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A spanning set for VOA modules

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Abstract

We develop a spanning set for weak modules of C_2 co-finite vertex operator algebras. This spanning set has certain finiteness properties that we use to show weak modules are C_n co-finite and $A_n(M)$ is finite dimensional.

1. Introduction

A vertex operator algebra, V, is C_2 co-finite if the subspace $\{u_{-2}v\colon\ u, v \in V\}$ has finite codimension. This condition, sometimes referred to as the C_2 condition or Zhu's finiteness condition [1], is important in the theory of vertex operator algebras. Zhu used it, as well as other assumptions, to show modular invariance of certain trace functions. One important feature of the C_2 condition is that it is an internal condition. Given a vertex operator algebra it is relatively easy to calculate if V is C_2 co-finite. The implications of the C_2 condition are wide ranging, however they are not completely understood. The C_2 condition implies that $A(V)$ is finite dimensional, there are a finite number of irreducible admissible modules up to equivalence [2], and irreducible admissible modules are ordinary [3]. The goal of this paper is to demonstrate new implications of the C_2 condition. Specifically, we develop new results about the modules of a vertex operator algebra that satisfies the C_2 condition. This information sheds new light on how this internal condition affects the structure of modules.

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In recent work, Gaberdiel and Neitzke [4] develop a spanning set for vertex operator algebras. They show that V is spanned by certain vectors of the form $x^1_{-n_1}x^2_{-n_2}\cdots x^k_{-n_k}\mathbf{1}$, where the modes are strictly decreasing and less than zero, i.e., $n_1 > n_2 > \cdots > n_k > 0$. The set $\{x^\alpha\}$ that generate the spanning set elements is finite if V is C_2 co-finite. The principle feature of this spanning set is this no repeat condition. Using this result, they prove C_2 co-finiteness implies C_n co-finiteness for $n \geqslant 2$, and the fusion rules for irreducible admissible modules are finite.

In this paper, we develop an analogous spanning set for modules of vertex operator algebras. We will show that under the C_2 condition, any weak module generated by w is spanned by certain elements of the form $x^1_{-n_1}x^2_{-n_2}\cdots x^k_{-n_k}w$. Here the modes will be decreasing, and each mode will be repeated at most a finite number of times. This finite repeat condition, though not as strong as a no repeat condition, still allows us to prove a number of results. With this new spanning set, we extend the results of Gabediel and Neitzke to modules. This means we will demonstrate $A_n(M)$ is finite dimensional and $C_n(M)$ is co-finite dimensional, for M finitely generated weak modules. In a future paper, we will use this module spanning set to show that rationality and C_2 co-finiteness imply regularity.

The following is a brief preview of the remaining sections of this paper. In the second section of this paper, we give the necessary definitions and notation conventions. We also present key results leading up to this paper. In the third section we develop the theory of so-called singular like vectors. With these vectors, we are able to reduce expressions with repeated modes. In the fourth section of this paper, we prove our main result which is the module spanning set with a finite repeat condition. The last section of this paper gives additional results that quickly follow from the main theorem.

2. Preliminaries

We make the assumption that the reader is somewhat familiar with the theory of vertex operator algebras (VOAs). We assume the definition of a vertex operator algebra as well as some basic properties. Good reference material is available in papers by Borcherds [5], Dong [6], and Frenkel et al. [7] and in a book by Frenkel et al. [8]. We begin with some definitions.

Definition 2.1. A vertex operator algebra, V, is of CFT type if $V = \bigoplus_{n\geqslant 0} V(n)$ and $V(0) = \mathrm{span}\{\mathbf{1}\}$.

Throughout out this paper, we assume V is of CFT type.

Definition 2.2. For V a VOA, $C_n(V) = \{v_{-n}w\colon\ v, w \in V\}$.

Definition 2.3. V, a VOA, is called C_n co-finite if $V/C_n(V)$ is finite dimensional. For $n=2$, this is often called the C_2 condition.

C_2 co-finiteness is an important assumption in Zhu's work demonstrating modularity of certain functions. This paper is about module spanning sets, and since there are a few different flavors of VOA modules, we now define weak, admissible, and ordinary modules.

Definition 2.4. A weak V module is a vector space M with a linear map $Y_M : V \to \mathrm{End}(M)[\![z, z^{-1}]\!]$ where $v \mapsto Y_M(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$, $v_n \in \mathrm{End}(M)$. In addition, Y_M satisfies the following:

(1) $v_n w = 0$ for $n \gg 0$ where $v \in V$ and $w \in M$;
(2) $Y_M(\mathbf{1}, z) = \mathrm{Id}_M$;
(3) the Jacobi identity:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_M(u,z_1)Y_M(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y_M(v,z_2)Y_M(u,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_M\big(Y(u,z_0)v, z_2\big). \tag{2.1}$$

There are two important consequences of this definition. Weak modules admit a Virasoro representation under the action of ω, the Virasoro vector. Also weak modules satisfy the $L(-1)$ derivation property. The distinctive feature of weak modules is that they have no grading is assumed. Admissible and ordinary module are both graded.

Definition 2.5. An admissible V module is a weak V module which carries a nonnegative integer grading, $M = \bigoplus_{n\geqslant 0} M(n)$, such that if $v \in V(r)$ then $v_m M(n) \subseteq M(n+r-m-1)$.

So for an admissible module, we have added a grading with a bottom level, and the action of V respects the grading.

Definition 2.6. An ordinary V module is a weak V module which carries a $\mathbb{C}$ grading, $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$, such that:

(1) $\dim(M_\lambda) < \infty$;
(2) $M_{\lambda+n=0}$ for fixed λ and $n \ll 0$;
(3) $L(0)w = \lambda w = wt(w)w$, for $w \in M$.

Although the definition of a $\mathbb{C}$ grading may seem weaker than a $\mathbb{Z}$ grading, the requirement that each graded piece of an ordinary modules must be finite dimensional is a strong condition. It turns out that any ordinary module is admissible. So we have this set of inclusions:

$$\{\text{ordinary modules}\} \subseteq \{\text{admissible modules}\} \subseteq \{\text{weak modules}\}.$$

In addition to the above definitions, we need to refer to results by Gaberdiel and Neitzke [4], and their work in determining a spanning set for vertex operator algebras. There are three pertinent results of theirs that are explained below. First we describe the generating set. Let $\{\bar{x}^\alpha\}_{\alpha\in I}$ be a basis of $V/C_2(V)$, where $\bar{x}^\alpha = x^\alpha + C_2(V)$, and x^α is a homogenous vector. So $\overline{X} = \{x^\alpha\}_{\alpha\in I}$ is a set of elements in V which are representatives of a basis for $V/C_2(V)$.

Theorem 2.7 [4]. *Let V be a vertex operator algebra, then V is spanned by elements of the form*

$$x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k} \mathbf{1}$$

where $n_1 > n_2 > \cdots > n_k > 0$ and $x^j \in \overline{X}$ for $1 \leqslant j \leqslant k$.

Note that a vector $x \in \overline{X}$ may appear any number of times in a spanning set element, but the mode x_{-n} may only appear once. For example x_{-m} and x_{-n} may both appear in a spanning set element as long as $m \neq n$. Throughout this paper will refer to the elements described in this theorem as VOA spanning set elements. This is in contrast to the module spanning set elements that are the goal of this paper. This theorem tells us the modes in this spanning set are strictly decreasing, and this is what we mean by a no repeat condition. This spanning set is especially usefully when we look at the C_n spaces. In fact the next result follows very quickly from this no repeat condition.

Theorem 2.8 [4]. *Suppose $V/C_2(V)$ is finite dimensional, then $V/C_n(V)$ is finite dimensional for $n \geqslant 2$.*

In their paper, Gaberdiel and Neitzke also formulate a spanning set for modules. Unfortunately, this formulation does not have a no repeat condition. Their repetition restriction is in terms of the weights of the modes. In their module spanning set, the weight of the modes are decreasing, but the inequality is not strict, which means that a particular mode could be repeated indefinitely. The following is the definition of the weight of the mode.

Definition 2.9. For $u \in V$, a homogeneous vector, and $n \in \mathbb{Z}$, $wt(u_n) = wt(u) - n - 1$.

Theorem 2.10 [4]. *Let M be an admissible V module. Then M is spanned by elements of the form*

$$x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k} u$$

where u is a lowest weight vector and $wt(x^1_{-n_1}) \geqslant wt(x^2_{-n_2}) \geqslant \cdots \geqslant wt(x^k_{-n_k}) > 0$ and $x^j \in \overline{X}$ for $1 \leqslant j \leqslant k$.

The goal of this paper is a module spanning set analogous to the VOA spanning set result by Gaberdiel and Neitzke under the additional condition of C_2 co-finiteness. In this new spanning set, modes are only allow to repeat an finite number of times, which is slightly weaker that the condition that modes can only be repeated once. It turns out however that this finite repeat condition is sufficient to demonstrate two nice finiteness properties for weak modules: $A_n(M)$ finite dimensionality and $C_n(M)$ co-finiteness.

In this paper the term, mode, is abused slightly. The term, mode, usually refers to an endomorphism, u_n, where $u \in V$ and $n \in \mathbb{Z}$. Sometimes, in this paper it refers to the indexing number n. For example, in the term, nonnegative mode, "mode" refers to a mode of the form u_n with $n \geqslant 0$. It should be apparent what "mode" is referring to by the context.

3. Singular like vectors

In order to limit the number of repeated modes, we need a method for shortening spanning set elements that have too many repetitions of a certain mode. To meet this end, we must develop the theory of so-called singular like vectors. The reason for this name is because the vectors described in this section are reminiscent of singular vectors in the Virasoro algebra [9].

The goal of this section is two-fold. First, we choose vectors of the form $x^1_{-1}x^2_{-1}\cdots x^k_{-1}\mathbf{1}$ in our VOA and rewrite them as a sum of spanning set elements, $x^{r_1}_{-n_{r_1}}x^{r_2}_{-n_{r_2}}\cdots x^{r_l}_{-n_{r_l}}\mathbf{1}$ where $l < k$. Second, we calculate the vertex operators of these singular like vectors. Isolating certain coefficients of these vertex operators is key to limiting the repetition of modes in the module spanning set that is the main result of this paper.

Henceforth, we are working under the assumption that V, our VOA, is C_2 co-finite. So now, $\overline{X}$ is a finite set homogenous of elements in V which are representatives of a basis for $V/C_2(V)$. We know that V is spanned by elements of the form $x^1_{-n_1}x^2_{-n_k}\cdots x^k_{-n_k}\mathbf{1}$ where $x^i \in \overline{X}$ for $1 \leqslant i \leqslant k$ and the modes are strictly decreasing.

We can simplify this $\overline{X}$ slightly. The vacuum, $\mathbf{1}$, is not an element of $C_2(V)$ so we could choose a basis $\overline{X}$ such that $\mathbf{1} \in \overline{X}$. The only mode for $\mathbf{1}$ which is nonzero is $\mathbf{1}_{-1}$ but this is the identity endomorphism. If we define $X = \overline{X} - \{\mathbf{1}\}$, the results of Gaberdiel and Neitzke still hold, but X is one element smaller. Note that this means the minimum weight of any vector in X is 1.

The first set of results in this section establish that we can rewrite a certain type of vector as a sum of VOA spanning elements of strictly shorter length. The proof involves comparing the weights of vectors. So we start the following definition.

Definition 3.1. Let $B = \max_{x \in X}\{wt(x)\}$.

This means B is the largest weight of any homogeneous vector not in $C_2(V)$.

Lemma 3.2. *Let $x^i \in X$ for $1 \leqslant i \leqslant k$, k a positive integer,*

$$wt\left(x^1_{-1}x^2_{-1}\cdots x^k_{-1}\mathbf{1}\right) \leqslant Bk. \tag{3.2}$$

Proof.

$$wt\left(x^1_{-1}x^2_{-1}\cdots x^k_{-1}\mathbf{1}\right) = \sum_{i=1}^{k} wt\left(x^i\right) \tag{3.3}$$

$$\leqslant k \max_{x \in X}\left\{wt(x)\right\} \tag{3.4}$$

$$= Bk. \qquad \square \tag{3.5}$$

Now that we have a maximum weight for a vector of the form $x^1_{-1}x^2_{-1}\cdots x^n_{-1}\mathbf{1}$, we compare its weight to a spanning set vectors weight. The next lemma will tell us the minimum weight of a VOA spanning set vector of length l.

Lemma 3.3. *Let $x^i \in X$ for $1 \leqslant i \leqslant l$, l a positive integer. If $n_1 > n_2 > \cdots > n_l > 0$ then,*

$$wt\left(x^1_{-n_1}x^2_{-n_2}\cdots x^l_{-n_l}\mathbf{1}\right) \geqslant \frac{l(l+1)}{2}. \tag{3.6}$$

Proof.

$$wt\left(x^1_{-n_1}x^2_{-n_2}\cdots x^l_{-n_l}\mathbf{1}\right) = \sum_{i=1}^{l} wt\left(x^i\right) + n_i - 1 \tag{3.7}$$

$$\geqslant \sum_{t=1}^{l} n_i \tag{3.8}$$

$$\geqslant \sum_{t=1}^{l} i \tag{3.9}$$

$$= \frac{l(l+1)}{2}. \qquad \square \tag{3.10}$$

Note here that the minimum weight of a VOA spanning set element increases quadratically as a function of length, while the maximum weight of a vector of the form

$$x^1_{-1}x^2_{-1}\cdots x^k_{-1}\mathbf{1}$$

increases linearly as a function of length. Using the previous two lemmas, we will now show that if we have a vector with a sufficient number of -1 modes, it can be rewritten as a sum of spanning set elements of strictly shorter length.

Lemma 3.4. *Let $x^i \in X$ for $1 \leqslant i \leqslant k$ and $y^j \in X$ for $1 \leqslant j \leqslant l$. If $2B-1 < k \leqslant l$ and $n_1 > n_2 > \cdots > n_l > 0$, then*

$$wt\left(x_{-1}^1 x_{-1}^2 \cdots x_{-1}^k \mathbf{1}\right) < wt\left(y_{-n_1}^1 y_{-n_2}^2 \cdots y_{-n_l}^l \mathbf{1}\right). \tag{3.11}$$

Proof. From the previous lemmas we have,

$$wt\left(x_{-1}^1 x_{-1}^2 \cdots x_{-1}^k \mathbf{1}\right) \leqslant Bk$$

and

$$wt\left(y_{-n_1}^1 y_{-n_2}^2 \cdots y_{-n_l}^l \mathbf{1}\right) \geqslant \frac{l(l+1)}{2}.$$

If $l \geqslant k$ then,

$$\frac{l(l+1)}{2} \geqslant \frac{k(k+1)}{2}. \tag{3.12}$$

Since $k > 2B-1$, then $k(k+1)/2 > Bk$. Thus,

$$wt\left(y_{-n_1}^1 y_{-n_2}^2 \cdots y_{-n_l}^l \mathbf{1}\right) > Bk,$$

and we have, finally, that

$$wt\left(x_{-1}^1 x_{-1}^2 \cdots x_{-1}^k \mathbf{1}\right) < wt\left(y_{-n_1}^1 y_{-n_2}^2 \cdots y_{-n_l}^l \mathbf{1}\right). \qquad \square$$

Before we continue, we need to recall a basic fact about C_2 co-finite vertex operator algebras.

Remark 3.5. If V is C_2 co-finite, and $V = \bigoplus_{i \geqslant 0} V_i$ is the weight space decomposition of V. Then for some $N > 0$, $\bigoplus_{i \geqslant N} V_i \subset C_2(V)$. We fix such an N.

Definition 3.6. Let $Q = \max\{N, 2B-1\} + 1$.

This Q will play an important role in the proof of the module spanning set. We start out, however, by showing that certain vectors long enough can be rewritten in terms of shorter vectors. In particular, if we have a vector composed of the product of Q, minus one modes acting on the vacuum, we can rewrite it in terms of VOA spanning set elements which only have at most $Q-1$ modes.

Proposition 3.7. *If $k \geqslant Q$,*

$$x_{-1}^1 x_{-1}^2 \cdots x_{-1}^k \mathbf{1} = \sum_{r \in R} x_{-n_{r_1}}^{r_1} x_{-n_{r_2}}^{r_2} \cdots x_{-n_{r_l}}^{r_l} \mathbf{1}$$

with $l < k$ and where $x^i \in X$ for $1 \leqslant i \leqslant k$; $n_{r_1} > n_{r_2} > \cdots > n_{r_l} > 0$ for fixed r; $x^{r_t} \in X$ for $1 \leqslant t \leqslant l$; and R a finite index set.

Proof. If $k > N$, then $x^1_{-1}x^2_{-1}\cdots x^k_{-1}\mathbf{1} \in C_2(V)$, and we can write

$$x^1_{-1}x^2_{-1}\cdots x^k_{-1}\mathbf{1} = \sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}}\mathbf{1} \tag{3.13}$$

where $n_1 \geqslant 2$ and $n_1 > n_2 > \cdots > n_l > 0$. So we have rewritten $x^1_{-1}x^2_{-1}\cdots x^k_{-1}\mathbf{1}$ as sum of spanning set elements in $C_2(V)$. Now assume to the contrary, that for all k and r, $l \geqslant k$. Now by Lemma 3.4, if $k > 2B-1$,

$$wt\bigl(x^1_{-1}x^2_{-1}\cdots x^k_{-1}\mathbf{1}\bigr) < wt\bigl(x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}}\mathbf{1}\bigr).$$

This is a contradiction since

$$wt\bigl(x^1_{-1}x^2_{-1}\cdots x^k_{-1}\mathbf{1}\bigr) = wt\bigl(x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{j_l}_{-n_{r_l}}\mathbf{1}\bigr).$$

Thus, if $k \geqslant Q$, $l < k$ for all r. □

We call vectors of the form (3.13) singular like because they are similar to singular vectors in the Virasoro algebra. The main point of this lemma is that there is a uniform length for which any product of -1 modes of that length can be rewritten as sum of strictly shorter spanning set elements.

The next step is to calculate the vertex operators of these singular like vectors. This will allow us later on to derive an identity that enables us to shorten repeated modes. In order to calculate the vertex operators we need to recall a formula.

Remark 3.8.

$$\begin{aligned} Y(u_n v, z) \;=\; & \operatorname{Res}_{z_1}\bigl\{(z_1 - z)^n Y(u, z_1)Y(v, z) \\ & \qquad - (-z + z_1)^n Y(v, z)Y(u, z_1)\bigr\}. \end{aligned} \tag{3.14}$$

By taking the residue, we obtain

$$\begin{aligned} Y(u_n v, z) \;=\; & \sum_{m\geqslant 0} \bigl(-1^m\bigr)\binom{n}{m} z^m u_{n-m} Y(v, z) \\ & - \sum_{m\geqslant 0} (-1)^{n-m}\binom{n}{m} z^{n-m} Y(v, z) u_m. \end{aligned} \tag{3.15}$$

In particular, if $n = -1$, we have

$$Y(u_{-1}v, z) = \sum_{m\geqslant 0} z^m u_{-1-m} Y(v, z) + \sum_{m\geqslant 0} z^{-1-m} Y(v, z) u_m. \tag{3.16}$$

To perform the next calculation, we will need to apply the above formula Q times to the left-hand side of our singular like vector. But before we can do the next calculation, we need to describe a certain indexing set. This set, in its ordering, will describe how operators are rearranged when we apply (3.16) multiple times.

Definition 3.9. $\Lambda_n^i = \{(k_1, k_2, \ldots, k_i)\colon \{k_1, k_2, \ldots, k_i\} \subseteq \{1, 2, \ldots, n\}, k_1 < k_2 < \cdots < k_i\}$. Now if $\lambda \in \Lambda_n^i$ then λ_j is the jth element of λ.

So what we have is an i element subset of $\{1, 2, \ldots, n\}$ where the order is specified. The elements are placed increasing order. Λ_n^i is then the set of these i element subsets.

Definition 3.10. If $\lambda \in \Lambda_n^i$ then $\bar{\lambda} = (k_{i+1}, k_{i+2}, \ldots, k_n)$ where $\{k_{i+1}, k_{i+2}, \ldots, k_n\}$ is the complement of $\{k_1, k_2, \ldots, k_i\}$ in $\{1, 2, \ldots, n\}$ and $k_{i+1} > k_{i+2} > \cdots > k_n$. Also $\bar{\lambda}_j$ is the jth element of $\bar{\lambda}$.

So in the complement to λ, the order is reversed. The elements are in decreasing order. For example, if $n = 8$, Λ_8^i is the set of i element subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. If $i = 3$, let $\lambda = \{2, 5, 8\}$, then $\bar{\lambda} = \{7, 6, 4, 3, 1\}$. If $i = 4$ and $\lambda = \{3, 4, 6, 7\}$, then $\bar{\lambda} = \{8, 5, 2, 1\}$.

Proposition 3.11. *Let $x^1, \ldots, x^n, v \in V$, then*

$$\begin{aligned} &Y\bigl(x_{-1}^1 x_{-1}^2 \cdots x_{-1}^n v, z\bigr) \\ &\quad= \sum_{i=0}^{n} \sum_{\lambda \in \Lambda_n^i} \sum_{m_i \geqslant 0} \left(\prod_{j=1}^{i} x_{-1-m_{\lambda_j}}^{\lambda_j} z^{m_{\lambda_j}} \right) Y(v, z) \left(\prod_{j=i+1}^{n} x_{m_{\bar{\lambda}_j}}^{\bar{\lambda}_j} z^{-1-m_{\bar{\lambda}_j}} \right). \end{aligned} \tag{3.17}$$

Proof. We proceed by induction. For the case where $n = 1$, we use (3.16). When $n = 1$ we have $\Lambda_1^0 = \{\emptyset\}$ and $\Lambda_1^1 = \{\{1\}\}$. So we have

$$Y(x_{-1} v, z) = \sum_{m \geqslant 0} z^m x_{-1-m} Y(v, z) + \sum_{m \geqslant 0} z^{-1-m} Y(v, z) x_m \tag{3.18}$$

$$= \sum_{i=0}^{1} \sum_{m \geqslant 0} \left(\prod_{j=1}^{i} x_{-1-m} z^m \right) \bigl(Y(v, z)\bigr) \left(\prod_{j=i+1}^{1} x_m z^{-1-m} \right). \tag{3.19}$$

Now assume true for $n - 1$, then

$$Y\left(x^1_{-1}x^2_{-1}\cdots x^n_{-1}v, z\right) \tag{3.20}$$

$$= \sum_{i=0}^{n-1} \sum_{\lambda\in\Lambda^i_{n-1}} \sum_{m_i\geqslant 0} \left(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}} z^{m_{\lambda_j}}\right)\left(Y\left(x^n_{-1}v, z\right)\right)$$

$$\times \left(\prod_{j=i+1}^{n-1} x^{\bar\lambda_j}_{m_{\lambda_j}} z^{-1-m_{\bar\lambda_j}}\right) \tag{3.21}$$

$$= \sum_{i=0}^{n-1} \sum_{\lambda\in\Lambda^i_{n-1}} \sum_{m_i\geqslant 0} \left(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}} z^{m_{\lambda_j}}\right) \tag{3.22}$$

$$\times \left(\sum_{m_n\geqslant 0} z^{m_n} x_{-1-m_n} Y(v, z) + \sum_{m_n\geqslant 0} z^{-1-m_n} Y(v, z) x_{m_n}\right) \tag{3.23}$$

$$\times \left(\prod_{j=i+1}^{n-1} x^{\bar\lambda_j}_{m_{\bar\lambda_j}} z^{-1-m_{\bar\lambda_j}}\right) \tag{3.24}$$

$$= \sum_{i=0}^{n} \sum_{\lambda\in\Lambda^i_{n}} \sum_{m_i\geqslant 0} \left(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}} z^{m_{\lambda_j}}\right) Y(v, z) \left(\prod_{j=i+1}^{n} x^{\bar\lambda_j}_{m_{\bar\lambda_j}} z^{-1-m_{\bar\lambda_j}}\right).$$

$$\square \tag{3.25}$$

So, in particular when $v = \mathbf{1}$, we have

$$Y\left(x^1_{-1}x^2_{-1}\cdots x^n_{-1}\mathbf{1}, z\right)$$

$$= \sum_{i=0}^{n} \sum_{\lambda\in\Lambda^i_{n}} \sum_{m_i\geqslant 0} \left(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}} z^{m_{\lambda_j}}\right)\left(\prod_{j=i+1}^{n} x^{\bar\lambda_j}_{m_{\bar\lambda_j}} z^{-1-m_{\bar\lambda_j}}\right). \tag{3.26}$$

Now we consider $x^1, \ldots, x^Q \in X$. By Proposition 3.7,

$$x^1_{-1}x^2_{-1}\cdots x^Q_{-1}\mathbf{1} = \sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}$$

where $l < Q$; $n_{r_1} > n_{r_2} > \cdots > n_{r_l} > 0$ for fixed r; $x^{r_t} \in X$ for $1 \geqslant t \geqslant l$; and R a finite index set. Substituting $\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}$ for $x^1_{-1}x^2_{-1}\cdots x^Q_{-1}\mathbf{1}$ in (3.26), we get:

$$Y\left(\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}, z\right)$$

$$= \sum_{i=0}^{Q} \sum_{\lambda\in\Lambda^i_{Q}} \sum_{m_i\geqslant 0} \left(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}} z^{m_{\lambda_j}}\right)\left(\prod_{j=i+1}^{Q} x^{\bar\lambda_j}_{m_{\bar\lambda_j}} z^{-1-m_{\bar\lambda_j}}\right) \tag{3.27}$$

where R is a finite index set, $x^1, \dots, x^Q, x^{r_1}, \dots, x^{r_l} \in X$, $n_{r_1} > n_{r_2} > \cdots > n_{r_l} > 0$ for fixed r, and $l < Q$.

4. A module spanning set

To reiterate our setting $V = (V, Y, \mathbf{1}, \omega)$ is a C_2 co-finite vertex operator algebra of CFT type. In this section we prove the main theorem of this paper. This theorem will give a spanning set for weak V modules using the set X. This spanning set will have mode repetition restrictions similar to those given by Gaberdiel and Neitzke for vertex operator algebras. We begin by formulating a few lemmas, which give us identities that we use to impose mode repetition restrictions on the module spanning set elements.

Remark 4.1. Borcherds's Identity [5]: Let $u, v \in V$ and $k, q, r \in \mathbb{Z}$

$$\sum_{i \geqslant 0} \binom{-k}{i} (u_{-r+i} v)_{-k-q-i} \tag{4.28}$$

$$= \sum_{i \geqslant 0} (-1)^i \binom{-r}{i} \left\{ u_{-k-r-i} v_{-q+i} - (-1)^{-r} v_{-q-r-i} u_{-k+i} \right\}. \tag{4.29}$$

This is the component form of the Jacobi identity.

We first recall two formulas resulting from Borcherds's Identity.

Lemma 4.2. *Let $u, v \in V$ and $k, q \in \mathbb{Z}$, then*

$$[u_{-k}, v_{-q}] = \sum_{i \geqslant 0} \binom{-k}{i} (u_i v)_{-k-q-i}.$$

Proof. In Borcherds's Identity, let $r = 0$. □

Lemma 4.3. *Let $u, v \in V$ and $r, q \in \mathbb{Z}$, then*

$$(u_{-r} v)_{-q} = \sum_{i \geqslant 0} (-1)^i \binom{-r}{i} u_{-r-i} v_{-q+i} - \sum_{i \geqslant 0} (-1)^{i-r} \binom{-r}{i} v_{-r-q-i} u_i.$$

Proof. In Borcherds's Identity, let $k = 0$. □

It is worth noting that in the previous two lemmas the weight of operators on either sides of the equations are equal. This can be verified by simple calculations. The third formula is a special case of the previous lemma.

Lemma 4.4. *Let $u, v \in V$ and $n \in \mathbb{Z}$, then*

$$u_{-n}v_{-n} = (u_{-1}v)_{-2n+1} - \sum_{i<0} u_{-1-i}v_{-2n+1+i} + \sum_{i\geqslant 0} v_{-2n-i}u_i.$$

Proof. This follows from the previous lemma with $q = 2n - l$ and $r = 1$. □

This third lemma will be the identity we use to limit repeated negative modes.

Definition 4.5. Let W be a weak V module, and given $w \in W$. Define L to be the smallest non-negative integer such that $x_m\omega = 0$ for all $x \in X$ and $m \geqslant L$. Note that L depends on ω.

When we are examining expressions involving products of modes, we do not need to look at modes L or larger, because they will kill the given w.

In addition to Lemma 4.4, we need another lemma that allow us to impose repetition restrictions on nonnegative modes. To obtain this new formula, we take a reside of equation in Proposition 3.11. The goal is to isolate certain coefficients of z where we have repeated positive modes. We will eventually find an expression for endomorphisms of the form $x_n^1 \cdots x_n^Q$ where $0 \leqslant n \leqslant L$.

Lemma 4.6. *Let $x_{-1}^1 \cdots x_{-1}^Q \mathbf{1} = \sum_{r\in R} x_{-n_{r_1}}^{r_1} x_{-n_{r_2}}^{r_2} \cdots x_{-n_{r_l}}^{r_l} \mathbf{1}$ where $x^i, x^{r_t} \in X$ for $1 \leqslant i \leqslant Q$ and $1 \leqslant t \leqslant l$; $l < Q$; and $n_{r_1} > n_{r_2} > \cdots > n_{r_l} > 0$ for fixed r. Then for $0 \leqslant k \leqslant L$,*

$$z^{Q(-1+k-L)} x_{L-k}^Q x_{L-k}^{Q-1} \cdots x_{L-k}^1 \tag{4.30}$$

$$= Y\left(\sum_{r\in R} x_{-n_{r_1}}^{r_1} x_{-n_{r_2}}^{r_2} \cdots x_{-n_{r_l}}^{r_l} \mathbf{1}, z\right) \tag{4.31}$$

$$- \sum_{i=1}^{Q} \sum_{\lambda\in\Lambda_Q^i} \sum_{m_i\geqslant 0} \left(\prod_{j=1}^{i} x_{-1-m_{\lambda_j}}^{\lambda_j} z^{m_{\lambda_j}}\right)\left(\prod_{j=i+1}^{Q} x_{m_{\bar{\lambda}_j}}^{\bar{\lambda}_j} z^{-1-m_{\bar{\lambda}_j}}\right) \tag{4.32}$$

$$- \sum_{m_j\geqslant 0,\, 1\leqslant j\leqslant Q} z^{-Q-\sum_{j=1}^{Q} m_j} \left(x_{m_Q}^Q \cdots x_{m_1}^1\right) \tag{4.33}$$

where, in (4.33), *for at least one j, $m_j \neq L - k$.*

Proof. From (3.27), we have

$$Y\left(\sum_{r\in R} x_{-n_{r_1}}^{r_1} x_{-n_{r_2}}^{r_2} \cdots x_{-n_{r_l}}^{r_l} \mathbf{1}, z\right)$$
$$= \sum_{i=0}^{Q} \sum_{\lambda\in\Lambda_Q^i} \sum_{m_i\geqslant 0} \left(\prod_{j=1}^{i} -x_{-1-m_{\lambda_j}}^{\lambda_j} z^{m_{\lambda_j}}\right)\left(\prod_{j=i+1}^{Q} x_{m_{\bar{\lambda}_j}}^{\bar{\lambda}_j} z^{-1-m_{\bar{\lambda}_j}}\right). \tag{4.34}$$

From this we will isolate the mode of the form $x^Q_{L-k}\cdots x^1_{L-k}$ where L is defined as above, and $0 \geqslant k > L$. That is, x_{L-k} is a nonnegative mode.

First observe that all nonnegative modes occur when $i = 0$. If $i \neq 0$ we will have at least one mode of the form $x^{\lambda_j}_{-1-m_{\lambda_j}}$. Now we examine the summand on the right-hand side of (4.34) when $i = 0$. Note that when $i = 0$, there is only one possible λ, and it is the empty set. So $\bar{\lambda} = \{Q, Q-1, \ldots, 1\}$. An expression with only positive modes has the form

$$\sum_{m_j \geqslant 0,\, 1 \leqslant j \leqslant Q} z^{-Q-\sum_{j=1}^{Q} m_{\lambda_j}} x^Q_{m_Q} x^{Q-1}_{m_{Q-1}} \cdots x^1_{m_1}.$$

Now we are able to isolate $x^Q_{L-k}\cdots x^1_{L-k} z^{Q(-1+k-L)}$, using (4.34),

$$z^{Q(-1+k-L)} x^Q_{L-k} x^{Q-1}_{L-k} \cdots x^1_{L-k} \tag{4.35}$$

$$= Y\left(\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}, z\right) \tag{4.36}$$

$$- \sum_{i=1}^{Q} \sum_{\lambda \in \Lambda^i_Q} \sum_{m_i \geqslant 0} \left(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}} z^{m_{\lambda_j}}\right)\left(\prod_{j=i+1}^{Q} x^{\bar{\lambda}_j}_{m_{\bar{\lambda}_j}} z^{-1-m_{\bar{\lambda}_j}}\right) \tag{4.37}$$

$$- \sum_{m_j \geqslant 0, 1 \leqslant j \leqslant Q} z^{-Q-\sum_{j=1}^{Q} m_j} \left(x^Q_{m_Q} \cdots x^1_{m_1}\right) \tag{4.38}$$

where in (4.38) for at least one j, $m_j \neq L-k$. This means we have isolated the only term for which all modes are $L-k$. In (4.41), there may be several modes equal to $L-k$, but at least one of the modes is not $L-k$. □

The next step is to take a residue of this result to isolate $x^Q_{L-k} x^{Q-1}_{L-k} \cdots x^1_{L-k}$. This will be our new identity that we will use to impose repetition restrictions on nonnegative modes.

Lemma 4.7. *Let* $x^1_{-1}\cdots x^Q_{-1}\mathbf{1} = \sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}$ *where* $x^i, x^{r_t} \in X$ *for* $1 \leqslant i \leqslant Q$ *and* $1 \leqslant t \leqslant l$, $l < Q$, *and* $n_{r_1} > n_{r_2} > \cdots > n_{r_l} > 0$ *for fixed* r. *Then*

$$x^Q_{L-k} x^{Q-1}_{L-k} \cdots x^1_{L-k} \tag{4.39}$$

$$= \operatorname{Res}_z \left\{ Y\left(\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}, z\right) z^{Q(-L-1+k)-1} \right\} \tag{4.40}$$

$$- \sum_{i=1}^{Q} \sum_{\lambda \in \Lambda^i_Q} \sum_{m_i \geqslant 0} \left(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}}\right)\left(\prod_{j=i+1}^{Q} x^{\bar{\lambda}_j}_{m_{\bar{\lambda}_j}}\right) \tag{4.41}$$

$$-\sum_{m_j\geqslant 0,\,1\leqslant j\leqslant Q} x^Q_{m_Q} x^{Q-1}_{m_{Q-1}} \cdots x^1_{m_1}. \tag{4.42}$$

Where in (4.41), $\sum_{j=1}^{i}(-1-m_{\lambda_j})+\sum_{j=i+1}^{Q}(m_{\bar{\lambda}_j})=Q(L-k)$, *and in* (4.42), $\sum_{j=1}^{Q} m_j = Q(L-k)$ *and* $m_j \neq L-k$ *for some* j.

When we take the residue, we impose some restrictions on the modes on the right-hand side of this identity. These restrictions are that the sum of the modes on the left-hand side of the equation equals the sum of the modes on the left-hand side of the equation. Since the vectors x are the same on both sides, this means the weight of the operators on both sides of the equations are equal.

Proof. We multiply the equation in the statement of Lemma 4.3 by $z^{Q(1-k+L)-1}$ and take the residue to obtain the following:

$$\operatorname{Res}_z\left\{z^{-1} x^Q_{L-k} x^{Q-1}_{L-k} \cdots x^1_{L-k}\right\} \tag{4.43}$$

$$= \operatorname{Res}_z\left\{ z^{Q(1-k+L)-1} Y\left(\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}, z\right)\right. \tag{4.44}$$

$$- z^{Q(1-k+L)-1} \sum_{i=1}^{Q} \sum_{\lambda\in\Lambda^i_Q} \sum_{m_i\geqslant 0} \left(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}} z^{m_{\lambda_j}}\right)$$

$$\times \left(\prod_{j=i+1}^{Q} x^{\bar{\lambda}_j}_{m_{\bar{\lambda}_j}} z^{-1-m_{\bar{\lambda}_j}}\right) \tag{4.45}$$

$$\left. - z^{Q(1-k+L)-1} \sum_{m_j\geqslant 0,\,1\leqslant j\leqslant Q} z^{-Q-\sum_{j=1}^{Q} m_j}\left(x^Q_{m_Q} \cdots x^1_{m_1}\right)\right\}. \tag{4.46}$$

Where in (4.46), for at least one j, $m_j \neq L-k$. Evaluating the residue we have,

$$x^Q_{L-k} x^{Q-1}_{L-k} \cdots x^1_{L-k} \tag{4.47}$$

$$= \operatorname{Res}_z\left\{ z^{Q(1-k+L)-1} Y\left(\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}, z\right)\right\} \tag{4.48}$$

$$- \sum_{i=1}^{Q} \sum_{\lambda\in\Lambda^i_Q} \sum_{m_i\geqslant 0} \left(\prod_{j=1}^{i} x^{\lambda_j}_{-1-m_{\lambda_j}}\right)\left(\prod_{j=i+1}^{Q} x^{\bar{\lambda}_j}_{m_{\bar{\lambda}_j}}\right) \tag{4.49}$$

$$- \sum_{m_j\geqslant 0,\,1\leqslant j\leqslant Q} \left(x^Q_{m_Q} \cdots x^1_{m_1}\right). \tag{4.50}$$

Where in (4.50), for at least one j, $m_j \neq L - k$. By taking the residue we impose the restriction $Q(1-k+L) - 1 - Q - \sum_{j=1}^{Q} m_j = -1$. So, we have the condition:

$$\sum_{j=1}^{Q} m_j = Q(L-k).$$

In addition in (4.49), since

$$Q(1-k+L) - 1 + \sum_{j=1}^{i} m_{\lambda_j} + \sum_{j=i+1}^{Q} (-m_{\bar{\lambda}_j} - 1) = -1,$$

we have the condition:

$$\sum_{j=1}^{i} (-1 - m_{\lambda_j}) + \sum_{j=i+1}^{Q} (m_{\bar{\lambda}_j}) = Q(L-k)$$

for $i \leqslant i \leqslant Q$. □

Now we have the two key lemmas. We will use Lemma 4.4 to limit the repetitions of negative modes and Lemma 4.7 to limit the repetitions of nonnegative modes.

Theorem 1. *Let V be a C_2 co-finite vertex operator algebra, and let W be an weak V module generated by $w \in W$. Then W is spanned by elements of the form*

$$x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k} w$$

where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k > -L$ and $x^i \in X$ for $1 \leqslant i \leqslant k$. In addition, if $n_j > 0$, then $n_j > n_{j'}$ for $j < j'$, and if $n_j \leqslant 0$ then $n_j = n_{j'}$ for at most $Q-1$ indices, j'.

These conditions say the following: For an element of the spanning set, all the modes are decreasing and strictly less than L. If the modes are negative then they are strictly decreasing. If the modes are nonnegative then they are not strictly decreasing. There may be repeats of nonnegative modes, but there are at most $Q-1$ repetitions. Here is a sort of picture of the mode restrictions:

strictly decreasing $< 0 \leqslant Q-1$ repetitions $< L$.

The statement of the theorem might be easier to understand in a simpler form.

Let W be an weak V module generated by $w \in W$. Then W is spanned by elements of the form

$$x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k} w$$

where the modes appear in decreasing order, are less than L, only finitely many of the same modes may appear in the expression of each spanning set, and $x^i \in X$ for $1 \leqslant i \leqslant k$.

This statement does not provide all the details, but it does give a clearer picture.

Proof. There are three main parts to this proof. In the first part, we define a filtration for the module and determine the properties of the filtration. The second part is the meat of proof, in which we use an induction argument and two of the lemmas to impose repetition restrictions on our spanning set elements. In the third part of the proof, we demonstrate our spanning set elements indeed span M, and they obey the stated repetition restrictions.

Part I: *Filtration and properties.*

First we define a filtration on W as follows:

$$W^{(0)} \subset W^{(1)} \subset \cdots \subset W^{(t)} \subset \cdots \subset W.$$

Here $W^{(t)} = \operatorname{span}\{u^1_{-n_1} u^2_{-n_2} \cdots u^s_{-n_s} w\}$ where $\sum_{i=1}^s wt(u^i) \leqslant t$, u^i a homogenous element of V. Now $W = \bigcup_t W^{(t)}$ since W is generated by w.

Let $u = u^1_{-n_1} \cdots u^s_{-n_s} w \in W^{(t)}$. Consider what happens where two modes are transposed:

$$\begin{aligned} &u^1_{-n_1} \cdots u^{i+1}_{-n_{i+1}} u^i_{-n_i} \cdots u^s_{-n_s} w \\ &\quad = u^1_{-n_1} \cdots u^s_{-n_s} w - u^1_{-n_1} \cdots \left[u^i_{-n_i}, u^{i+1}_{-n_{i+1}}\right] \cdots u^s_{-n_s} w. \end{aligned}$$

Now $u^1_{-n_1} \cdots [u^i_{-n_i}, u^{i+1}_{-n_{i+1}}] \cdots u^s_{-n_s} w \in W^{(t-1)}$, because by Lemma 4.2

$$\left[u^i_{-n_i}, u^{i+1}_{-n_{i+1}}\right] = \sum_{j \geqslant 0} \binom{wt(u^i)}{j} \left(u^i_j u^{i+1}\right)_{-n_i - n_{i+1} - j}$$

and $wt(u^i_j u^{i+1}) = wt(u^i) + wt(u^{i+1}) - j - 1 < wt(u^i) + wt(u^{i+1})$. So,

$$u^1_{-n_1} \cdots u^{i+1}_{-n_{i+1}} u^i_{-n_i} \cdots u^s_{-n_s} w = u^1_{-n_1} \cdots u^s_{-n_s} w + v$$

where $v \in W^{(t-1)}$. This allows us to reorder modes of a vector in $W^{(t)}$ without changing the vector modulo $W^{(t-1)}$.

Using the same u as above, where $u \in W^{(t)}$. Now consider what happens when we replace u^i by its representative modulo $C_2(V)$. Let $u^i = x^i + c$ where $x^i \in X$ and $c \in V/C_2(V)$. Without loss of generality we can assume $c = v_{-2} y$.

$$\begin{aligned} &u^1_{-n_1} \cdots u^i_{-n_i} \cdots u^s_{-n_s} w \\ &\quad = u^1_{-n_1} \cdots \left(x^i + c\right)_{-n_i} \cdots u^s_{-n_s} w \\ &\quad = u^1_{-n_1} \cdots x^i_{-n_i} \cdots u^s_{-n_s} w + u^1_{-n_1} \cdots c_{-n_i} \cdots u^s_{-n_s} w \\ &\quad = u^1_{-n_1} \cdots x^i_{-n_i} \cdots u^s_{-n_s} w + u^1_{-n_1} \cdots (v_{-2} y)_{-n_i} \cdots u^s_{-n_s} w. \end{aligned}$$

Now $wt(v_{-2}y) = wt(v) + wt(y) + 1$, but by Lemma 4.3, we can replace $(v_{-2}y)_{-n_i}$ by

$$\sum_{k\geqslant 0}(-1)^k\binom{-2}{k}v_{-2-k}y_{-n_i+k} - \sum_{i\geqslant 0}(-1)^{k-2}\binom{-2}{k}y_{-2-n_i-k}v_k$$

where this only contributes $wt(y) + wt(v)$ to the filtration level, which is less than the $wt(v) + wt(y) + 1$ that $v_{-2}y$ contributes. Thus we have

$$u^1_{-n_1}\cdots u^i_{-n_i}\cdots u^s_{-n_s}w = u^1_{-n_1}\cdots x^j_{-n_i}\cdots u^s_{-n_s}w + b$$

where $b \in W^{(t-1)}$. This means that we can replace u^i by its representative modulo $C_2(V)$ without changing the original vector u modulo $W^{(t-1)}$.

The upshot of these observations is that we can reorder modes and replace vectors modulo $C_2(V)$ with out increasing the filtration of the vector. In fact, under reordering and replacement the vector stays the same modulo a lower filtration level.

Part II: *Induction.*

To proceed with the proof, we now use induction on the pairs (t, K) with the ordering $(t, K) > (t', K')$ if $t > t'$ or $t = t'$ and $K > K'$. The inductive hypothesis is:

$$W^{(t)} = \operatorname{span}\left\{x^1_{-n_1}x^2_{-n_2}\cdots x^n_{-n_k}w\colon\ x^i \in X \text{ for } 1 \leqslant i \leqslant Q\right\}$$

where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k > -L$ and, $\sum_i wt(x^i) \leqslant t$. In addition, if $K \geqslant n_j > 0$ then $n_j > n_{j+1}$. Also, for $n_j \leqslant 0$ and $n_j < K$ then $n_j = n_{j'}$ for at most $Q - 1$ indices, j'. Basically the induction hypothesis is saying that there are repetition restrictions on modes greater than $-K$, but no restrictions on modes less than $-K$. The restrictions on modes greater than $-K$ are the following: Negative modes greater than $-K$ are strictly increasing, and positive modes greater than $-K$ can have at most $Q - 1$ repetitions. This picture gives an idea of what is going on with the modes in the induction hypothesis:

$$\text{decreasing} \leqslant -K < \text{strictly decreasing} < 0 \leqslant \text{at most } Q-1 \text{ repetitions} < L.$$

We proceed in the following manner: We start with the base case $(0, -L)$, then show the induction hypothesis holds for all pairs $(t, -L)$, and then finally move on to the general case, (t, K).

Our base case is $(0, -L)$, that is there are no repetition restrictions for any modes. Let $u = u^1_{-n_1}\cdots u^s_{-n_s}w \in W^{(0)}$. Since V is CFT, $u^i = c_i\mathbf{1}$ where $c_i \in \mathbb{C}$. If $-n_i \neq -1$ then $u = 0$, because $\mathbf{1}_p = \mathrm{Id}_V$ if and only if $p = -1$. If $u \neq 0$ then $u = c(\mathbf{1}_{-1})^s w = cw$, $c \in \mathbb{C}$. And w is in our set of spanning elements. So this case is finished.

Now consider the pairs $(t, -L)$. This our intermediate step in the induction proof. This step will show that we can reorder and replace modes so that we can rewrite any vector as the sum of elements of the form $x^1_{-n_1}x^2_{-n_2}\cdots x^l_{-n_l}w$ where the modes are decreasing. Let $u = u^1_{-n_1} \cdots u^k_{-n_k} w \in W^{(t)}$. By using the results from Lemmas 4.2 and 4.3, we can reorder the modes and replace u^i by x^i, its representative in X, to obtain

$$u = x^1_{-1} \cdots x^k_{-n_k} w + v$$

where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$ and $v \in W^{(t-1)}$. Now if $-n_s \geqslant L$, then $x^k_{-n_k} w = 0$. This means we are left with v which is in $W^{(t-1)}$, and by the induction hypothesis for $(t-1, -L)$, $v = \sum_{s \in S} x^{s_1}_{-n_{s_1}} x^{s_2}_{-n_{s_2}} \cdots x^{s_q}_{-n_{s_q}} w$ where $n_{s_1} \geqslant n_{s_2} \geqslant \cdots \geqslant n_{s_q} > -L$, and S is a finite index set. So if $-n_s \geqslant L$, we have $u = v$, and v is a spanning set element for the induction hypothesis $(t-1, -L)$.

If $-n_k < L$,

$$u = x^1_{-n_1} \cdots x^k_{-n_k} w + v$$

where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k > -L$ and $v \in W^{(t-1)}$. Again using the induction hypothesis for $(t-1, -L)$, $v = \sum_{s \in S} x^{s_1}_{-n_{s_1}} x^{s_2}_{-n_{s_2}} \cdots x^{s_q}_{-n_{s_q}} w$ where $n_{s_1} \geqslant n_{s_2} \geqslant \cdots \geqslant n_{s_q} > -L$, and S is a finite index set. Then

$$u = x^1_{-n_1} \cdots x^k_{-n_k} w + \sum_{s \in S} x^{s_1}_{-n_{s_1}} x^{s_2}_{-n_{s_2}} \cdots x^{s_q}_{-n_{s_q}} w$$

where all the modes are decreasing. This completes our intermediate step to show that the induction hypothesis holds for all pairs $(t, -L)$.

This intermediate step ensures our spanning set has decreasing modes. The next step is to show that the modes greater than $-K$ can only repeat an finite number of times. It is now that we move to the general case, where we try to show that the induction hypothesis hold for all pairs (t, K).

We begin by assuming that the inductive hypothesis holds for all pairs strictly less than (t, K). Let $u = u^1_{-n_1} \cdots u^k_{-n_k} w \in W^{(t)}$. There are two cases; $K \leqslant 0$ and $K > 0$. For the case where $K > 0$, this corresponds to placing repetition restrictions on negative modes, and we use Lemma 4.4 to do this. When $K \leqslant 0$, we use Lemma 4.7 to impose repetition restriction on nonnegative modes.

Case 1. $K \leqslant 0$. In this case we are working on modes between 0 and L, trying to impose repetition restrictions. Again, let $u = u^1_{-n_1} \cdots u^k_{-n_k} w \in W^{(t)}$. By rearranging and replacing terms by Lemmas 4.2 and 4.3, and by applying the inductive hypothesis for $(t, K-1)$, we get u is the sum of vectors of the form

$$\dot{x}^1_{-n_1} \cdots \dot{x}^m_{-n_m} x^1_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w + v \tag{4.51}$$

where $\dot{x}, x, y \in X$ for all indices; $n_1 \geqslant \cdots \geqslant n_m > K > l_1 \geqslant \cdots \geqslant l_s > -L$; $l_i = l_{i'}$ for at most $Q-1$ indices; and $v \in W^{t-1}$. If $p < Q$, that is we have less that Q modes that are $-K$, then

$$\dot{x}^1_{-n_1} \cdots \dot{x}^m_{-n_m} x^1_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w$$

is in the proper form. That is, it is a spanning set element for $W^{(t)}$, as given in the induction hypothesis. We finish this subcase off by applying the induction hypothesis for $(t-1, K)$ to v to obtain:

$$v = \sum_{s \in S} x^{s_1}_{-n_{s_1}} x^{s_2}_{-n_{s_2}} \cdots x^{s_q}_{-n_{s_q}} w$$

where $n_{s_1} \geqslant n_{s_2} \geqslant \cdots \geqslant n_{s_q} > -L$; $n_{s_j} = n_{s_{j+1}}$ is only allowed for $n_{s_j} > K$ and $n_{s_j} > 0$; for $n_{s_j} \leqslant 0$ and $n_{s_j} < -K$ then $n_{s_j} = n_{s_{j'}}$ for at most $Q-1$ indices, j'; and S is a finite index set. Thus

$$u = \dot{x}^1_{-n_1} \cdots \dot{x}^m_{-n_m} x^1_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w + \sum_{s \in S} x^{s_1}_{-n_{s_1}} x^{s_2}_{-n_{s_2}} \cdots x^{s_q}_{-n_{s_q}} w$$

and u is the sum of spanning set elements with mode restrictions fitting the induction hypothesis for (t, K).

Note that for the rest of the cases we deal with in this proof, we can place v in the desired form using the same method as above. That is we can apply the induction hypothesis for $(t-1, K)$ to v to place it in the desired form. This is when v is a vector with filtration level $t-1$ that comes from reordering and replacing the modes of a vector with filtration level t. Because of this fact, in the remaining part of the proof we will not worry about this v that comes from reordering and replacing.

If $p \geqslant Q$ and $m \neq 0$, we have more than Q modes that are $-K$ in our expression. We need to find a way to reduce the number of repetitions of the mode $-K$ to $Q-1$. Since $\sum_{i=1}^m wt(\dot{x}^i) > 0$, we can apply the induction hypothesis for $(t-1, K)$ to $x^1_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w$. When we do this we obtain a sum of vectors of the form

$$\dot{x}^1_{-n_1} \cdots \dot{x}^m_{-n_m} x'^1_{-K} \cdots x'^{p'}_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w$$

where $p' < Q$, and thus satisfies the proper repetition restrictions. The key here is that there are modes less than $-K$ at the front of this vector. Since each $x \in X$ has a positive weight. We can use the induction case for $(t-1, K)$ to the back part of this vector to give the modes $-K$ and higher the proper repetition restrictions. We will use this method for reduction a few more times.

If $m = 0$, then we are dealing with a vector of the form

$$x^1_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w$$

with $p \geqslant Q$. Now we apply Lemma 4.7 where $L - k = -K$ to get:

$$\left(\prod_{j=1}^{Q} x^j_{-K}\right) x^{Q+1}_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w \tag{4.52}$$

$$= \left(\mathrm{Res}_z \left\{ Y\left(\sum_{r \in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}, z\right) \cdot z^{Q(K-1)} \right\}\right) \tag{4.53}$$

$$\times x_{-K}^{Q+1} \cdots x_{-K}^{p} y_{-l_1}^{1} \cdots y_{-l_s}^{s} w \tag{4.54}$$

$$-\left(\sum_{i=1}^{Q} \sum_{\lambda \in \Lambda_Q^i} \sum_{m_i \geqslant 0} \left(\prod_{j=1}^{i} x_{-1-m_{\lambda_j}}^{\lambda_j}\right)\left(\prod_{j=i+1}^{Q} x_{m_{\bar{\lambda}_j}}^{\bar{\lambda}_j}\right)\right) \tag{4.55}$$

$$\times x_{-K}^{Q+1} \cdots x_{-K}^{p} y_{-l_1}^{1} \cdots y_{-l_s}^{s} w \tag{4.56}$$

$$-\left(\sum_{m_j \geqslant 0,\, 1 \leqslant j \leqslant Q} x_{m_Q}^{Q} \cdots x_{m_1}^{1}\right) x_{-K}^{Q+1} \cdots x_{-K}^{p} y_{-l_1}^{1} \cdots y_{-l_s}^{s} w. \tag{4.57}$$

Where $\sum_{j=1}^{i}(-m_{\lambda_j} - 1) + \sum_{j=i+1}^{Q} m_{\bar{\lambda}_j} = -QK$, in (4.55). And $m_j \neq -K$ for some j and $\sum_{j=1}^{Q} m_j = -QK$, in (4.57).

Let's look at the following term:

$$-\left(\sum_{m_j \geqslant 0,\, 1 \leqslant j \leqslant Q} x_{m_Q}^{Q} \cdots x_{m_1}^{1}\right) x_{-K}^{Q+1} \cdots x_{-K}^{p} y_{-l_1}^{1} \cdots y_{-l_s}^{s} w.$$

In this term, $\sum_{j=1}^{Q} m_j = -QK$ and $m_j \neq -K$ for some j. We need to show that in this term there is a mode less than $-K$. If there is we can reorder the modes so that there is a mode less than $-K$ at the front of the vector. As before we can finish off this case by applying the induction hypothesis on $(t-1, K)$ to the back of the vector where the modes are greater than or equal to $-K$. So $\frac{1}{Q}\sum_{j=1}^{Q}(m_j) = -K$ where one of the $m_j \neq -K$. If one of the modes is not $-K$, then one of the modes is less than $-K$ or greater than $-K$. If the mode is less than $-K$, we are done. If the mode is greater than $-K$, since the average of the modes is $-K$ then there must be another mode less than $-K$, and we can apply the induction hypothesis. So we are done with this term.

Next we examine,

$$\left(\prod_{j=1}^{i} x_{-1-m_{\lambda_j}}^{\lambda_j}\right)\left(\prod_{j=i+1}^{Q} x_{m_{\bar{\lambda}_j}}^{\bar{\lambda}_j}\right) x_{-K}^{Q+1} \cdots x_{-K}^{p} y_{-l_1}^{1} \cdots y_{-l_s}^{s} w.$$

In this term, since $i \neq 0$, there will a mode of the form $x_{-1-m_{\lambda_j}}^{\lambda_j}$. In this mode, $-1 - m_{\lambda_j}$ is negative, and thus it will be less than $-K$. Again by reordering we are in the case where there is a mode less than $-K$ at the front of the vector. So by the method described previously we are done with this term.

The final term we need to examine for this case is:

$$\left(\mathrm{Res}_z\left\{Y\left(\sum_{r \in R} x_{-n_{r_1}}^{r_1} x_{-n_{r_2}}^{r_2} \cdots x_{-n_{r_l}}^{r_l} \mathbf{1}, z\right) z^{Q(K-1)}\right\}\right)$$
$$\times x_{-K}^{Q+1} \cdots x_{-K}^{p} y_{-l_1}^{1} \cdots y_{-l_s}^{s} w.$$

In this term, we must determine what happens when we evaluate

$$\mathrm{Res}_z\left\{Y\left(\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}, z\right) z^{Q(K-1)}\right\}.$$

After evaluating the residue we will get a sum of products of r_l modes. We must examine what happens for each product of r_l modes. If the product of modes has at least one negative mode or at least one mode less than $-K$, we can place this vector in the proper form by rearranging the modes and applying the induction hypothesis for $(t-1, K)$ the back part of this vector.

What now remains are the products of modes for which each mode is greater than of equal to $-K$. But since $l < Q$,

$$\left(\mathrm{Res}_z\left\{Y\left(\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}, z\right) z^{Q(K-1)}\right\}\right) \times x^{Q+1}_{-K} \cdots x^{p}_{-K} y^{1}_{-l_1} \cdots y^{s}_{-l_s} w$$

has strictly less modes equal to $-K$, then

$$\left(\prod_{j=1}^{Q} x^{p+1-j}_{-K}\right) x^{Q+1}_{-K} \cdots x^{p}_{-K} y^{1}_{-l_1} \cdots y^{s}_{-l_s} w$$

does. So this means that we can repeat the process of applying Lemma 4.7, now to terms in

$$\left(\mathrm{Res}_z\left\{Y\left(\sum_{r\in R} x^{r_1}_{-n_{r_1}} x^{r_2}_{-n_{r_2}} \cdots x^{r_l}_{-n_{r_l}} \mathbf{1}, z\right) z^{Q(K-1)}\right\}\right) \times x^{Q+1}_{-K} \cdots x^{p}_{-K} y^{1}_{-l_1} \cdots y^{s}_{-l_s} w.$$

By repeating this process we eventually, reduce the number of modes equal to $-K$ to $Q-1$. This finishes off the case when $K \leqslant 0$.

Case 2. $K > 0$. In this case we show that the negative modes must be strictly decreasing. Start with u as above. Using the inductive hypothesis for $(t, K-1)$, we get u is the sum of vectors of the form

$$\dot{x}^{1}_{-n_1} \cdots \dot{x}^{m}_{-n_m} x^{1}_{-K} \cdots x^{p}_{-K} y^{1}_{-l_1} \cdots y^{s}_{-l_s} w$$

where $n_1 \geqslant \cdots \geqslant n_m > K > l_1 \geqslant \cdots \geqslant l_s > -L$; $l_i = l_{i+1}$ only if $l_i \leqslant 0$; and if $n_j \leqslant 0$ then $n_j = n_{j'}$ for at most $Q-1$ indices, j'. Now if $m \neq 0$, then we are in the case where the is a mode at the front of the vector that is less than $-K$. As we have done before, we can apply the induction hypothesis for $(t-1, K)$ to

$$x^{1}_{-K} \cdots x^{p}_{-K} y^{1}_{-l_1} \cdots y^{s}_{-l_s} w.$$

This will remove the repeats at $-K$, giving us a vector of the form

$$\dot{x}^{1}_{-n_1} \cdots \dot{x}^{m}_{-n_m} x^{1}_{-l_1} \cdots x^{p}_{-l_p} w$$

where $n_1 \geqslant \cdots \geqslant n_m > K \geqslant l_1 \geqslant \cdots \geqslant l_s > -L$; $l_i = l_{i+1}$ only if $l_i \leqslant 0$; and if $n_j \leqslant 0$ then $n_j = n_{j'}$ for at most $Q-1$ indices, j'. These are the mode restrictions for the induction hypothesis for (t, K), so this vector is in the required form. Again this case where there is a mode less than $-K$ at the front of the vector is an important case, and will be used again.

So now we are reduced to considering what happens if $m = 0$. That is, we are dealing with a vector of the form

$$x^1_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w$$

where $K > l_1 \geqslant \cdots \geqslant l_s > -L$ and $l_i = l_{i+1}$ only if $l_i \leqslant 0$. Now we will use Lemma 4.4:

$$x^1_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w \tag{4.58}$$

$$= \left(x^1_{-K} x^2\right)_{-2K+1} x^3_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w \tag{4.59}$$

$$- \sum_{i>0} (-1)^i \binom{-K}{i} x^1_{-1-i} x^2_{-2K+1+i} x^3_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w \tag{4.60}$$

$$+ \sum_{i\geqslant 0} (-1)^{i-K} \binom{-K}{i} x^2_{-2K-i} x^1_i x^3_{-K} \cdots x^p_{-K} y^1_{-l_1} \cdots y^s_{-l_s} w. \tag{4.61}$$

In the first term, $2K - 1 > K$ if and only if $K > 1$. We deal with the case where $K = 1$ in a moment. For now assume $K > 1$. In the second term, it is easy to check that either $-1 - i < -K$ or $-2K + 1 + i < -K$ for all $i \geqslant 0$ and $i \neq K - 1$. In the third term, $-2K + 1 + i < -K$ for all $i \geqslant 0$. So now, we have reduced this to the case where there is a mode at the front of each vector that is less than $-K$. We have shown previously that vectors of this form can be place in the required form.

Now, we deal with the case where $K = 1$. This case requires us to repeatedly apply Lemma 4.4. We are dealing with a vector of the form

$$x^1_{-1} \cdots x^p_{-1} y^1_{-l_1} \cdots y^s_{-l_s} w.$$

By applying Lemma 4.4, we obtain

$$x^1_{-1} \cdots x^p_{-1} y^1_{-l_1} \cdots y^s_{-l_s} w \tag{4.62}$$

$$= \left(x^1_{-1} x^2\right)_{-1} x^3_{-1} \cdots x^p_{-1} y^1_{-l_1} \cdots y^s_{-l_s} w \tag{4.63}$$

$$- \sum_{i>0} (-1)^i \binom{-1}{i} x^1_{-1-i} x^2_{-1+i} x^3_{-1} \cdots x^p_{-1} y^1_{-l_1} \cdots y^s_{-l_s} w \tag{4.64}$$

$$+ \sum_{i\geqslant 0} (-1)^{i-1} \binom{-1}{i} x^2_{-2-i} x^1_i x^3_{-1} \cdots x^p_{-1} y^1_{-l_1} \cdots y^s_{-l_s} w. \tag{4.65}$$

For (4.64) and (4.65), we have a mode at the front of the vector which is strictly less than -1. As before, we can apply the induction hypothesis for $(t-1,1)$ the latter $s-1$ modes to place these vectors in the required form. For (4.63), we can replace $(x^{k_1}_{-1}x^{k_2})_{-1}$ by its representative modulo $C_2(V)$. The we have an expression with on less -1 mode. By repeating the process, we eventually eliminate all repetitions of the -1 mode.

Part III: *A spanning set.*

Finally, we must show that given any element of W we can rewritten it a sum of spanning set elements, as claimed. We know that M is spanned by elements of the form

$$u^1_{-m_1}u^2_{-m_2}\cdots u^k_{-m_k}w.$$

We must show that this vector can be written as a sum of elements of the form

$$x^1_{-n_1}x^2_{-n_2}\cdots x^l_{-n_l}w$$

where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k > -L$. In addition, if $n_j > 0$, then $n_j > n_{j'}$ for $j < j'$, and if $n_j \leqslant 0$ then $n_j = n_{j'}$ for at most $Q-1$ indices, j'.

We can make the assumption that u^i is homogenous for $1 \leqslant i \leqslant k$. Any vector is a sum of homogenous vectors and $(u+v)_n = u_n + v_n$ where $u, v \in V$ and $n \in \mathbb{Z}$. Let $D = wt(u^1_{-m_1}u^2_{-m_2}\cdots u^k_{-m_k}) = \sum_{i=1}^k wt(u^i) + m_i - 1$ and let $t = \sum_{i=1}^k wt(u^i)$, the filtration level of $u^1_{-m_1}u^2_{-m_2}\cdots u^k_{-m_k}w$. Consider the induction hypothesis for the pair $(t, D+(t-1)L+1)$. By the induction hypothesis for this pair we have

$$u^1_{-m_1}u^2_{-m_2}\cdots u^k_{-m_k}w = \sum_{r\in R} x^{r_1}_{-n_{r_1}}\cdots x^{r_l}_{-n_{r_l}}w$$

where $n_{r_1} \geqslant n_{r_2} \geqslant \cdots \geqslant n_{r_l} > -L$, $\sum_i wt(x^i) \leqslant t$, and $n_{r_j} = n_{r_{j+1}}$ is only allowed for $n_{r_j} > D+(t-1)L+1$ and $n_j > 0$. Also, for $n_j \leqslant 0$ and $n_j < D+(t-1)L+1$ then $n_{r_j} = n_{r_{j'}}$ for at most $Q-1$ indices, $r_{j'}$. Note that

$$wt\big(u^1_{-m_1}u^2_{-m_2}\cdots u^k_{-m_k}\big) = wt\big(x^{i_1}_{-n_{r_1}}\cdots x^{i_l}_{-n_{r_l}}\big).$$

This is true because the identities that we use to prove the induction hypothesis all preserve the weights of operators. Now we have that

$$D = wt\big(x^{r_1}_{-n_{r_1}}\cdots x^{r_l}_{-n_{r_l}}\big) \tag{4.66}$$

$$= \sum_{i=1}^{l}\big(wt\big(x^{r_i}\big) + n_{r_i} - 1\big). \tag{4.67}$$

Now we attempt to calculate an lower bound for n_{r_1}:

$$n_{r_1} = D - wt\big(x^{r_1}\big) - 1 - \sum_{i=2}^{l}\big(wt\big(x^{r_i}\big) + n_{r_i} - 1\big) \tag{4.68}$$

$$\leqslant D - \sum_{i=2}^{l} n_{r_i} \tag{4.69}$$

$$\leqslant D + (l-1)L. \tag{4.70}$$

But $\sum_{i=1}^{l} wt(x^{r_i}) \leqslant t$ implies that $l \leqslant t$, since $wt(x^i) > 0$. So we have

$$n_{r_1} \leqslant D + (t-1)L.$$

Since we used the induction hypothesis for the pair $(t, D + (t-1)L + 1)$, we have

$$u^1_{-m_1} u^1_{-m_2} \cdots u^1_{-m_k} w = \sum_{r \in R} x^{r_1}_{-n_{r_1}} \cdots x^{r_l}_{-n_{r_l}} w$$

where $n_{r_1} \geqslant n_{r_2} \geqslant \cdots \geqslant n_{r_l} > -L$. In addition, if $n_j > 0$, then $n_{r_j} > n_{r_{j'}}$ for $j < j'$ and if $n_j \leqslant 0$ then $n_j = n_{j'}$ for at most $Q - 1$ indices, j'. This show that $u^1_{-m_1} u^1_{-m_2} \cdots u^1_{-m_k} w$ can be rewritten in terms of spanning set vectors of the desired form. □

So this theorem accomplishes the goal of imposing a finite repeat condition on the modes of the module spanning set. This tells us that limiting the number of positive modes, and thus we limit the number of modes with negative weights. The idea here is that modes with negative weights push our vector w down. Limiting the number of positive modes means we can only push w down so far. Now since there is no grading on weak modules, this is not a precise statement, but it is the picture of what is going on.

5. Additional results

In this section we look at some applications of this module spanning set. So all of these results assume the V is C_2 co-finite.

In the work of Li [10], he defined $C_n(M)$ and used it to show that the fusion rules are finite for admissible modules.

Definition 5.1. Let M be a weak V module, then define $C_n(M) = \{v_{-n}m\colon m \in M, v \in V\}$.

As with vertex operator algebras, we can look at the quotient space $M/C_n(M)$.

Definition 5.2. A weak V module, M, is called C_n co-finite if $M/C_n(M)$ is finite dimensional.

Just as C_2 co-finiteness implies C_n co-finiteness for V, the C_2 co-finiteness implies the co-finiteness of $C_n(M)$.

Corollary 5.3. *If M, a irreducible weak V module, is C_2 co-finite then M is C_n co-finite for $n \geqslant 2$.*

Proof. Using our new modules spanning set, we see that $M/C_n(M)$ is spanned by elements of the form

$$x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k} w$$

where $n > n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k > -L$. In addition, if $n_j > 0$, then $n_j > n_{j'}$ for $j < j'$, and if $n_j \leqslant 0$ then $n_j = n_{j'}$ for at most $Q - 1$ indices, j'. Since M is C_2 co-finite, and there are only finitely many vectors that we add to a spanning set of $M/C_2(M)$ to get a spanning set for $M/C_n(M)$, then M is C_n co-finite. □

In [11], they show that if V is C_2 co-finite then $A(V)$ is finite dimensional. Using the module spanning set, we can extend this result to cover $A_n(M)$. The $A(V)$ bimodule $A(W)$ appears first in [12]. We can extend the definition of $A(W)$ to $A_n(W)$, just as the definition of $A(V)$ is extended to $A_n(V)$ in [11].

Definition 5.4. Let

$$u \circ_n w = \operatorname{Res}_z Y(u,z) w \frac{(1+z)^{wt(u)+n}}{z^{2n+2}}.$$

For homogenous u, this can be rewritten as

$$u \circ_n w = \sum_{j \geqslant 0} \binom{wt(u)+n}{j} u_{j-2n-2} w$$

where $u \in V$ and $w \in M$. Then let $O_n(M) = \{u \circ_n w \colon u \in V,\ w \in M\}$. Finally we define $A_n(M) = M/O_n(M)$. We use the notation convention that $A_0(M) = A(M)$.

Corollary 5.5. *If M, an irreducible weak V module, is C_2 co-finite then $A_n(M)$ is finite dimensional.*

Proof. First if V is C_2 co-finite then M is C_{2n+2} finite. Let u be a module spanning set element, that is $u = x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k} w$. Now define N to be the smallest integer such that if $wt(x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k}) \geqslant N$ then $n_1 \geqslant 2n+2$. We will show that each element of M can be written as an element in $O_n(M)$ plus an element in a finite-dimensional space. Specifically we will show for any module spanning element u that $u = x + y$ where $y \in O_n(M)$ and $x = \sum_{r \in R} x^{r_1}_{-n_{r_1}} \cdots x^{r_l}_{-n_{r_l}} w$ where $wt(x^{r_1}_{-n_{r_1}} \cdots x^{r_l}_{-n_{r_l}}) < N$. That is, x is in a finite-dimensional subspace of M.

We prove this by induction on $r = wt(x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k}) - N$. If $r \leqslant 0$ then $u = x + y$ where $y = 0$. Now consider $u = x^1_{-n_1} x^2_{-n_2} \cdots x^k_{-n_k} w$ where

$wt(x^1_{-n_1}x^2_{-n_2}\cdots x^k_{-n_k}) - N = r > 0$. This means that $u \in C_{2n+2}(M)$. So, $n_1 \geqslant 2n+2$. We use the fact that $(L(-1)v)_{-n} = nv_{-n-1}$, we can rewrite $x^1_{-n_1}$ as $(\frac{1}{s!}L(-1)^s x^1)_{-2n-2}$ for some positive s. So now

$$
\begin{aligned}
u &= x^1_{-n_1}x^2_{-n_2}\cdots x^k_{-n_k}w && (5.71)\\
&= \left(\frac{1}{s!}L(-1)^s x^1\right)_{-2n-2} x^2_{-n_2}\cdots x^k_{-n_k}w && (5.72)\\
&= \left(\frac{1}{s!}L(-1)^s x^1\right) \circ_n \left(x^2_{-n_2}\cdots x^k_{-n_k}w\right) && (5.73)\\
&\quad - \sum_{j\geqslant 1}\binom{wt(L(-1)^s x^1)+n}{j}\left(\frac{1}{s!}L(-1)^s x^1\right)_{j-2n-2} x^2_{-n_2}\cdots x^k_{-n_k}w && (5.74)\\
&= \left(\frac{1}{s!}L(-1)^s x^1\right) \circ_n \left(x^2_{-n_2}\cdots x^k_{-n_k}w\right) && (5.75)\\
&\quad - \sum_{r\in R} x^{r_1}_{-n_{r_1}}\cdots x^{r_l}_{-n_{r_l}}w && (5.76)
\end{aligned}
$$

where $wt(x^{r_1}_{-n_{r_1}}\cdots x^{r_l}_{-n_{r_l}}) - N < r$. In the last step we rewrite

$$\sum_{j\geqslant 1}\binom{wt(L(-1)^s x^1)+n}{j}\left(\frac{1}{s!}L(-1)^s x^1\right)_{j-2n-2} x^2_{-n_2}\cdots x^k_{-n_k}w$$

in terms of the module spanning set. We know that

$$\left(\frac{1}{s!}L(-1)^s x^1\right) \circ_n \left(x^2_{-n_2}\cdots x^k_{-n_k}w\right) \in O_n(M).$$

We can apply induction argument to

$$\sum_{r\in R} x^{r_1}_{-n_{r_1}}\cdots x^{r_l}_{-n_{r_l}}w$$

so that we can place it in the form $x + y$ as described above. Now since $A_n(M) = M/O_n(M)$, $A_n(M)$ is spanned by these elements $x + O_n(M)$ where x is in a finite-dimensional subspace of M. □

Corollary 5.6. *Let $W = \bigoplus_{n\geqslant 0} W(n)$ be an admissible V module generated by a finite dimensional $W(i)$, then W is an ordinary V module.*

Proof. To prove this corollary, it is sufficient to show that each graded piece of W is finite dimensional. Let $\{w^j\}$ with $1 \leqslant j \leqslant s$ be a basis for $W(i)$. For each w^j we can look at the module spanning set associated to w^j. Since $W(i)$ generates W, these spanning sets combined will form a spanning set for W. Since W is admissible, if $wt(x^1_{-n_1}x^2_{-n_2}\cdots x^k_{-n_k}) = N$ then $x^1_{-n_1}x^2_{-n_2}\cdots x^k_{-n_k}w^j \in W(i+N)$. This means that $W(n)$ will be spanned by module spanning set elements of the form

$$x^1_{-n_1}x^2_{-n_2}\cdots x^k_{-n_k}w^j$$

where $wt(x^1_{-n_1}x^2_{-n_2}\cdots x^k_{-n_k}) = n - i$. Because of the mode restrictions, this spanning set will be finite. □

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