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Multigraph decomposition into stars and into multistars

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Abstract

We study the decomposition of multigraphs with a constant edge multiplicity into copies of a fixed star $H = K_{1,t}$: We present necessary and sufficient conditions for such a decomposition to exist where $t = 2$ and prove NP-completeness of the corresponding decision problem for any $t \geqslant 3$. We also prove NP-completeness when the edge multiplicity function is not restricted either on the input G or on the fixed multistar H.

Keywords: Decomposition; Multigraphs; NPC; Stars

1. Introduction

Given two graphs H and G, an H-decomposition of G is a partition of the edge set of G into disjoint isomorphic copies of H. The study of graph decomposition started back in the mid-19th century, with the seminal concept of *Steiner triple systems* [13], and has since become the subject of some hundreds of research papers, with active research still carried out today. Wilson's fundamental theorem [14] states that for any fixed graph H there exists an H-decomposition of the complete graph K_n if the obvious necessary divisibility conditions hold and n is *large enough*. A considerable amount of research was indeed devoted to thoroughly studying the existence of H-decompositions of complete graphs for specific graphs H, such as some small graphs, complete graphs, complete multipartite graphs, and

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paths and cycles (a finite problem for every fixed graph H, in light of Wilson's theorem). For a review of methods and results see e.g. [2,4].

Hopes for similar accurate results where H-decomposition of a general graph G is considered are slim, due to the following negative result:

Theorem 1.1. *Deciding whether there exists an H-decomposition of an input graph G that is NP-complete for any fixed simple graph H which contains a connected component with at least three edges.*

The above was conjectured by Holyer [9] in 1981 and proved sixteen years later in [7]. On the other hand, the existence of a polynomial time algorithm to decide H-decomposability of an input G, where every component of H consists of at most two edges, was proved (though not in terms of an explicit necessary and sufficient condition) in [10].

In this article we study *multigraph decomposition*, which is the case where multiple edges are allowed in both graphs H and G. Although Theorem 1.1 was not (yet?) generalized to multigraphs, a graph decomposition decision problem most probably remains at least as hard when extended to multigraphs.

Within this article we prove two instances of a generalization of Theorem 1.1: The first instance is restricted to the case in which H is a star—Theorem 2.2 which states that the problem of decomposing a multigraph with an identical multiplicity on each of its edges, to copies of a star of at least three edges, is complete in NP: The second instance is restricted to the case in which H is a multistar—Theorem 3.1, which proves NP-completeness for the decomposition problem of multigraphs, to copies of a given multistar, that is, a given multigraph derived from a star by replacing each of its edges by multiple edges.

In an attempt to find the conditions for decomposability of a general "input" multigraph G into a "fixed" connected multigraph H, serious hopes for results are limited, in light of the theorems above, to the case in which every connected component of H consists of at most two underlying edges.

If every connected component of H consists of exactly one edge, that is, if H is a matching, then the problem is polynomial. The decomposition of multigraphs in that case is an extension based on the result of [1].

In the following section we deal with the case in which H is a simple path with two edges and G has a constant multiplicity. Separate articles are dedicated to the more involved case in which the multiplicities on the edges of H and G are not restricted [12,11].

The following terminology sets the frame for a more formal and rigorous treatment of the subject.

1.1. Notation

- *A multigraph* (V, E, w), also denoted by (G, w), consists of a simple *underlying graph* $G=(V, E)$ and a *multiplicity function* $w : E \to N$, where N is the set of natural numbers (unless explicitly stated otherwise, the multiplicity of an edge is strictly positive).
- The multigraph on an underlying graph G with a constant multiplicity λ is denoted by $\lambda \cdot G$.
- When referring to a simple graph G as a multigraph, we mean $1 \cdot G$

- An isomorphism between multigraphs is an isomorphism between their underlying simple graphs, which preserves edge multiplicity.
- A *subgraph* H of a multigraph G is a multigraph H whose underlying graph is a subgraph of that of G and its multiplicity function is dominated by the multiplicity function of G, that is, the multiplicity of an edge in H does not exceed its multiplicity in G.
- An *H-subgraph* of G is a subgraph of a multigraph G, isomorphic to a multigraph H.
- Let G and H be two multigraphs. An H-decomposition of G is a set D of H-subgraphs of G, such that the sum of $w(e)$ over all graphs in D which include an edge e equals the multiplicity of e in G, for all edges e in G.
- An *H-λ decomposition* of a simple graph G is an H-decomposition of the multigraph $\lambda \cdot G$. If it exists we say that G is H-λ decomposable, or that G admits an H-λ decomposition.
- The *t-star*, S_t (also commonly denoted by $K_{1,t}$), is a simple graph, consisting of t edges which share one common vertex, referred to as the *center* of the star, and are otherwise disjoint.
- The *multistar* $S^{w_1,\ldots,w_t}$ is the multigraph, whose underlying graph is a t-star, and the multiplicities of its t edges are $w_1, \ldots, w_t$.
- Associated with a fixed multigraph H is the H-decomposition computational problem: *Does an input multigraph M admit an H-decomposition*?
- In particular, associated with a fixed multigraph H and a natural number λ is the H-λ decomposition computational problem: *Does an input simple graph G admit an H-λ decomposition*?

2. Star-decomposition of multigraphs

2.1. S_2-λ decomposition

S_2-λ decomposition of a graph G is clearly equivalent to a perfect matching in *the λ line graph*, $L_\lambda(G)$, consisting of λ vertices for each edge of G, where two vertices are adjacent if they stand for adjacent, distinct edges of G. Thus, S_2-λ decomposition is solvable in polynomial time.

However, a maximum matching algorithm is not really essential here, as we can give a simple explicit characterization of S_2-λ decomposable graphs.

Some more terminology is first required:

- A single edge is a *z-tree* and any graph obtained by identifying a leaf of S_2 with a vertex of a z-tree is also a z-tree.
- If a graph G includes a vertex v of degree 2, adjacent to another vertex x, of degree 1, we say that the S_2-subgraph, centered at v, is *loose*. The edge incident to x is referred to as the *remote* edge of that subgraph.

Using this notation, a z-tree is either a single edge, or it is obtained by appending a new loose S_2-subgraph to a smaller z-tree.

Lemma 2.1. *The edge set of a connected odd (that is, of odd size) graph with more than three edges, which is neither a z-tree nor a simple cycle, can be partitioned into two connected subgraphs, neither of which is a z-tree.*

Proof. Let G be a graph as stated above. If G is the simple star S_{2n+1}, $n > 1$, then G can be partitioned into the two stars S_{2n-1} and S_2, and neither of them is a z-tree; thus we may assume that G is not a star. Moreover, we can assume that G admits no loose S_2-subgraph H; otherwise H and $G - H$ (obtained by removing all edges of H and vertices which thus become isolated) would form the required partition (since G is not a z-tree, neither is $G - H$).

If G is a tree we focus on a vertex v, which is adjacent to at least one vertex of degree 1 and to exactly one vertex of degree greater than 1. No loose S_2 exists and hence, $d(v) \geqslant 3$. The partition is formed by splitting G into a three-star S, centered at v, and an even connected graph $G - S$. (Notice that a z-tree is always odd.)

If G is not a tree, let C be a simple cycle in G. Consider a connected nonempty component B of $G - C$; there exists one, since the entire graph G is not a cycle. If B is not a z-tree, then B and $G - B$ provide the required partition ($G - B$ contains the cycle C and hence it is not a z-tree).

If B is a z-tree with at least three edges, then, since no loose S_2-subgraph of B is loose in G (see above), there exists a loose S_2-subgraph of B with a remote edge e, such that $C \cup \{e\}$ is connected. The partition, in that case, consists of the even subgraph $B' = B - \{e\}$ and the odd connected subgraph $G - B'$ (which, again, contains the cycle C and thus is not a z-tree).

If B is a single edge then it is contained in a three-star S, where the other two edges are from C, and $G - S$ is an even connected graph. □

We can state now the main result of this section:

Theorem 2.1. *A connected graph $G = (V, E)$ admits an S_2-λ decomposition if and only if $\lambda|E|$ is even or G is not a z-tree.*

Proof. Any connected even graph can easily be partitioned into S_2-subgraphs. For a formal proof see [5]. S_2 is clearly S_2-λ decomposable for any value of λ. Theorem 2.1 thus holds where G is even.

Let G be an odd connected graph which is not a z-tree. Repeatedly applying Lemma 2.1, it suffices to consider the case in which G is either a simple odd cycle or it has three edges. C_{2n+1}, $n > 0$, and S_3 are S_2-2 decomposable and hence they admit an S_2-λ decomposition for every even λ. The only graph with three edges, which is neither a cycle nor a star, is a z-tree.

It remains to show that a z-tree T does not admit any S_2-λ decomposition. This is obviously true where T consists of a single edge. We proceed by induction on the size of T: Let T be obtained by appending a loose S_2-subgraph H to a smaller z-tree T'. The only S_2-subgraph which includes the remote edge of H is H itself. An S_2-λ decomposition of T, then, contains λ copies of H and an S_2-λ decomposition of T', the existence of which contradicts the induction hypothesis. □

The $\lambda = 2$ instance of Theorem 2.1 is also a part of a previous result of Bondy [3].

2.2. Intractability of S_t-λ decomposition for $t \geqslant 3$

In this section we prove the following:

Theorem 2.2. *S_t-λ decomposition is* NP*-complete, for every star of size $t \geqslant 3$ and every multiplicity λ.*

In the rest of this chapter, t and λ are the parameters defined above. The following notation makes the formulation of our proof easier:

- A *module* is a graph $M = (V, E)$ with a predefined set $C \subseteq E$ of *connectors*. Each connector has a *connecting endvertex* of degree 1 and an *inner endvertex* of degree $\geqslant t$. The edges in $E \backslash C$ are *inner edges*.
- A *modular decomposition* of a module M is an S_t-decomposition of a multigraph on the underlying graph M with multiplicity λ on every inner edge and at most λ on the connectors.

Along the proof, whenever we combine modules to form a larger graph, we make sure that any S_t-λ decomposition of that graph would necessarily be a union of modular decompositions.

Lemma 2.2. *Theorem* 2.2 *can be derived from a polynomial construction of a splitting module SP, with two connectors labeled c_1 and c_2* (*out of a possibly larger set of connectors*), *which satisfies the following*:

- *Every modular decomposition of SP induces multiplicities λ on one of the two connectors c_1 or c_2 and $\lambda - 1$ on the other one.*
- *For each one of the two connectors c_1 and c_2 there exists a modular decomposition of SP which induces multiplicities λ on that one and $\lambda - 1$ on the other.*

Proof. Let G be an input graph for the (NP-complete) S_t-decomposition problem. We replace every edge (x, y) of G by a splitting module with x and y as the connecting endvertices of c_1 and c_2, to construct a new graph G'. Let us first assume that c_1 and c_2 are the only connectors of SP. In that case any partition of the original edge set of G into stars can be obtained, once a modular decomposition of every splitting module is removed from $\lambda \cdot G'$. Consequently, S_t-decomposition of G is equivalent to S_t-λ decomposition of G'.

Unfortunately, when we get to the actual construction, additional connectors necessarily exist, unless t and λ are relatively prime. Such extra connectors, with their connecting endvertices left *loose*, might remain with multiplicity smaller than λ. To overcome this difficulty we construct t disjoint copies of G' and contract the t copies of each such loose connecting vertex into a single vertex, which can be the center of as many t-stars as required to complete the decomposition. □

The construction of SP is now reduced to that of another module, which satisfies a relaxed set of requirements.

Lemma 2.3. *Theorem* 2.2 *can be derived from a polynomial construction of an isolating module IS, which contains a connector* c_0 (*and possibly also other connectors*), *and satisfies the following*:

- *Every modular decomposition of IS induces multiplicity* $\lambda - 1$ *on* c_0, *and at least one modular decomposition of IS does exist.*

Proof. Starting from *IS*, we construct *SP* by attaching an additional *t*-star centered at the connecting endvertex *x* of c_0. Two of the *t* additional edges are arbitrarily selected to be the connectors c_1 and c_2.

In a modular decomposition of *SP*, that way built, λ copies of S_t, centered at *x*, are required to cover the other $t-2$ new edges. One edge of one of these stars is c_0, completing its multiplicity from $\lambda - 1$ to λ. The multiplicities on c_1 and c_2 are then clearly as stated in Lemma 2.2. □

We use three additional modules as building blocks for the construction of *IS*:

The equalizer *EQ* consists of a circuit of length $2(\lambda + 1)$, where every vertex along the circuit is also the inner endvertex of $t - 2$ connectors. The degree of each inner vertex of *EQ* is then exactly *t*. In order to achieve multiplicity λ on the inner edges, the number of *t*-stars centered at each pair of consecutive inner vertices should sum up to λ. Let the set of connectors incident to vertices in even location along the circuit be denoted by C_a and the set of connectors incident to odd vertices of the circuit be denoted by $C_{\bar{a}}$. The following characterization of modular decompositions of *EQ* immediately follows:

- In every modular decomposition of *EQ*, every connector in C_a has the same multiplicity, say *a*, and every connector in $C_{\bar{a}}$ has multiplicity $\lambda - a$. In particular, there exists such a modular decomposition of *EQ* with $a = 1$.

The *lower bound LB* is a mere $(t - 1 + \lambda)$-star, where λ of its edges form a set C_l of connectors. At least λ (exactly two if $\lambda = 2$) copies of S_t are required to cover the other $t-1$ edges and at least one edge of each of these stars must be a connector. Consequently:

- In every modular decomposition of *LB*, the average multiplicity of a connector in C_l is at least 1 (exactly 1 if $\lambda = 2$), and there exists a modular decomposition where the multiplicity of each connector in C_l equals 1.

The *upper bound UB* is the empty graph if $\lambda = 2$ and it otherwise consists of $\lambda - 3 + (\lambda - 2)(t - 2)$ inner edges and a set C_u of λ connectors, arranged as follows: $\lambda - 3$ edges which form a simple path on $\lambda - 2$ vertices, a $(t - 2)$-star centered at each vertex of the path, two connectors incident with each end of the path and one with every other vertex along the path (notice that if $\lambda = 3$ then *UB* is a star of $t + 1$ edges, such that three of them are connectors). To take care of the $(t - 2)$-stars, in a modular decomposition of *UB*, each vertex of the path should be the center of at least λ copies of S_t. Removing from the obtained total of $(\lambda - 2)\lambda t$, the multiplicity $\lambda(\lambda - 3 + (\lambda - 2)(t - 2))$ required for the inner edges, there is an excess of at least $\lambda(\lambda - 1)$ for the connectors. This leads to:

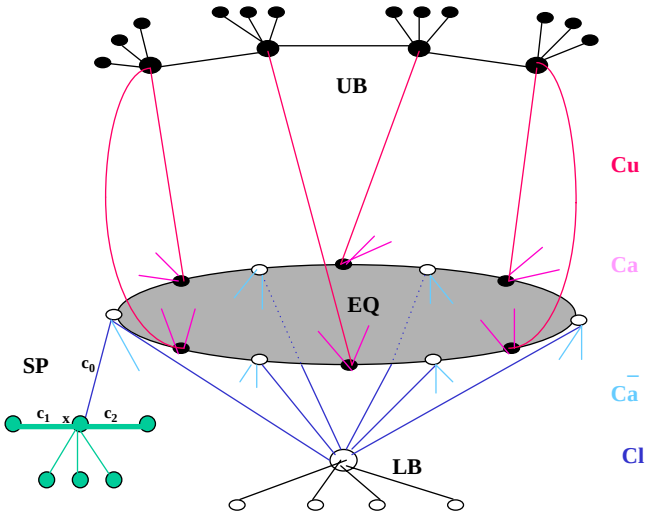

Fig. 1. The splitting module SP.

- In every modular decomposition of *UB*, the average multiplicity of a connector in C_u is at least $\lambda - 1$.

If indeed λ copies of S_t are placed at each vertex on the path, of which multiplicity $\lambda - 1$ is allocated to each connector, one can check that the remaining edges can be distributed on the path to get:

- There exists a modular decomposition of *UB* where the multiplicity of every connector in C_u is $\lambda - 1$.

We complete now the proof of Theorem 2.2 with an explicit construction of the isolating module: *IS* is the union of *EQ*, *LB* and *UB*, which share connectors, such that $C_u \subset C_a$, $C_l \subset C_{\bar{a}}$ and are otherwise disjoint. Each endvertex of every common connector serves as the connecting endvertex in one module and the inner endvertex in the other module (see Fig. 1). To avoid parallel edges, distinct endvertices in *EQ* are selected for connectors with the same endvertex in either *LB* or *UB* (the circuit in *EQ* is large enough to make this possible). The shared connectors thus become inner edges of *IS*. Their multiplicity in a modular decomposition of *IS* is then λ. The contribution of a modular decomposition of *EQ* to the multiplicity of an edge from C_u is a and hence $\lambda - a$ is its multiplicity in *UB*. Since the average contribution of *UB* to the multiplicity of these edges is at least $\lambda - 1$, we obtain $a \leqslant 1$. Similar analysis of the multiplicity on members of C_l leads to $a \geqslant 1$ and to the conclusion $a = 1$. Since $|C_{\bar{a}}| = (t-2)(\lambda+1)$ and $|C_l| = \lambda$, $C_{\bar{a}} \backslash C_l \neq \phi$ and the connector c_0 of *IS* is arbitrarily selected from that set. Due to the properties of *EQ*, the multiplicity induced on c_0 in a modular decomposition of *IS* is indeed $\lambda - a = \lambda - 1$ as required. $\square$

3. Intractability of multigraph decompositions into multistars on at least three underlying edges

H-decomposition when H is a connected *simple* graph on at least three edges is always NPC [7]. We strongly believe this remains true when "simple" is deleted. We only prove, however, the following partial result for multistar decomposition:

Theorem 3.1. *$S^{\alpha_1,\alpha_2,\ldots,\alpha_n}$-decomposition is NPC, unless $n = 1$, or $n = 2$ and $\alpha_1 = \alpha_2$, in which case the problem is polynomially solvable.*

Proof. This proof is an almost straightforward adaptation of the proof for the case in which H is a simple star, presented in [6]. The known NPC problem for which we show polynomial reduction into $S^{\alpha_1,\alpha_2,\ldots,\alpha_n}$-decomposition is "Exact Hitting Set for 3-subsets" (3-EHS, for short), defined as follows: Given a finite set U and a collection Ω of three-element subsets of U, is there a subset $X \subset U$ such that $|X \cap A| = 1$ for every $A \in \Omega$? The problem is also known as "One in three 3sat without negated literals" (see [8, p. 259 Lo4]).

Let $I = (U, \Omega)$ be an instance of 3-EHS. We construct (polynomially) a multigraph H for which $S^{\alpha_1,\alpha_2,\ldots,\alpha_n}$-decomposition is equivalent to 3-EHS on I. We first describe the underlying graph $G_{S_n}(I)$ of H and then define the multiplicity function.

Let every three-tuple $A \in \Omega$ be represented by two disjoint stars: a copy of S_{n-1} and a copy of S_{2n-2}. The centers of these stars are denoted by A^+ and A^-, respectively. Let $k(x)$ denote the number of three-tuples $A \in \Omega$ which contain the element x of U. For every $x \in U$, construct an $(n-1)$ -regular, connected, bipartite graph $G_x(S_n)$ on two independent sets S_x^+ and S_x^-, each consisting of $k' = \max\{k(x), n\}$ vertices. Label the vertices of S_x^+ by $v_1, \ldots, v_{k'}$ and those of S_x^- by $u_1, \ldots, u_{k'}$. In case $n > k$, choose for every $i > k$, non-adjacent u_i and v_i and then add the edge (u_i, v_i) to the graph. For every pair (x, A), where $x \in A$, choose a distinct pair of vertices u_i and v_i from $G_x(S_n)$, and add one edge, $e^+_{x,A}$, with end vertices (A^+, v_i) and another, $e^-_{x,A}$, with end vertices (A^-, u_i). Three additional edges are appended this way to every vertex A^+ and A^-. The obtained graph is $G_{S_n}(I)$.

Let us now define the multiplicity of every edge and then verify that $S^{\alpha_1,\alpha_2,\ldots,\alpha_n}$-decomposition of the obtained multigraph H is equivalent to 3-EHS on I: The degree in $G_{S_n}(I)$ of every vertex of $S_x^+ \cup S_x^-$ is n. We set the multiplicity function such that the multistar centered at each of these vertices would be an $S^{\alpha_1,\alpha_2,\ldots,\alpha_n}$-subgraph of H. We also make sure that all the edges between vertices in $S_x^+ \cup S_x^-$ and the vertices A^+ and A^- have the same multiplicity, say, α_1. This goal is achieved by partitioning each $(n-1)$-regular bipartite $G_x(S_n)$ into $n-1$ matchings and assigning multiplicity α_i to all edges of the ith matching, for $i = 2, 3, \ldots, n$. The remaining edge incident with each vertex in $S_x^+ \cup S_x^-$ is assigned with multiplicity α_1. For every $A \in \Omega$, we assign multiplicities $\alpha_2, \ldots, \alpha_n$ to the remaining $n-1$ edges of the star centered at A^+ and repeat this sequence of multiplicities twice on the $2n-2$ edges of the star centered at A^-. The multistar centered at A^+ consists now of an $S^{\alpha_1,\alpha_2,\ldots,\alpha_n}$-subgraph and two extra edges of multiplicity α_1. The multistar centered at A^-consists of two copies of $S^{\alpha_1,\alpha_2,\ldots,\alpha_n}$, and one extra edge of multiplicity α_1.

Let us denote $S^{\alpha_1,\alpha_2,\ldots,\alpha_n}$ in short by S. Assume $X \subset U$ is a solution of I. That is, for every $A \in \Omega$, exactly one element of A belongs to X and two elements do not. For every $x \in X$,

remove the copies of S centered at every $u \in S_x^-$. For every $y \notin X$, remove the copies of S centered at every $v \in S_y^+$.

Take any $A \in \Omega$. Since $|A \cap X| = 1$, the removed stars contain exactly one edge $e_{x,A}^-$ with multiplicity α_1 for a certain $x \in X$, incident with A^-, and two such edges $e_{y,A}^+$ and $e_{z,A}^+$, $y, z \notin X$, incident with A^+. Thus, the remaining edges form a copy of S centered at every vertex A^+ and two such copies centered at every A^-. Together with the removed multistars, an S-decomposition of H is completed.

On the other hand, assume that an S-decomposition of H exists. Take an edge e of one of the bipartite subgraphs $G_x(S_n)$. Since e is covered by the decomposition, at least one of its end vertices is the center of an S-subgraph which belongs to the decomposition (we call such a vertex *a center of the decomposition* or simply *a center*). The edges incident with every vertex of $G_x(S_n)$ form an exact copy of S; hence only one of the end vertices of e can be a center. The set of centers is thus independent in $G_x(S_n)$ and covers all its edges with their full multiplicities. $G_x(S_n)$ is connected, so there are exactly two possibilities: either every $v \in S_x^+$ is a center and no $u \in S_x^-$ is such, or every $u \in S_x^-$ and no $v \in S_x^+$ is a center. Define $X = \{x \in U \mid u \in S_x^- \Rightarrow u \text{ is a center}\}$. Focus on a three-tuple $A \in \Omega$. The vertex A^- is necessarily the center of two decomposition multistars. The extra edge of multiplicity α_1, incident to A^-, belongs to a decomposition star centered at a vertex $u \in S_x^-$ for some $x \in X$. Thus A contains exactly one element $x \in X$ and hence X is indeed a solution of I.

Some extra caution is required to verify the validity of that construction when $n = 2$ and $\alpha_1 \neq \alpha_2$. Notice that if the two multiplicities are equal, it can happen that three edges incident with A^- are covered by stars centered at $G_x(S_n)$ and the remaining two edges form another decomposition star, or that the four edges, of equal multiplicity, incident with A^+ are matched into two stars. Indeed, when the multiplicities are equal the problem is polynomially solvable by Theorem 2.1 (after dividing λ by the edge multiplicity of H). $\square$

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