1)} = R^2\), and \(R^{(k-1)} = (R^{(k)})^2\). We say that \(R\) is solvable of index \(n\) if \(n\) is the least integer such that \(R^{(n)} = 0\). As usual \(R^n\) denotes the set of all finite sums of products of \(n\) elements of \(R\) regardless of how they are associated, and \((R^+)^n\) denotes the set of all finite sums of products of \(n\) elements in the attached ring \(R^+\) regardless of how they are associated. \(R\) is nilpotent of index \(n\) if \(n\) is the least positive integer such that \(R^n = 0\).

**Lemma 2.** For any ring \(R\), \(R^{3^n} \subseteq (R^+)^{(n)}\). In particular, if \(R^+\) is solvable of index \(n\) then \(R\) is nilpotent of index \(\leq 3^n\).

*Proof.* We proceed by induction on \(n\). Let \(x, y, z \in R\). By Lemma 1, \(R \cdot R\) is an ideal of \(R\) so that we may compute \(xyz = -zxy = xzy \equiv -xyz \mod R \cdot R\). Therefore, \(2xyz = 0 \mod R \cdot R\) so that \(R^3 \subseteq R \cdot R\). Thus, the result holds for \(n = 1\). Suppose now that \(R^{3^k} \subseteq C(R^+)^{(k)}\) for some integer \(k\) and let \(B = (R^+)^{(k)}\). Since by Lemma 1 \(B\) is an associative ring in its own right, we apply the earlier argument to get \(B^3 \subseteq B \cdot B\). Thus, we have \((R^3)^3 \subseteq ((R^+)^{(k)})^3 \subseteq (R^+)^{(k^2 + 1)}\), which gives \(R^{3^{k+1}} \subseteq (R^+)^{(k+1)}\) to complete the induction. Therefore, if \((R^+)^{(n)} = 0\), it follows that \(R^{3^n} = 0\).

**Remark.** Since in any associative ring solvability and nilpotency are equivalent and since in any ring nilpotency implies solvability, from Lemma 2 we have: \(R\) solvable \(\iff\) \(R\) nilpotent \(\iff\) \(R^+\) solvable \(\iff\) \(R^+\) nilpotent.

**Lemma 3.** \(R\) is finitely generated if and only if \(R^+\) is finitely generated.

*Proof.* Clearly, \(R^+\) finitely generated implies that \(R\) is finitely generated. Conversely, let \(F\) be the free associative ring generated by \(x_1, x_2, \ldots, x_n\). Then by Lemma 2, \(F^3 \subseteq F \cdot F\). We claim that \(F^+\) is generated by \(x_1, x_2, \ldots, x_n\).
and the elements $x_i, x_j, i, j = 1, 2, \ldots, n$. For otherwise there is a monomial $u$ of least degree in $F^+$, which is not generated by the $x_i$ and the $x_i x_j$. Clearly, degree $u \geq 3$. Therefore, $u = \Sigma a_i \cdot b_i$ with $a_i$ and $b_i$ having smaller degree than $u$. Therefore, $a_i$ and $b_i$ are generated by the $x_i$ and the $x_i x_j$ in $F^+$ so that $u$ is also. This contradicts the choice of $u$. Therefore, $F^+$ is finitely generated.

Now $R \simeq F/K$ for some ideal $K$ of $F$, so that $R^+ \simeq (F/K)^+ \simeq F^+/K^+$. Since $F^+$ is finitely generated so is $F^+/K^+$. Thus, $R^+$ is finitely generated.

**Remark.** A ring $R$ is called locally nilpotent if every finitely generated subring is nilpotent. An algebra $R$ is called locally finite if every finitely generated subalgebra of $R$ is finite dimensional. Golod and Shafarevitch [4] have solved the Kurosh problem for associative algebras by showing that over any countable field there exists a nil, finitely generated associative algebra that is not finite dimensional (hence, not nilpotent). Observe that a similar result holds for Jordan algebras. For let $A$ be a nil, finitely generated associative algebra that is not finite dimensional over a countable field of characteristic $\neq 2$, as constructed by Golod and Shafarevitch. Then by Lemma 3, $A^+$ is a nil, finitely generated Jordan algebra that is not finite dimensional, hence, not nilpotent.

The Levitzki radical $\mathcal{L}(R)$ of an associative ring $R$ is the maximal locally nilpotent ideal of $R$ [2]. Tsai [12] has shown the existence of the Levitzki radical $\mathcal{L}(J)$ in any Jordan ring $J$ for which $2a = b$ has a unique solution. He also shows that the basic properties of the Levitzki radical hold for $\mathcal{L}(J)$ (e.g., $\mathcal{L}(J)$ contains all locally nilpotent ideals, $J/\mathcal{L}(J)$ is Levitzki semisimple, and $\mathcal{L}(J)$ is the intersection of the prime ideals $P$ of $J$ such that $J/P$ is Levitzki semisimple). It is also not hard to see that if $B$ is an algebra (associative or Jordan) then $\mathcal{L}(B)$ is the same whether $B$ is treated as a ring or as an algebra.

It is known that for $R$ an associative ring, $J(R) = J(R^+)$, $N(R) = N(R^+)$ [9], and $P(R) = P(R^-)$ [3], where $J$, $N$, and $P$ represent the Jacobson, nil, and prime radicals, respectively. Our goal now is to show that the same result holds for the Levitzki radical. We begin with a result known for associative rings.

**Lemma 4.** If $A$ is a Jordan ring, then $\mathcal{L}(A)$ is the intersection of all ideals $Q$ of $A$ such that $A/Q$ is Levitzki semisimple.

**Proof.** Recall that $\mathcal{L}(A)$ is known to be the intersection of all prime ideals $P$ of $A$ such that $\mathcal{L}(A/P) = 0$. Let $K$ be the intersection of all ideals $Q$ of $A$ such that $\mathcal{L}(A/Q) = 0$. Then clearly $K \subseteq \mathcal{L}(A)$. For the other inclusion, assume that $T$ is an ideal of $A$ such that $\mathcal{L}(A/T) = 0$. Since $(\mathcal{L}(A) + T)/T \simeq \mathcal{L}(A)/(\mathcal{L}(A) \cap T)$ it follows that $(\mathcal{L}(A) + T)/T$ is locally
nilpotent. But $A/T$ is Levitzki semisimple. Therefore, $\mathcal{L}(A) + T \subseteq T$ or $\mathcal{L}(A) \subseteq T$. Thus, $\mathcal{L}(A) \subseteq K$.

**Theorem 1.** If $R$ is an associative ring then $\mathcal{L}(R) = \mathcal{L}(R^+)$.  

**Proof.** It is immediate that $\mathcal{L}(R) \subseteq \mathcal{L}(R^+)$. For $\mathcal{L}(R)$ is an ideal of $R^+$ and if $x_1, x_2, \ldots, x_n$ is a finite set of elements in $\mathcal{L}(R)$, then the subring of $R$ generated by $x_1, x_2, \ldots, x_n$ is nilpotent. Therefore, the subring of $R^+$ generated by $x_1, x_2, \ldots, x_n$ is nilpotent. Thus, $\mathcal{L}(R)$ is a locally nilpotent ideal of $R^+$ from which it follows that $\mathcal{L}(R) \subseteq \mathcal{L}(R^+)$.  

In order to show set inclusion in the other direction, it is sufficient to show that if $R$ is Levitzki semisimple then $R^+$ is Levitzki semisimple. For, in case $R$ is not Levitzki semisimple then, since $R/\mathcal{L}(R)$ is Levitzki semisimple, it would follow that $(R/\mathcal{L}(R))^+ \cong R^+/\mathcal{L}(R)^+$ is Levitzki semisimple. Thus, by Lemma 4 it would follow that $\mathcal{L}(R^+) \subseteq \mathcal{L}(R)$. We assume henceforth, without loss of generality, that $\mathcal{L}(R) = 0$.  

To complete the proof we rely on the result of Herstein [6] that if $U$ is an ideal of $R^+$ and $R$ has no nilpotent ideals, then there is a nonzero ideal $B$ of $R$ such that $B \subseteq U$. Since $\mathcal{L}(R) = 0$, $R$ has no nilpotent ideals so the result can be applied. Suppose that $\mathcal{L}(R^+) \neq 0$. Let $B$ be a nonzero ideal of $R$ such that $B \subseteq \mathcal{L}(R^+)$. We show that $B$ is locally nilpotent. For if $C$ is a finitely generated subring of $B$ then by Lemma 3, $C^+$ is finitely generated. Since $C^+ \subseteq \mathcal{L}(R^+)$, $C^+$ is nilpotent. Then by Lemma 2, $C$ is nilpotent. Therefore, $B$ is a nonzero locally nilpotent ideal of $R$. Thus, $B \subseteq \mathcal{L}(R)$, which contradicts the fact that $\mathcal{L}(R) = 0$. Therefore, $\mathcal{L}(R^+) = 0$ and the proof is completed.  

**Remark.** We could actually complete the proof of the theorem without Lemma 2. For since $C$ is finitely generated and $C^+$ is nilpotent, it follows that $C$ is nil of bounded index. Therefore, by a result of Levitzki [8], $C$ is nilpotent.

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Henceforth, we shall assume that $R$ is equipped with an involution $*$; i.e., $*$ is an antiautomorphism of $R$ of period 2. If $R$ is an algebra over a field $F$ then we assume that $R$ has an involution as a ring and that there is an automorphism of $F$ of period 2 such that $(ax)^* = \overline{ax}$ for all $a \in F$ and $x \in R$. When we speak of a subring (ideal) of $R$ we shall mean a subring (ideal) of $R$ which is invariant under $*$. Let $S = \{x \in R \mid x^* = x\}$. Then $S$ is a Jordan subring of $R^+$. We shall be interested in some relationships between $S$ and $R$ — in particular the relationship between $\mathcal{L}(S)$ and $\mathcal{L}(R)$. It is known that $\mathcal{L}(R)$ is in fact a $*$-invariant ideal. For $x \in \mathcal{L}(R)$ if and only if to each choice of $x_1, x_2, \ldots, x_n$ in $(x)$, the principal ideal generated by $x$,
there is an integer \( k(n) > 0 \) such that any product of \( k(n) \) of the \( x_i \) is zero. Therefore, if \( x \in \mathcal{L}(R) \), then \( x^* \in \mathcal{L}(R) \). For if we pick any \( n \) elements \( x_1, x_2, \ldots, x_n \in \langle x \rangle \), then \( x_1^*, x_2^*, \ldots, x_n^* \in \langle x \rangle \). Thus, if we choose any product \( y \) of \( k(n) \) of the \( x_i \), it follows that \( y^* = 0 \). Thus \( y = 0 \). Therefore, \( x^* \in \mathcal{L}(R) \) and \( \mathcal{L}(R) \) is \( * \)-invariant.

It has been shown by Osborn [10] that if \( R \) is an algebra over an uncountable field \( F \), then if \( S \) is a nil algebra then \( R \) is a nil algebra. Without the assumption that \( F \) is uncountable this remains an open question. The following results relate to this question and will have a bearing on our Theorem 3. If \( \alpha, \beta \in R \) and \( n, k \) are positive integers we define \( \alpha \beta(n, k) \) to be the sum of all monomials of degree \( n \) in \( \alpha \) and degree \( k \) in \( \beta \). For convenience we let \( \alpha \beta(0, 0) = 1 \).

The following Lemma is valid for a ring \( R \) of any characteristic.

**Lemma 5.** Let \( x \) be in \( R \), \( \alpha = x + x^* \), and \( \beta = -x^*x \). Then

\[
x^{2n} = \left[ \sum_{k=0}^{n-1} \alpha \beta(2n - 2k, 1, k) \right] x + \left[ \sum_{k=0}^{n-1} \alpha \beta(2k, n - k, k) \right] \beta.
\]

**Proof.** The proof is by induction on \( n \). Since \( x^2 = \alpha x + \beta \) the result holds for \( n = 1 \). Assume that the result holds for \( n \). Then \( x^{2n} = yx + \delta \) where

\[
y = \sum_{k=0}^{n-1} \alpha \beta(2n - 2k, 1, k) \quad \text{and} \quad \delta = \sum_{k=0}^{n-1} \alpha \beta(2n - 2k, 1, k) \beta.
\]

Thus \( x^{2n+1} = (yx + \delta) x + \gamma \beta \) and \( x^{2n+2} = (yx^2 + \delta x + \gamma \beta)x + (yx + \delta) \beta \). We wish to show that

1. \( \gamma x^2 + \delta x = \sum_{k=0}^{n} \alpha \beta(2n - 2k + 1, k) \)
2. \( \gamma x + \delta = \sum_{k=0}^{n} \alpha \beta(2k, n - k) \)

to complete the induction. For (a), note that \( \gamma x^2 \) consists of all terms of

\[
\sum_{k=0}^{n} \alpha \beta(2n - 2k + 1, k)
\]

which end in \( \alpha^2 \), \( \delta x \) consists of all terms of (4) which end in \( \beta x \), and \( \gamma \beta \) consists
of all terms of (4) which end in \( \beta \). Thus, (a) holds. Similarly we note that
\[
\sum_{k=0}^{n} \alpha \beta (2k, n - k)
\]
(5)
that end in \( \alpha \) while \( \delta \) consists of all terms of (5) that end in \( \beta \). Thus, (b) is verified also. Thus, (3) holds in the case \( n + 1 \) also and so is true in all cases.

**Lemma 6.** Let \( \alpha = x + x^* \), \( \beta = -x^*x \). If the Jordan subring of \( S \) generated by \( \alpha, \beta \) is nilpotent of index \( n \), then \( x \) is nilpotent of index \( \leq 2n \).

**Proof.** Let \( K \) represent the Jordan subring of \( S \) generated by \( \alpha, \beta \). We show that \( \hat{x} \beta(m, t) \in K^{m+t} \) for all choices of \( m, t \geq 0 \). (\( K^{m+t} \) represents Jordan multiplication in \( S \).) In view of (3), this is sufficient to prove the lemma. We proceed by induction on \( m + t \). The result is certainly true if \( n + t = 1 \).

Suppose it holds true whenever \( m + t < l \) and consider \( \alpha \beta(m, t) \) where \( m + t = l \). Now \( \alpha \beta(m, t) \) consists of terms which (1) begin and end with \( \alpha \), (2) begin and end with \( \beta \), (3) begin with \( \alpha \) and end with \( \beta \), and (4) begin with \( \beta \) and end with \( \alpha \).

Thus, (1) is just \( \alpha[\alpha \beta(m - 2, t)] \alpha \), (2) is \( \beta[\alpha \beta(m, t - 2)] \beta \), (3) is \( \alpha[\alpha \beta(m - 1, t - 1)] \alpha \), and (4) is \( \beta[\alpha \beta(m - 1, t - 1)] \beta \). But \( \alpha[\alpha \beta(m - 2, t)] \alpha - 2[\alpha \beta(m - 2, t) \cdot \alpha] \cdot \alpha = \alpha \beta(m - 2, t) \cdot \alpha^2 \). Since by the inductive hypothesis, \( \alpha \beta(m - 2, t) \in K^{m+t-2} \) it follows that \( \alpha[\alpha \beta(m - 2, t)] \alpha \in K^{m+t} \). Similarly for \( \beta[\alpha \beta(m - 2, t)] \beta \).

Finally
\[
\alpha[\alpha \beta(m - 1, t - 1)] \beta + \beta[\alpha \beta(m - 1, t - 1)] \alpha
\]
\[
= 2[\alpha \beta(m - 1, t - 1) \cdot \alpha] \cdot \beta + 2[\alpha \beta(m - 1, t - 1) \cdot \beta] \cdot \alpha
\]
\[
- 2[\alpha \beta(m - 1, t - 1) \cdot (\alpha \cdot \beta)].
\]
Since \( \alpha \beta(m - 1, t - 1) \in K^{m+t-2} \) it follows that \( (3) + (4) \in K^{m+t} \). Thus \( \alpha \beta(m, t) \in K^{m+t} \) for all \( m, t \geq 0 \) and \( \alpha^{2n} - 0 \).

**Corollary.** If \( S \) is Jordan nilpotent of index \( n \), then \( R \) is nil of index \( \leq 2n \).

Amitsur has shown that if the symmetric elements of a ring \( R \) with involution satisfy a polynomial identity of degree \( d \), then \( R \) satisfies a power of the standard identity of degree \( 2d \) [1]. It is an open question whether in this case \( R \) actually satisfies a polynomial of degree \( 2d \). The following theorem, however, is a step in the right direction.
For simplicity the theorem is stated under the assumption that \( \bar{\alpha} = \alpha \) for all \( \alpha \in F \). In the more general case the theorem is valid when \( F \) has at least \( n^2 \) elements. For if \( F_0 = \{ \alpha \in F \mid \bar{\alpha} = \alpha \} \), then \( [F : F_0] = 2 \) so that \( F_0 \) has at least \( n \) elements and the proof carries through if \( \lambda \) is restricted to \( F_0 \). Finally, note that the theorem is valid for a ring \( R \) of any characteristic.

**Theorem 2.** Let \( R \) be an associative algebra with involution \( * \) over a field \( F \), where \( F \) has at least \( n \) elements. Then if \( S \) is nil with bounded nilindex \( n \), \( R \) is nil with bounded nilindex \( \leq 2n \).

**Proof.** If \( x \in R \), \( \alpha = x + x^* \), \( \beta = -x^*x \), then by hypothesis \( (\alpha + \lambda_i \beta)^n = 0 \) for all \( \lambda_i \in F \). Then, by the standard van der Monde determinant argument the coefficients of \( \lambda_i^j \) must be zero for all \( i, j \) \([5, 11]\). On the other hand, the coefficient of \( \lambda_i^j \) is \( \hat{\omega}(n - j, j) \) for all \( i, j \). Thus \( \hat{\omega}(r, s) = 0 \) for all \( r, s \) such that \( r + s = n \). It is easy to see that this implies that \( \hat{\omega}(r, s) = 0 \) for all \( r, s \) such that \( r + s \geq n \). Thus, (3) reduces to \( x^{2n} = 0 \).

It is known that \( P(S) = S \cap P(R) \) \([3, 13]\) and that \( J(S) = S \cap J(R) \) \([9]\). Whether it is true that \( N(S) = S \cap N(R) \) is still an open question, although it is implicit in the literature that the answer is affirmative if \( R \) is an algebra over an uncountable field \([9, 10]\). In the following we look at the relationship between \( \mathcal{L}(S) \) and \( S \cap \mathcal{L}(R) \). Our result is based on a striking result recently proved by J. M. Osborn (private communication).

**Theorem (Osborn).** If \( R \) is an associative finitely generated ring with involution \( * \), then the set \( S \) of \( * \)-symmetric elements of \( R \) is a finitely generated Jordan ring.

We are now able to prove our:

**Theorem 3.** If \( R \) is an associative ring with involution \( * \), then \( \mathcal{L}(S) = S \cap \mathcal{L}(R) \).

**Proof.** It is easy to see that \( S \cap \mathcal{L}(R) \subseteq \mathcal{L}(S) \). For \( S \cap \mathcal{L}(R) \) is an ideal of \( S \) and if \( B \) is a finitely generated subring of \( S \cap \mathcal{L}(R) \) generated by \( x_1, x_2, \ldots, x_n \), we consider the subring \( B' \) of \( R \) generated by \( x_1, x_2, \ldots, x_n \). \( B' \) is \( * \)-invariant and contained in \( \mathcal{L}(R) \). Thus, \( B' \) is nilpotent. Hence, \( B \) is nilpotent. Thus, \( S \cap \mathcal{L}(R) \) is locally nilpotent and \( S \cap \mathcal{L}(R) \subseteq \mathcal{L}(S) \).

We can reduce the question of set inclusion in the other direction to the case in which \( R \) is Levitzki semisimple. For since \( \mathcal{L}(R) \) is \( * \)-invariant, \( \bar{R} = R/\mathcal{L}(R) \) inherits an involution \( \bar{*} \) from \( R \). Denote the set of \( \bar{*} \)-symmetric elements in \( \bar{R} \) by \( S \). Then, as pointed out by McCrimmon \([9]\), if \( \bar{x} \in \bar{S} \), then \( \bar{x} = \bar{y} \), where \( y \in S \). Therefore, if \( x \in \mathcal{L}(S) \) but \( x \notin \mathcal{L}(R) \), then the ideal of \( S \) generated by \( x \) is locally nilpotent. Thus, the ideal of \( S \) generated
by $\bar{x}$ is locally nilpotent. Hence, $\bar{x} \in \mathcal{L}(S)$. Thus, we would have $\mathcal{L}(\bar{R}) = 0$ but $\mathcal{L}(\bar{S}) \neq 0$. Thus, there is no loss in assuming henceforth that $\mathcal{L}(R) = 0$.

Since $\mathcal{L}(S)$ is an ideal of $S$, an application of a second result of Herstein [7] shows that if $\mathcal{L}(S) \neq 0$ there is a nonzero $^*$-invariant ideal $B$ of $R$ such that $R \cap S \subseteq \mathcal{L}(S)$. Let $S' = R \cap S$. Then $B$ is in its own right an associative ring with involution such that $S'$, the $^*$-symmetric elements of $B$, is a locally nilpotent Jordan ring. Thus, $B$ is also a nilring. For if $x \in B$ and $\alpha = x + x^*$ and $\beta = x^*x$ then the subring of $S'$ generated by $\alpha$ and $\beta$ is nilpotent. Thus, by Lemma 6, $x$ is nilpotent. We show now that $B$ is a locally nilpotent ideal of $R$. For if $K$ is a finitely generated subring of $B$, then by Osborn's theorem it follows that $K \cap S$ is a finitely generated Jordan ring. Since $K \cap S \subseteq \mathcal{L}(S)$ we see that $K \cap S$ is nilpotent. Thus by the corollary to Lemma 5, $K$ is nil of bounded index so that, by Levitzki's theorem, $K$ is nilpotent. Thus, $B$ is a locally nilpotent ideal of $R$ in contradiction to $\mathcal{L}(R) \neq 0$. It follows that $\mathcal{L}(S) \subseteq S \cap \mathcal{L}(R)$ to complete the proof.

Note added in proof. The results of Lemma 3 through Theorem 2 can be extended to an alternative ring $R$ without any characteristic assumptions.

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