A vector potential KdV equation and vector Ito equation: soliton solutions, bilinear Bäcklund transformations and Lax pairs

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Abstract
A vector potential KdV equation and vector Ito equation are proposed based on their bilinear forms. Soliton solutions expressed by Pfaffians are obtained. Bilinear Bäcklund transformations and the corresponding Lax pairs for the vector potential KdV equation and the vector Ito equation are derived.

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1. Introduction
In the literature, several approaches have been developed to search for various integrable coupled versions for the celebrated Korteweg–de Vries equation

\[ u_t + 6uu_x + u_{xxx} = 0. \] (1)

One of them is the bilinear approach. It is known that the KdV equation (1) can be transformed into the bilinear form

\[ D_x(D_t + D_x^3)f \cdot f = 0 \] (2)
by the dependent variable transformation
\[ u = 2 (\ln f)_{xx}, \]
where the bilinear operators \( D^m_x D^k_t \) are defined by [1,2]
\[
D^m_x D^k_t a \cdot b \equiv \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^k a(x, t) b(x', t') \bigg|_{x' = x, t' = t}.
\]
From [3], we know that the extension of the bilinear equation (2) into a coupled bilinear form
\[
(D_t + D^3_x) f \cdot f = g^2, \quad (D_t - 2D^3_x) f \cdot g = 0
\]
leads to the following Hirota–Satsuma coupled KdV equation:
\[
\begin{align*}
  u_t + 6u u_x + u_{xxx} &= 2v v_x, \\
  v_t - 2v_{xxx} - 6u v_x &= 0,
\end{align*}
\]
where \( u = 2(\ln f)_{xx}, \ v = g/f \). Besides, we note that the KdV equation (1) can be transformed into another bilinear form
\[
(D_t + D^3_x) G \cdot F = 0, \quad D^2_x F \cdot F = 2G F
\]
through the rational transformation \( u = G/F \). As a result, another coupled KdV equation is obtained by extending Eqs. (8) and (9) to a coupled form
\[
\begin{align*}
  (D_t + D^3_x) G_j \cdot F &= 0, \quad j = 1, 2, \ldots, M, \\
  D^2_x F \cdot F &= 2 \left( \sum_{j=1}^{N} G_j \right) F,
\end{align*}
\]
which is transformed into the ordinary form
\[
\begin{align*}
  \frac{\partial u_j}{\partial t} + 6 \left( \sum_{k=1}^{M} u_k \right) \frac{\partial u_j}{\partial x} + \frac{\partial^3 u_j}{\partial x^3} &= 0, \quad j = 1, 2, \ldots, M,
\end{align*}
\]
through the dependent variable transformation \( G_j = u_j F \). Equation (12) has been considered by Yoneyama in [4].
Based on the fact that the bilinear KdV equation (2) can be rewritten as \( (D_t + D^3_x) f_x \cdot f = 0 \), we now give the third bilinear form for the KdV equation (1),
\[
\begin{align*}
  (D_t + D^3_x) g \cdot f &= 0, \\
  D^2_x f \cdot f &= 2D_x g \cdot f,
\end{align*}
\]
which has a natural coupled form
\[(D_t + D_x^3)g_j \cdot f = 0, \quad j = 1, 2, \ldots, M,\]  \hspace{1cm} (15)

\[D_t^2 f \cdot f = 2D_t \left( \sum_{j=1}^M g_j \right) \cdot f.\]  \hspace{1cm} (16)

By the dependent variable transformation \(v_j = \frac{2g_j}{f}\), Eqs. (15) and (16) can be transformed into an \(M\)-component potential KdV equation

\[\frac{\partial v_j}{\partial t} + 3 \left( \sum_{k=1}^M \frac{\partial v_k}{\partial x} \right) \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0, \quad j = 1, 2, \ldots, M,\]  \hspace{1cm} (17)

or its vector form

\[v_t + 3(c \cdot v)_x v_x + v_{xxx} = 0,\]  \hspace{1cm} (18)

where \(v = (v_1, v_2, \ldots, v_M), c = (1, 1, \ldots, 1)\) and the inner product \(c \cdot v\) is defined by

\[c \cdot v = \sum_{i=1}^M v_i.\]

Set \(u_i = v_i x, u = v_x\); then we obtain an \(M\)-component KdV equation

\[\frac{\partial u_i}{\partial t} + 3 \left( \sum_{k=1}^M u_k \right) \frac{\partial u_i}{\partial x} + 3 \left( \sum_{k=1}^M \frac{\partial u_k}{\partial x} \right) u_i + \frac{\partial^3 u_i}{\partial x^3} = 0, \quad i = 1, 2, \ldots, M,\]  \hspace{1cm} (19)

or its equivalent vector form

\[u_t + 3\left[ (c \cdot u) u \right]_x + u_{xxx} = 0.\]  \hspace{1cm} (20)

It is remarked that the above vector KdV equation (20) has been previously proposed in [5,6] in different ways. In particular, if we choose \(M = 2\), then Eq. (19) becomes the one given in [7]. The success of extension from Eq. (2) to the coupled form (15) and (16) motivates one to propose a natural coupled form

\[(D_t + D_x^3)g_j \cdot f = 0, \quad j = 1, 2, \ldots, M,\]  \hspace{1cm} (21)

\[D_t^2 f \cdot f = 2D_t \left( \sum_{j=1}^M g_j \right) \cdot f\]  \hspace{1cm} (22)

for the Ito equation [8]

\[D_t (D_t + D_x^3) f \cdot f = 0.\]  \hspace{1cm} (23)

Equations (21) and (22) may be rewritten in an equivalent form

\[(D_t + D_x^3)g_j \cdot f = 0, \quad j = 1, 2, \ldots, M,\]  \hspace{1cm} (24)

\[\sum_{j=1}^M g_j = f,\]  \hspace{1cm} (25)

which becomes
\[ u_t = \sum_{i=1}^{M} v_i, \quad \text{(26)} \]
\[ v_i + 3(uv_i)_x + v_{i xxx} = 0, \quad i = 1, \ldots, M, \quad \text{(27)} \]
or its vector form
\[ u_t = (e \cdot v)_x, \quad \text{(28)} \]
\[ v_t + 3(uv)_x + v_{xxx} = 0 \quad \text{(29)} \]
by the dependent variable transformation
\[ v_i = 2 \frac{D_x g_t \cdot f}{f^2}, \quad u = 2(\ln f)_{xx}. \]

We call the system (28)–(29) the vector Ito equation.

The purpose of this paper is to study the vector potential KdV equation and the vector Ito equation. We will give soliton solutions and bilinear Bäcklund transformations for (15)–(16) and (24)–(25), and Lax pairs for Eqs. (20) and (28)–(29).

This paper is organized as follows. Firstly, soliton solutions expressed by Pfaffians are found for (15)–(16) in Section 2. Secondly, in Section 3 we give a bilinear Bäcklund transformation (BT) for (15)–(16). Furthermore, the corresponding Lax pair for Eq. (20) is obtained from the bilinear BT. Next, soliton solutions expressed by Pfaffians are found for (24)–(25) in Section 4. Section 5 is devoted to deriving a bilinear BT and the corresponding Lax pair for the vector Ito equation (28)–(29). Finally, conclusion and discussions are given in Section 6.

2. N-soliton solution for the coupled KdV equation

We have transformed the vector potential KdV equation into the bilinear form,
\[ (D_t + D_x^3)g_\mu \cdot f = 0 \quad \text{for } \mu = 1, 2, \ldots, M, \quad \text{(30)} \]
\[ D_x^2 f \cdot f = 2 \sum_{\mu=1}^{M} D_x g_\mu \cdot f. \quad \text{(31)} \]

We note that the second bilinear form (31) is transformed into the linear equation
\[ \frac{\partial f}{\partial x} = \sum_{\mu=1}^{M} g_\mu. \quad \text{(32)} \]

Using a perturbational method we obtain a 3-soliton solution to the 4-coupled equations \((M = 4)\), which is expressed as follows:
\[
\begin{align*}
  f &= 1 + \exp[\eta_1] + \exp[\eta_2] + \exp[\eta_3] \\
  &+ c_0(1, 2) \exp[\eta_1 + \eta_2] + c_0(1, 3) \exp[\eta_1 + \eta_3] + c_0(2, 3) \exp[\eta_2 + \eta_3] \\
  &+ c_0(1, 2, 3) \exp[\eta_1 + \eta_2 + \eta_3].
\end{align*}
\]
\[ g_\mu = c_\mu(1) \exp[\eta_1] + c_\mu(2) \exp[\eta_2] + c_\mu(3) \exp[\eta_3] \\
+ c_\mu(1, 2) \exp[\eta_1 + \eta_2] + c_\mu(1, 3) \exp[\eta_1 + \eta_3] + c_\mu(2, 3) \exp[\eta_2 + \eta_3] \\
+ c_\mu(1, 2, 3) \exp[\eta_1 + \eta_2 + \eta_3] \quad \text{for } \mu = 1, 2, 3, 4, \]

where
\[
\frac{\partial}{\partial x} \exp[\eta_j] = p_j \exp[\eta_j],
\]
\[
\frac{\partial}{\partial t} \exp[\eta_j] = -p_j^3 \exp[\eta_j] \quad \text{for } j = 1, 2, 3,
\]
\[
c_0(j, k) = \frac{(p_j - p_k)^2}{(p_j + p_k)^2} \quad \text{for } j, k = 1, 2, 3,
\]
\[
c_0(1, 2, 3) = c_0(1, 2)c_0(1, 3)c_0(2, 3),
\]

and
\[
c_\mu(j, k) = (c_\mu(j) - c_\mu(k))(p_j - p_k)/(p_j + p_k)
\]
\quad \text{for } \mu = 1, 2, 3, 4 \text{ and for } j, k = 1, 2, 3,
\]
\[
c_\mu(1, 2, 3) = N_\mu(1, 2, 3)D(1, 2, 3),
\]
\[
N_\mu(1, 2, 3) = c_\mu(1)(p_2 - p_3)(p_1 + p_2)(p_1 + p_3)
\]
\[
+ c_\mu(2)(p_3 - p_1)(p_2 + p_3)(p_2 + p_1)
\]
\[
+ c_\mu(3)(p_1 - p_2)(p_3 + p_1)(p_3 + p_2) \quad \text{for } \mu = 1, 2, 3, 4,
\]
\[
D(1, 2, 3) = \frac{(p_1 - p_2)(p_1 - p_3)(p_2 - p_3)}{(p_1 + p_2)^2(p_1 + p_3)^2(p_2 + p_3)^2},
\]
\[
\sum_{\mu=1}^4 c_\mu(j) = p_j \quad \text{for } j = 1, 2, 3,
\]

where \( p_j, c_1(j), c_2(j), c_3(j) \) for \( j = 1, 2, 3 \) are free parameters.

These expressions suggest that \( N \)-soliton solution to the equations is expressed by Pfaffians. In fact we find that
\[
f = \text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_0),
\]
\[
g_\mu = \text{pf}(d_0, a_1, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu) \quad \text{for } \mu = 1, 2, \ldots, M,
\]

where the entries of the Pfaffians are defined as follows:
\[
\text{pf}(d_0, a_j) = \exp(\eta_j), \quad \text{pf}(d_0, b_j) = -1, \quad \text{pf}(d_0, \beta_0) = 1,
\]
\[
\text{pf}(a_j, a_k) = -a_{j,k} \exp(\eta_j + \eta_k), \quad \text{pf}(a_j, b_k) = \delta_{j,k}, \quad \text{pf}(a_j, \beta_0) = 0,
\]
\[
\text{pf}(b_j, b_k) = a_{j,k}, \quad \text{pf}(d_0, \beta_\mu) = 0, \quad \text{pf}(b_j, \beta_0) = 1,
\]
\[
\text{pf}(a_j, \beta_\mu) = 0, \quad \text{pf}(b_j, \beta_\mu) = c_\mu(j)
\]
\quad \text{for } j, k = 1, 2, \ldots, N \text{ and for } \mu = 1, 2, \ldots, M,
\]

where
\[ \eta_j = p_j x - \eta_0, \quad (39) \]
\[ a_{j,k} = (p_j - p_k)/(p_j + p_k), \quad (40) \]
\[ \delta_{j,k} = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases} \quad (41) \]

where \( p_j \) and \( \eta_0 \) are free parameters, and \( c_{\mu}(j) \) are parameters satisfying a condition
\[ \sum_{\mu=1}^{M} c_{\mu}(j) = p_j. \quad (42) \]

First we prove that \( f \) and \( g_{\mu} \) satisfy the linear equation (32). Expanding \( g_{\mu} \) with respect to the final character \( \beta_{\mu} \), we obtain
\[ g_{\mu} = \mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_{\mu}) \quad (43) \]
\[ = \sum_{j=1}^{N} \mathrm{pf}(\beta_{\mu}, b_j)(-1)^{N+j-1} \mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, \hat{b}_j, \ldots, b_N) \quad (44) \]
\[ = \sum_{j=1}^{N} c_{\mu}(j)(-1)^{N+j} \mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, \hat{b}_j, \ldots, b_N). \quad (45) \]

where \( \hat{\cdot} \) indicates deletion of the character under it. The sum of \( g_{\mu} \) over \( \mu \) gives
\[ \sum_{\mu=1}^{M} g_{\mu} = \sum_{j=1}^{N} p_j (-1)^{N+j} \mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, \hat{b}_j, \ldots, b_N), \quad (46) \]

which is expressed, introducing a new character \( \beta_1 \), by a Pfaffian
\[ \sum_{\mu=1}^{M} g_{\mu} = \mathrm{pf}(d_0, \beta_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N), \quad (47) \]

where new entries are defined by
\[ \mathrm{pf}(d_0, \beta_1) = 0, \quad \mathrm{pf}(\beta_1, a_j) = 0, \quad \mathrm{pf}(\beta_1, b_j) = -p_j. \quad (48) \]

We show in Appendix A the following relations:
\[ \mathrm{pf}(d_0, \beta_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N) = - \mathrm{pf}(d_0, d_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N) \quad (49) \]
\[ = \frac{\partial f}{\partial x}. \quad (50) \]

Accordingly we have shown that \( f \) and \( g_{\mu} \) satisfy the linear equation
\[ \frac{\partial f}{\partial x} = \sum_{\mu=1}^{M} g_{\mu}. \quad (51) \]

Next we show that \( f \) and \( g_{\mu} \) satisfy the bilinear equation (30).
The bilinear equation is rewritten as

\[
\left( \frac{\partial g_\mu}{\partial t} + \frac{\partial^3 g_\mu}{\partial x^3} \right) f - g_\mu \left( \frac{\partial f}{\partial t} + \frac{\partial^3 f}{\partial x^3} \right) - 3 \left( \frac{\partial^2 g_\mu}{\partial x^2} \frac{\partial f}{\partial x} - \frac{\partial g_\mu}{\partial x} \frac{\partial^2 f}{\partial x^2} \right) = 0. \tag{52}
\]

Using a method described in Appendix A we obtain the following differential formulae:

\[
f = \text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_0), \tag{53}
\]

\[
\frac{\partial f}{\partial x} = -\text{pf}(d_0, d_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N), \tag{54}
\]

\[
\frac{\partial^2 f}{\partial x^2} = -\text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N), \tag{55}
\]

\[
\frac{\partial^3 f}{\partial x^3} = -\text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_0), \tag{56}
\]

\[
\frac{\partial f}{\partial t} = \text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N)
- 2\text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_0), \tag{57}
\]

\[
g_\mu = \text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu), \tag{58}
\]

\[
\frac{\partial g_\mu}{\partial x} = \text{pf}(d_0, d_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu, \beta_0), \tag{59}
\]

\[
\frac{\partial^2 g_\mu}{\partial x^2} = \text{pf}(d_0, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu, \beta_0), \tag{60}
\]

\[
\frac{\partial^3 g_\mu}{\partial x^3} = \text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu, \beta_0)
- \text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu), \tag{61}
\]

\[
\frac{\partial g_\mu}{\partial t} = -\text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu, \beta_0)
- 2\text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu). \tag{62}
\]

Substituting these relations into Eq. (52), we find that the bilinear equation is reduced to the Pfaffian identity described in Appendix A,

\[
\text{pf}(d_1, \ldots) \text{pf}(d_2, \beta_\mu, \beta_0, \ldots) - \text{pf}(d_2, \ldots) \text{pf}(d_1, \beta_\mu, \beta_0, \ldots)
+ \text{pf}(\beta_\mu, \ldots) \text{pf}(d_1, d_2, \beta_0, \ldots) - \text{pf}(\beta_0, \ldots) \text{pf}(d_1, d_2, \beta_\mu, \ldots) = 0, \tag{63}
\]

where the list \{\ldots\} represents

\[
\{d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N\}. \tag{64}
\]

Accordingly \(f\) and \(g_\mu\) satisfy the bilinear equation (30).
3. Bilinear Bäcklund transformation and Lax pair for (20)

In this section, we will firstly present a bilinear Bäcklund transformation for Eqs. (15)–(16). In fact, concerning Eqs. (15)–(16), we have the Bäcklund transformation

\[
D_x(g_i \cdot f' - f \cdot g_i') - \lambda_i D_x f \cdot f' = 0, \quad i = 1, 2, \ldots, M, \tag{65}
\]

\[
(D_t + D_x^3) f \cdot f' = 0, \tag{66}
\]

\[
(D_t + D_x^3)(g_i \cdot f' + f \cdot g_i') = 0, \quad i = 1, 2, \ldots, M, \tag{67}
\]

between Eqs. (15)–(16) and

\[
(D_t + D_x^3) g_j' \cdot f' = 0, \quad j = 1, 2, \ldots, M, \tag{68}
\]

\[
D^2_x f' \cdot f' = 2D_x \left( \sum_{j=1}^{M} g_j' \right) \cdot f', \tag{69}
\]

where we have assumed that

\[
g_1 + g_2 + \cdots + g_M = f_x, \quad g_1' + g_2' + \cdots + g_M' = f'_x \tag{70}
\]

such that Eqs. (16) and (69) are satisfied automatically and \(\lambda_i \ (i = 1, 2, \ldots, M)\) are arbitrary constants. Starting from (65)–(67) with (70), we can derive a Lax pair for (20). To this end, set

\[
f = \phi f', \quad g_i = \psi_i f' + \phi g_i', \quad u_i = \frac{2D_x g_i' \cdot f'}{f'^2},
\]

\[
c \cdot u = \frac{D^2_x f' \cdot f'}{f'^2}, \quad \sum_{i=1}^{M} \lambda_i = \lambda.
\]

Then from Bäcklund transformation (65)–(67) with (70), we can deduce that

\[
\left( \begin{array}{c} \tilde{\psi} \\ \phi \\ \phi_x \end{array} \right)_x = U \left( \begin{array}{c} \tilde{\psi} \\ \phi \\ \phi_x \end{array} \right) = \left( \begin{array}{ccc} 0 & -u^T & 0 \\ 0 & 0 & 1 \\ 0 & -c \cdot u & \lambda \end{array} \right) \left( \begin{array}{c} \tilde{\psi} \\ \phi \\ \phi_x \end{array} \right), \tag{71}
\]

\[
\left( \begin{array}{c} \tilde{\psi} \\ \phi \\ \phi_x \end{array} \right)_t = V \left( \begin{array}{c} \tilde{\psi} \\ \phi \\ \phi_x \end{array} \right), \tag{72}
\]

where

\[
V = \left( \begin{array}{cc} 0_{M \times M} & ((c \cdot u)_x + \lambda(c \cdot u))K + u^T \cdot u - 2(c \cdot u)u^T \cdot u - 2(c \cdot u) - \lambda \cdot u \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & (c \cdot u)_x + \lambda(c \cdot u) + \lambda^2(c \cdot u) + \lambda^2(c \cdot u) \end{array} \right),
\]

\[
u = (u_1, u_2, \ldots, u_M), \quad 0 = (0, 0, \ldots, 0),
\]

\[K = (\lambda_1, \lambda_2, \ldots, \lambda_M)^T, \quad \psi = (\psi_1, \psi_2, \ldots, \psi_M)^T,\]

and \(0_{M \times M}\) is an \(M \times M\) zero matrix. We can check that their compatibility condition

\[U_t - V_x + UV - VU = 0\]
gives Eq. (20). It is also remarked that from Bäcklund transformation (65)–(67) with (70) and by the dependent variable transformation
\[ w_i = g_i/f - g_i'/f', \]
we can derive the following coupled equation:
\[
\begin{align*}
  w_{it} + w_{ixxx} - 3 \left( \sum_{k=1}^{M} w_k \right) \left( w_i - \lambda_i \right) \sum_{k=1}^{M} w_k x &= 0, \\
  3 \left( \sum_{k=1}^{M} \lambda_k \right) \left( \sum_{k=1}^{M} w_k \right) x &= 0.
\end{align*}
\]
(73)

In particular, if we choose \( \lambda_i = 0 \) (\( i = 1, \ldots, M \)), then (73) becomes a coupled mKdV equation
\[
\begin{align*}
  w_{it} + w_{ixxx} - 3 \left( \sum_{k=1}^{M} w_k \right) \left( w_i \sum_{k=1}^{M} w_k \right) x &= 0.
\end{align*}
\]
(74)

4. \( N \)-soliton solution of the vector Ito equation

We have transformed the vector Ito equation into the following equations:
\[
\begin{align*}
  (D_t + D_x^3)g_\mu \cdot f &= 0 \quad \text{for } \mu = 1, 2, \ldots, M, \\
  \frac{\partial f}{\partial t} &= \sum_{\mu=1}^{M} g_\mu.
\end{align*}
\]
(75)\hspace{1cm}(76)

We find that \( N \)-soliton solution to the equations is expressed by Pfaffians of the same forms as those of the vector potential KdV equations (30)–(31),
\[
\begin{align*}
  f &= \text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu), \\
  g_\mu &= \text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu) \quad \text{for } \mu = 1, 2, \ldots, M,
\end{align*}
\]
(77)\hspace{1cm}(78)
where the entries of the Pfaffians are the same as those of Section 2 except the entries \( \text{pf}(b_j, b_k) \),
\[
\begin{align*}
  \text{pf}(d_0, a_j) &= \exp(\eta_j), \\
  \text{pf}(d_0, b_j) &= -1, \\
  \text{pf}(d_0, \beta_0) &= 1, \\
  \text{pf}(a_j, a_k) &= -a_j \cdot k \exp(\eta_j + \eta_k), \\
  \text{pf}(a_j, b_k) &= \delta_{j,k}, \\
  \text{pf}(a_j, \beta_\mu) &= 0, \\
  \text{pf}(b_j, b_k) &= b_{j,k}, \\
  \text{pf}(b_j, \beta_0) &= 1, \\
  \text{pf}(d_0, \beta_\mu) &= 0, \\
  \text{pf}(a_j, \beta_\mu) &= 0, \\
  \text{pf}(b_j, \beta_\mu) &= c_\mu(j)
\end{align*}
\]
for \( j, k = 1, 2, \ldots, N \) and for \( \mu = 1, 2, \ldots, M \),
where
\[
\begin{align*}
  \eta_j &= p_j x - p_j^3 t + \eta_0, \\
  a_{j,k} &= (p_j - p_k)/(p_j + p_k), \\
  b_{j,k} &= (p_j^2 - p_k^2)/(p_j^2 + p_k^2), \\
  \delta_{j,k} &= \begin{cases} 
  1 & \text{for } j = k, \\
  0 & \text{for } j \neq k.
\end{cases}
\end{align*}
\]
First we show that \( f \) and \( g_\mu \) satisfy the bilinear equation (75).

The bilinear equation (75) is the same as that of Section 2 which is rewritten as

\[
\left( \frac{\partial g_\mu}{\partial t} + \frac{\partial^3 g_\mu}{\partial x^3} \right) f - \frac{\partial f}{\partial t} \left( \frac{\partial f}{\partial x} + \frac{\partial^3 f}{\partial x^3} \right) - 3 \left( \frac{\partial^2 g_\mu}{\partial x^2} \frac{\partial f}{\partial x} - \frac{\partial g_\mu}{\partial x} \frac{\partial^2 f}{\partial x^2} \right) = 0. \tag{80}
\]

We find that the differential formulae of \( f \) and \( g_\mu \) are the same as those of Section 2,

\[
f = \text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_0), \tag{81}
\]

\[
\frac{\partial f}{\partial x} = - \text{pf}(d_0, d_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N), \tag{82}
\]

\[
\frac{\partial^2 f}{\partial x^2} = - \text{pf}(d_0, d_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N), \tag{83}
\]

\[
\frac{\partial^3 f}{\partial x^3} = - \text{pf}(d_0, \beta_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N) \tag{84}
\]

\[
\frac{\partial f}{\partial t} = \text{pf}(d_0, d_3, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N)
- \text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_0), \tag{85}
\]

\[
g_\mu = \text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu), \tag{86}
\]

\[
\frac{\partial g_\mu}{\partial x} = \text{pf}(d_0, d_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu, \beta_0), \tag{87}
\]

\[
\frac{\partial^2 g_\mu}{\partial x^2} = \text{pf}(d_0, d_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu, \beta_0), \tag{88}
\]

\[
\frac{\partial^3 g_\mu}{\partial x^3} = \text{pf}(d_0, d_3, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu, \beta_0)
- \text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu), \tag{89}
\]

\[
\frac{\partial g_\mu}{\partial t} = - \text{pf}(d_0, d_3, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu, \beta_0)
- 2 \text{pf}(d_0, d_1, d_2, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu). \tag{90}
\]

because the values of the entries \( \text{pf}(b_j, b_k) \) (= constant of \( x, t \)) are irrelevant to these differential formulæ. They, however, play important roles in proving the linear equation (76). Substituting these relations into Eq. (80), we find that the bilinear equation is reduced to the Pfaffian identity in Section 2,

\[
\text{pf}(d_1, \ldots) \text{pf}(d_2, \beta_\mu, \beta_0, \ldots) - \text{pf}(d_2, \ldots) \text{pf}(d_1, \beta_\mu, \beta_0, \ldots) \\
+ \text{pf}(\beta_\mu, \ldots) \text{pf}(d_1, d_2, \beta_0, \ldots) - \text{pf}(\beta_0, \ldots) \text{pf}(d_1, d_2, \beta_\mu, \ldots) = 0. \tag{91}
\]
where the list \{\ldots\} represents
\[\{d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N\}.\] (92)

Accordingly \(f\) and \(g_n\) satisfy the bilinear equation (75).

Next we prove that \(f\) and \(g_n\) satisfy the linear equation (76).

A new expression of \(\partial f/\partial t\) is obtained by using the procedures in Appendix A. We introduce a new character \(\beta_0'\) defined by
\[pf(d_0, \beta_0') = 0, \quad pf(d_0, \beta_0) = 1, \quad pf(a_j, \beta_0') = 0, \quad pf(a_j, \beta_0) = 1 \quad \text{for } j = 1, 2, \ldots, N,\] (93)
and \(pf(b_j, \beta_0') = pf(b_j, \beta_0) = 1\) for \(j = 1, 2, \ldots, N\), (94)
in order to utilize the expansion formulae in Appendix A.

Then \(f\) is expressed by
\[f = f_0 + f',\]
\[f_0 = pf(a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N),\]
\[f' = pf(d_0, \beta_0', a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N).\] (95) (96)

We note that \(f_0\) and \(f'\) are invariant under the transformation
\[a_j \rightarrow a_j' \quad (= a_j \exp (-\eta_j)),\]
\[b_j \rightarrow b_j' \quad (= b_j \exp (\eta_j)),\] so that
\[f_0 = pf(a_1', a_2', \ldots, a_N', b_1', b_2', \ldots, b_N'),\]
\[f' = pf(d_0, \beta_0', a_1, a_2, \ldots, a_N, b_1', b_2', \ldots, b_N'),\] with the entries,
\[pf(a_j', a_k') = -a_j, \quad pf(a_j', b_k') = \delta_{j,k}, \quad pf(b_j', b_k') = b_j \exp (\eta_j + \eta_k),\]
\[pf(d_0, \beta_0') = 0, \quad pf(d_0, a_j') = 1, \quad pf(d_0, b_j') = -\exp (\eta_j),\]
\[pf(\beta_0', a_j') = 0, \quad pf(\beta_0', b_j') = -\exp (\eta_j).\]
Furthermore we introduce another character \(d_0' (= d_0 - \beta_0')\) so that
\[pf(d_0', a_j') = pf(d_0, a_j') - pf(\beta_0', a_j') = 1,\]
\[pf(d_0', b_j') = pf(d_0, b_j') - pf(\beta_0', b_j') = 0.\]

Then we find the differential formulae
\[\frac{\partial}{\partial t} pf(a_j', a_k') = 0 = pf(\alpha_3, \beta_0', a_j', a_k'),\]
\[\frac{\partial}{\partial t} pf(a_j', b_k') = 0 = pf(\alpha_3, \beta_0', a_j', b_k'),\]
\[\frac{\partial}{\partial t} pf(b_j', b_k') = -(p_j^3 - p_k^3) \exp (\eta_j + \eta_k) = pf(\alpha_3, \beta_0', b_j', b_k').\]
and
\[
\frac{\partial}{\partial t} \phi(d_0, \beta_0', a_j', a_k') = 0 = \phi(\alpha_3, d_0', a_j', a_k'),
\]
\[
\frac{\partial}{\partial t} \phi(d_0, \beta_0', a_j', b_k') = -p_j^3 \exp(\eta_j) = \phi(\alpha_3, d_0', a_j', b_k'),
\]
\[
\frac{\partial}{\partial t} \phi(d_0, \beta_0', b_j', b_k') = 0 = \phi(\alpha_3, d_0', b_j', b_k'),
\]
where the new entries are defined by
\[
\phi(\alpha_3, \beta_0) = 0, \quad \phi(\alpha_3, b_j') = -p_j^3 \exp(\eta_j),
\]
\[
\phi(\alpha_3, d_0) = 0, \quad \phi(\alpha_3, a_j') = 0.
\]
Let us introduce new characters \( c_j' \) \((j = 1, 2, \ldots, 2N)\) defined by
\[
c_j' = a_j, \quad c_{N+j}' = b_j \quad \text{for} \quad j = 1, 2, \ldots, N,
\]
in order to shorten the expressions of \( f_0, f' \) and the differential formulae, which are now expressed by
\[
f_0 = \phi(c_1', c_2', \ldots, c_{2N}), \quad (97)
\]
\[
f' = \phi(d_0, \beta_0', c_1', c_2', \ldots, c_{2N}), \quad (98)
\]
\[
\frac{\partial}{\partial t} \phi(c_j', c_k') = \phi(\alpha_3, \beta_0', c_j', c_k'), \quad (99)
\]
\[
\frac{\partial}{\partial t} \phi(d_0, \beta_0', c_j', c_k') = \phi(\alpha_3, d_0', c_j', c_k'). \quad (100)
\]
Following the procedures described in Appendix A, we obtain the following differential forms of \( f_0 \) and \( f' \):
\[
\frac{\partial}{\partial t} f_0 = \phi(\alpha_3, \beta_0', c_1', c_2', \ldots, c_{2N}), \quad (101)
\]
\[
\frac{\partial}{\partial t} f' = \phi(\alpha_3, d_0', c_1', c_2', \ldots, c_{2N}). \quad (102)
\]
Accordingly we find
\[
\frac{\partial}{\partial t} f = \frac{\partial}{\partial t} (f_0 + f') = \phi(\alpha_3, d_0', c_1', c_2', \ldots, c_{2N}). \quad (103)
\]
We expand Eq. (103) with respect to the first character \( \alpha_3 \),
\[
\frac{\partial}{\partial t} f = \sum_{j=1}^{N} -p_j^3 (-1)^{N+j} \exp(\eta_j) \phi(d_0, c_1', c_2', \ldots, c_N', c_{N+j}', \ldots, c_{2N})
\]
\[
= \sum_{j=1}^{N} -p_j^3 (-1)^{N+j} \phi(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_j, \ldots, b_N). \quad (104)
\]
On the other hand expanding \( g_{\mu} \) with respect to the final character \( \beta_\mu \), we obtain
\[ g_\mu = \text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu) \]  
\[ = \sum_{j=1}^{N} c_\mu(j)(-1)^{N+j}\text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, \hat{b}_j, \ldots, b_N). \]  

The sum of \( g_\mu \) over \( \mu \) gives
\[ \sum_{\mu=1}^{M} g_\mu = -\sum_{j=1}^{N} p_j^3(-1)^{N+j}\text{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, \hat{b}_j, \ldots, b_N), \]  
which is equal to Eq. (105). Accordingly we have shown that \( f \) and \( g_\mu \) satisfy the linear equation
\[ \frac{\partial f}{\partial t} = \sum_{\mu=1}^{M} g_\mu. \]  

5. Bilinear Bäcklund transformation and Lax pair for (28)–(29)

Same as in Section 3, we will firstly present a bilinear Bäcklund transformation for Eqs. (24)–(25). In fact, concerning Eqs. (24)–(25), we have the following Bäcklund transformation
\[ D_x(g_i \cdot f' - f \cdot g_i') - \lambda_i D_x f \cdot f' = 0, \quad i = 1, 2, \ldots, M, \]  
\[ (D_t + D_x^3)f \cdot f' = 0, \]  
\[ (D_t + D_x^3)(g_i \cdot f' + f \cdot g_i') = 0, \quad i = 1, 2, \ldots, M, \]  
between Eqs. (24)–(25) and
\[ (D_t + D_x^3)g_j' \cdot f' = 0, \quad j = 1, 2, \ldots, M, \]  
\[ \sum_{j=1}^{M} g_j' = f_t', \]  
where we have assumed that \( g_j \) and \( g_j' \) are always chosen to satisfy (25) and (114) and \( \lambda_i \) \( (i = 1, 2, \ldots, M) \) are arbitrary constants. Starting from (110)–(112) with (25) and (114), we can derive a Lax pair for (28) and (29). To this end, set
\[ f = \phi f', \quad g_i = \psi_i f' + \phi g_i', \quad v_i = \frac{2D_x g_i' \cdot f'}{f'^2}, \]  
\[ c \cdot v = \frac{D_x D_t f' \cdot f'}{f'^2}, \quad u = 2(\ln f')_{xx}, \quad \sum_{i=1}^{M} \lambda_i = \lambda. \]  
Then from Bäcklund transformation (110)–(112) with (25) and (114), we can deduce that
\[
\left( \begin{array}{c}
\vec{\psi} \\
\phi \\
\phi_x \\
\phi_{xx} \\
\phi_{xxx}
\end{array} \right)_x = U \left( \begin{array}{c}
\vec{\psi} \\
\phi \\
\phi_x \\
\phi_{xx} \\
\phi_{xxx}
\end{array} \right),
\]
where
\[
V = \begin{pmatrix}
0_{M \times M} & -u^T & 0^T \\
0 & K & 0^T \\
0 & 0 & 0
\end{pmatrix},
\]
and \(0_{M \times M}\) is an \(M \times M\) zero matrix. We can check that their compatibility condition
\[
U_t - V_x + UV - VU = 0
\]
gives the vector Ito equation (28)–(29). It is also remarked that from Bäcklund transformation (110)–(112) with (25) and (114), and by the dependent variable transformation
\[
w_i = \frac{g_i}{f} - \frac{g'_i}{f'}, \quad p = (\ln f/f')_x, \quad q = (\ln f f')_x,
\]
we can derive the following coupled system:
\[
p_t = \sum_{k=1}^M w_k x, \quad (117)
\]
\[
q_t = p \sum_{k=1}^M (\lambda_k - w_k), \quad (118)
\]
\[
\frac{w_{it} + w_{ixx} + 3\lambda_i p p_x - 3 p p_x w_i + 3 w_{ix} q_x}{x} = 0. \quad (119)
\]

6. Conclusion and discussions

In this paper, Hirota’s bilinear formalism has been utilized to generate a vector potential KdV equation and vector Ito equation. Soliton solutions expressed by Pfaffians have been obtained. Besides, bilinear Bäcklund transformations and the corresponding Lax pairs for the vector potential KdV equation and the vector Ito equation have been derived. It is noted that Eqs. (12) and (19) are two special cases of the coupled KdV equation.
\[
\frac{\partial u_i}{\partial t} + 6a \left( \sum_{k=1}^{N} u_k \right) \frac{\partial u_i}{\partial x} + 6(1-a) \left( \sum_{k=1}^{N} \frac{\partial u_k}{\partial x} \right) u_i + \frac{\partial^3 u_i}{\partial x^3} = 0, \quad i = 1, 2, \ldots, N. \quad (120)
\]

Now it is quite natural for us to ask if there is any other choice of \(a\) such that the corresponding system (120) is integrable. A detailed calculation and analysis show that the system (120) passes Painlevé test [9] iff \(a = 1\) or \(a = 1/2\) or \(a = 3/2\). Obviously, the system (120) with \(a = 1\) and \(a = 1/2\) corresponds to Eqs. (12) and (19), respectively. However, it remains an open problem as to whether Eq. (120) with \(a = 3/2\) has a BT and Lax pair. Besides, we can also show that the vector Ito equation (28)–(29) passes Painlevé test. Finally, it is remarked that the bilinear procedure to generate the vector potential KdV and vector Ito equations in this paper may be utilized to derive integrable coupled versions for some other integrable equations. The work in this direction is in progress.

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Appendix A. Properties of Pfaffians

A.1. Pfaffian identity

We show the following identity of Pfaffians:

\[
\text{pf}(\beta_0, c_0, c_1, \ldots, c_N) \text{pf}(\beta_1, \beta_2, \beta_3, c_0, c_1, \ldots, c_N) \\
- \text{pf}(\beta_1, c_0, c_1, \ldots, c_N) \text{pf}(\beta_0, \beta_2, \beta_3, c_0, c_1, \ldots, c_N) \\
+ \text{pf}(\beta_2, c_0, c_1, \ldots, c_N) \text{pf}(\beta_0, \beta_1, \beta_3, c_0, c_1, \ldots, c_N) \\
- \text{pf}(\beta_3, c_0, c_1, \ldots, c_N) \text{pf}(\beta_0, \beta_1, \beta_2, c_0, c_1, \ldots, c_N) = 0 \quad (A.1)
\]

for an even integer \(N\), where all entries \(\text{pf}(\beta_{\mu}, \beta_{\nu})\), \(\text{pf}(\beta_{\mu}, c_j)\) and \(\text{pf}(c_j, c_k)\) for \(\mu, \nu = 0, 1, 2, 3\) and \(j, k = 0, 1, 2, \ldots, N\), are defined to be arbitrary without any conditions on them.

The identity is proved by using the simple identity after Ohta (Y. Ohta, Bilinear theory of soliton, Ph.D. thesis, Faculty of Engineering, Tokyo University, 1992):

\[
\sum_{j=0}^{M} (-1)^j \text{pf}(b_0, b_1, \ldots, \hat{b}_j, \ldots, b_M) \text{pf}(b_j, c_0, c_1, \ldots, c_N) \\
= \sum_{k=0}^{N} (-1)^k \text{pf}(b_0, b_1, \ldots, b_M, c_k) \text{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N), \quad (A.2)
\]
which is proved simply as follows. Expanding Pfaffian \( \text{pf}(b_j, c_0, c_1, \ldots, c_N) \) on the left-hand side with respect to the first character \( b_j \), and \( \text{pf}(b_0, b_1, \ldots, b_M, c_k) \) on the right-hand side with respect to the final character \( c_k \), we obtain

\[
\sum_{j=0}^{M} (-1)^j \sum_{k=0}^{N} (-1)^k \text{pf}(b_0, b_1, \ldots, b_M, c_k) \text{pf}(b_j, c_k) \text{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N)
= \sum_{k=0}^{N} (-1)^k \sum_{j=0}^{M} (-1)^j \text{pf}(b_0, b_1, \ldots, \hat{b}_j, \ldots, b_M) \text{pf}(b_j, c_k) \text{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N),
\]

which is nothing but a trivial identity obtained by interchanging the order of the summations over \( j \) and \( k \). Accordingly, the identity (A.2) is proved.

The identity (A.2) is known to generate a variety of Pfaffian identities by selecting numbers \( M, N \) and characters \( b_j, c_k \) appropriately [10].

In the present case we choose \( M = N + 4 \) and

\[
b_j = \beta_j \text{ for } j = 0, 1, 2, 3, \\
b_{j+4} = c_j \text{ for } j = 0, 1, 2, \ldots, N.
\]

Then Eq. (A.2) results in the equation

\[
\sum_{j=0}^{3} (-1)^j \text{pf}(\beta_0, \ldots, \hat{\beta_j}, \ldots, \beta_3, c_0, c_1, \ldots, c_N) \text{pf}(\beta_j, c_0, c_1, \ldots, c_N)
+ \sum_{j=0}^{N} (-1)^j \text{pf}(\beta_0, \beta_1, \beta_2, \beta_3, c_0, c_1, \ldots, \hat{c}_j, \ldots, c_N) \text{pf}(c_j, c_0, c_1, \ldots, c_N)
= \sum_{k=0}^{N} (-1)^k \text{pf}(\beta_0, \beta_1, \beta_2, \beta_3, c_0, c_1, \ldots, c_N, c_k) \text{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N).
\]

The above choice makes summands of Eq. (A.4) 0 except the first term,

\[
\sum_{j=0}^{3} (-1)^j \text{pf}(\beta_0, \ldots, \hat{\beta_j}, \ldots, \beta_3, c_0, c_1, \ldots, c_N) \text{pf}(\beta_j, c_0, c_1, \ldots, c_N) = 0,
\]

which is the identity (A.1).

A.2. Expansion formulae

Expansion formulae for Pfaffians are described in [10]. We present them for the completeness of the present paper.

The Pfaffian \( \text{pf}(\alpha_1, \alpha_2, c_1, c_2, \ldots, c_{2n}) \) is, if \( \text{pf}(\alpha_1, \alpha_2) = 0 \), expanded in the following forms:
Next, the Pfaffians $pf(\alpha_1, \alpha_2, c_1, c_2, \ldots, c_{2n})$
\[= \sum_{1 \leq j < k \leq 2n} (-1)^{j+k-1} pf(\alpha_1, \alpha_2, c_1, c_2, \cdots, \hat{c}_j, \cdots, \hat{c}_k, \cdots, c_{2n}),\]

(ii) $pf(\alpha_1, \alpha_2, c_1, c_2, \ldots, c_{2n})$
\[= \sum_{j=2}^{2n} (-1)^j \left[ pf(\alpha_1, \alpha_2, c_1, c_j) pf(c_2, \cdots, \hat{c}_j, \cdots, c_{2n}) \right. \]
\[\left. + pf(c_1, c_j) pf(\alpha_1, \alpha_2, c_2, \cdots, \hat{c}_j, \cdots, c_{2n}) \right].\]

The expansion formula (i) is shown as follows. Expanding the Pfaffian $pf(\alpha_1, \alpha_2, c_1, c_2, \ldots, c_{2n})$ first with respect to $\alpha_2$ and next to $\alpha_1$, we have
\[pf(\alpha_1, \alpha_2, c_1, c_2, \ldots, c_{2n})\]
\[= \sum_{j=1}^{2n} \sum_{k=1}^{2n} (-1)^{j+k} pf(\alpha_1, c_j) pf(\alpha_2, c_k) \epsilon_{jk} pf(c_1, c_2, \cdots, \hat{c}_j, \cdots, \hat{c}_k, \cdots, c_{2n})\]
\[= \sum_{1 \leq j < k \leq 2n} (-1)^{j+k} \left[ pf(\alpha_1, c_j) pf(\alpha_2, c_k) - pf(\alpha_1, c_k) pf(\alpha_2, c_j) \right] \]
\[\times pf(c_1, c_2, \cdots, \hat{c}_j, \cdots, \hat{c}_k, \cdots, c_{2n}),\]

where
\[\epsilon_{j,k} = \begin{cases} 
1 & \text{for } j < k, \\
0 & \text{for } j = k, \\
-1 & \text{for } j > k.
\end{cases}\]

Noticing the relation $pf(\alpha_1, \alpha_2) = 0$, we obtain
\[= \sum_{1 \leq j < k \leq 2n} (-1)^{j+k-1} pf(\alpha_1, \alpha_2, c_j, c_k) pf(c_1, c_2, \cdots, \hat{c}_j, \cdots, \hat{c}_k, \cdots, c_{2n}).\]

The expansion formula (ii) is obtained by expanding the Pfaffian $pf(\alpha_1, \alpha_2, c_1, c_2, \ldots, c_{2n})$ with respect to the character $c_1$
\[pf(\alpha_1, \alpha_2, c_1, c_2, \ldots, c_{2n})\]
\[= pf(c_1, \alpha_1) pf(\alpha_2, c_2, \ldots, c_{2n}) - pf(c_1, \alpha_2) pf(\alpha_1, c_2, \ldots, c_{2n})\]
\[+ \sum_{j=2}^{2n} (-1)^j pf(c_1, c_j) pf(\alpha_1, \alpha_2, c_2, \cdots, \hat{c}_j, \cdots, c_{2n}).\]

Next, the Pfaffians $pf(\alpha_2, c_2, \ldots, c_{2n})$, $pf(\alpha_1, c_2, \ldots, c_{2n})$ are expanded as
\[= pf(c_1, \alpha_1) \sum_{j=2}^{2n} (-1)^j pf(\alpha_2, c_j) pf(c_2, \cdots, \hat{c}_j, \cdots, c_{2n})\]
\[ - pf(c_1, \alpha_2) \sum_{j=2}^{2n} (-1)^j pf(\alpha_1, c_j) pf(c_2, \cdots, \hat{c}_j, \cdots, c_{2n})\]
\[ + \sum_{j=2}^{2n} (-1)^j \text{pf}(c_1, c_j) \text{pf}(\alpha_1, \alpha_2, c_2, c_3, \ldots, \hat{c}_j, \ldots, c_{2n}). \]

Noticing the relation \( \text{pf}(\alpha_1, \alpha_2) = 0 \), we obtain

\[ = \sum_{j=2}^{2n} (-1)^j \left[ \text{pf}(\alpha_1, \alpha_2, c_1, c_j) \text{pf}(c_2, \ldots, \hat{c}_j, \ldots, c_{2n}) \right. \]
\[ + \text{pf}(c_1, c_j) \text{pf}(\alpha_1, \alpha_2, c_2, \ldots, \hat{c}_j, \ldots, c_{2n}) \].

If we replace the Pfaffian \( \text{pf}(c_1, c_2, \ldots, c_{2n}) \) in the expansion formulae (i) by a Pfaffian \( \text{pf}(\beta_1, \beta_2, c_1, c_2, \ldots, c_{2n}) \), this formula can be generalized as follows:

(iii) \[ \text{pf}(\alpha_1, \alpha_2, \beta_1, \beta_2, c_1, c_2, \ldots, c_{2n}) \]
\[ = \sum_{1 \leq j, k \leq 2n} (-1)^{j+k-1} \text{pf}(\alpha_1, \alpha_2, c_j, c_k) \]
\[ \times \text{pf}(\beta_1, \beta_2, c_1, c_2, \ldots, \hat{c}_j, \ldots, \hat{c}_k, \ldots, c_{2n}), \]

provided that \( \text{pf}(\alpha_j, \beta_k) = 0 \) for \( j, k = 1, 2 \) and \( \text{pf}(\beta_1, \beta_2) = 0 \).

A.3. Transformation of the Pfaffian

We show the relation

\[ \text{pf}(d_0, d_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N) = -\text{pf}(d_0, \beta_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N), \]  

(A.5)

where the entries are defined by

- \( \text{pf}(d_0, d_i) = 0 \), \( \text{pf}(d_0, a_j) = \exp \eta_j \), \( \text{pf}(d_0, b_j) = -1 \),
- \( \text{pf}(d_1, a_j) = p_j \exp \eta_j \), \( \text{pf}(d_1, b_j) = 0 \),
- \( \text{pf}(a_j, a_k) = -a_{j,k} \exp (\eta_j + \eta_k) \), \( \text{pf}(a_j, b_k) = \delta_{j,k} \),
- \( \text{pf}(b_j, b_k) = a_{j,k} \), \( a_{j,k} = (p_j - p_k)/(p_j + p_k) \),
- \( \text{pf}(d_0, \beta_1) = 0 \), \( \text{pf}(\beta_1, a_k) = 0 \), \( \text{pf}(\beta_1, b_j) = -p_j \).

Let us introduce a new Pfaffian expressed by

\[ P = \text{pf}(d'_0, d'_1, a'_1, a'_2, \ldots, a'_N, b'_1, b'_2, \ldots, b'_N). \]  

(A.6)

Put

\[ d'_0 = i d_0, \quad i = \sqrt{-1}, \]  
\[ d'_1 = i d_1, \]  
\[ a'_j = i a_j \exp (-\eta_j) \quad \text{for} \quad j = 1, 2, \ldots, N, \]  
\[ b'_j = i b_j \exp \eta_j \quad \text{for} \quad j = 1, 2, \ldots, N. \]  

(A.7) \quad (A.8) \quad (A.9) \quad (A.10)

Then we find the entries of the new Pfaffian are given by

\[ + \sum_{j=2}^{2n} (-1)^j \text{pf}(c_1, c_j) \text{pf}(\alpha_1, \alpha_2, c_2, c_3, \ldots, \hat{c}_j, \ldots, c_{2n}). \]
\( \text{pf}(d_0', d_1') = 0 = \text{pf}(d_0, \beta_1) \), \hspace{1cm} (A.11) \\
\( \text{pf}(d_0', a_j') = -1 = \text{pf}(d_0, b_j) \), \hspace{1cm} (A.12) \\
\( \text{pf}(d_0', b_j') = \exp \eta_j = \text{pf}(d_0, a_j) \), \hspace{1cm} (A.13) \\
\( \text{pf}(a_j', b_j') = -\delta_{j,k} = \text{pf}(b_j, a_j) \), \hspace{1cm} (A.14) \\
\( \text{pf}(a_j', a_j') = -p_j = \text{pf}(\beta_1, b_j) \), \hspace{1cm} (A.15) \\
\( \text{pf}(d_1', b_j') = 0 = \text{pf}(\beta_1, a_j) \). \hspace{1cm} (A.16)

which shows that the roles played by the characters \( d_0', d_1', a_j', b_j' \) are the same as those played by the characters \( d_0, \beta_1, b_j, a_j \), respectively. Accordingly \( P \) is written as

\[
P = \text{pf}(d_0', d_1', a_1', a_2', \ldots, a_N', b_1', b_2', \ldots, b_N').
\]

On the other hand, substituting the expressions (A.7)–(A.10) directly into \( P \) we obtain

\[
P = (-1)^N \text{pf}(d_0, \beta_1, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N).
\]

Hence, we have shown the relation (A.5).

\subsection*{A.4. Differential formulae}

Let us introduce new characters \( c_j \ (j = 1, 2, \ldots, 2N) \) defined by

\[
c_j = a_j, \hspace{1cm} c_{N+j} = b_j \quad \text{for} \ j = 1, 2, \ldots, N,
\]

in order to shorten the expression of \( f \), which is expressed now by

\[
f = \text{pf}(d_0, c_1, c_2, \ldots, c_{2N}, \beta_0).
\]

Furthermore we introduce a new character \( \beta_0' \) defined by

\[
\text{pf}(d_0, \beta_0') = 0 = \text{pf}(d_0, \beta_0) \hspace{1cm} (\text{pf}(d_0, \beta_0) = 1),
\]

\[
\text{pf}(c_j, \beta_0') = \text{pf}(c_j, \beta_0) \hspace{1cm} \text{for} \ j = 1, 2, \ldots, 2N,
\]

in order to utilize the expansion formulae (i)–(iii). Then \( f \) is expressed by

\[
f = f_0 + f',
\]

\[
f_0 = \text{pf}(c_1, c_2, \ldots, c_{2N}),
\]

\[
f' = \text{pf}(d_0, \beta_0', c_1, c_2, \ldots, c_{2N}).
\]

Let us introduce another character \( d_0' \ (= d_0 - \beta_0') \) so that

\[
\text{pf}(d_0', \ldots) = \text{pf}(d_0, \ldots) - \text{pf}(\beta_0', \ldots)
\]

for an arbitrary list \( \{ \ldots \} \).
We investigate firstly the differential formulae of the entries \( pf(c_j, c_k) \), \( pf(d_0, \beta'_0, c_j, c_k) \) and \( pf(d_0, \beta'_\mu, c_j, c_k) \) before studying the differential formulae of \( f \) and \( g_\mu \).

We find the following differential formulae hold:

(i) \[ \frac{\partial}{\partial x} pf(c_j, c_k) = -pf(d'_0, d_1, c_j, c_k), \]

(ii) \[ \frac{\partial}{\partial t} pf(c_j, c_k) = pf(d'_0, d_3, c_j, c_k) - 2pf(d_1, d_2, c_j, c_k), \]

(iii) \[ \frac{\partial}{\partial x} pf(d_0, \beta'_0, c_j, c_k) = pf(d_1, \beta'_0, c_j, c_k), \]

(iv) \[ \frac{\partial}{\partial t} pf(d_0, \beta'_0, c_j, c_k) = -pf(d_3, \beta'_0, c_j, c_k), \]

(v) \[ \frac{\partial}{\partial x} pf(d_0, d_1, c_j, c_k) = pf(d_1, c_j, c_k) - pf(d_0, d_2, c_j, c_k), \]

(vi) \[ \frac{\partial}{\partial t} pf(d_0, d_2, c_j, c_k) = pf(d_0, d_3, c_j, c_k) + pf(d_1, d_2, c_j, c_k), \]

(vii) \[ \frac{\partial}{\partial x} pf(d_0, \beta'_\mu, c_j, c_k) = pf(d_1, \beta'_\mu, c_j, c_k), \]

(viii) \[ \frac{\partial}{\partial t} pf(d_0, \beta'_\mu, c_j, c_k) = -pf(d_3, \beta'_\mu, c_j, c_k), \]

(ix) \[ \frac{\partial}{\partial x} pf(d_1, \beta'_\mu, c_j, c_k) = pf(d_2, \beta'_\mu, c_j, c_k), \]

(x) \[ \frac{\partial}{\partial x} pf(d_2, \beta'_\mu, c_j, c_k) = pf(d_3, \beta'_\mu, c_j, c_k) \]

for \( j, k = 1, 2, \ldots, 2N \), where the new entries of the Pfaffians are defined as follows:

\[
\begin{align*}
pf(d_j, d_k) &= 0, \\
pf(d_j, \beta'_0) &= \delta_{j,0}, \\
pf(d_j, \beta'_\mu) &= 0, \\
pf(d_j, a_k) &= p_j^k \exp \eta_k, \\
pf(d_j, b_k) &= -\delta_{j,0}
\end{align*}
\]

(A.26)

for \( j = 0, 1, 2, 3, k = 1, 2, \ldots, N \) and \( \mu = 1, 2, \ldots, M \).

These differential formulae are shown very simply considering the individual cases separately.

Take the formula (i), for example.

(a) We have, for \( j, k = 1, 2, \ldots, N \),

\[ pf(c_j, c_k) = pf(a_j, a_k) = -a_{j,k} \exp (\eta_j + \eta_k). \]

Differentiating it we obtain

\[ \frac{\partial}{\partial x} pf(a_j, a_k) = -(p_j - p_k) \exp (\eta_j + \eta_k). \]

On the other hand the r.h.s. of (i) is reduced, by noting \( pf(\beta'_0, a_j) = 0 \), to

\[
- pf(d_0, d_1, a_j, a_k) + pf(\beta'_0, d_1, a_j, a_k)
\]

\[
= pf(d_0, a_j) pf(d_1, a_k) - pf(d_0, a_k) pf(d_1, a_j)
\]

\[
= -(p_j - p_k) \exp (\eta_j + \eta_k).
\]

Hence the formula (i) holds for \( j, k = 1, 2, \ldots, N \).
(b) We have, for \( j = 1, 2, \ldots, N \) and \( k = N + 1, N + 2, \ldots, 2N \),
\[
\text{pf}(c_j, c_k) = \text{pf}(a_j, b_k) = \delta_{j,k}.
\]
Accordingly
\[
\frac{\partial}{\partial x} \text{pf}(a_j, b_k) = 0.
\]
On the other hand the r.h.s. of (i) is reduced, by noting \( \text{pf}(d_0, b_k) = \text{pf}(\beta'_0, b_k) = -1 \), to
\[
-\text{pf}(d'_0, d_1, a_j, b_k) = -\text{pf}(d_0, d_1, a_j, b_k) + \text{pf}(\beta'_0, d_1, a_j, b_k) = 0.
\]
Hence the formula (i) holds for \( j = 1, 2, \ldots, N \) and \( k = N + 1, N + 2, \ldots, 2N \).
(c) We have for \( j, k = N + 1, N + 2, \ldots, 2N \),
\[
\text{pf}(c_j, c_k) = \text{pf}(b_j, b_k) = a_{j,k}.
\]
Accordingly
\[
\frac{\partial}{\partial x} \text{pf}(b_j, b_k) = 0.
\]
On the other hand the r.h.s. of (i) vanishes because \( \text{pf}(d_1, b_k) = 0 \). Hence the formula (i) holds for \( j, k = N + 1, N + 2, \ldots, 2N \).

Thus we have shown that the formula (i) holds for \( j, k = 1, 2, \ldots, 2N \).

The other formulae (ii)–(x) are shown by using the same procedure.

The differential formulae of \( f_0 \), \( f' \) and \( g_\mu \) are obtained by the following procedure (see [10] for other equations).

(1) Expand the Pfaffian using the expansion formulae.
(2) Express a derivative of the expanded Pfaffian in terms of Pfaffians using the formulae (i)–(x) and employing an induction if necessary.
(3) Simplify the derivative of the Pfaffian using the expansion formula.

We take a differential formula of \( f_0 = \text{pf}(c_1, c_2, \ldots, c_{2n}) \) with respect to \( x \) as an example.

(1) We expand \( f_0 \) with respect to letter \( c_1 \),
\[
f_0 = \sum_{j=2}^{2n} \text{pf}(c_1, c_j)(-1)^j \text{pf}(c_2, \ldots, \hat{c}_j, \ldots, c_{2n}). \tag{A.28}
\]
(2) A derivative of \( f_0 \) is
\[
\frac{\partial}{\partial x} f_0 = \sum_{j=2}^{2n} \left\{ \frac{\partial}{\partial x} \text{pf}(c_1, c_j) \right\} (-1)^j \text{pf}(c_2, \ldots, \hat{c}_j, \ldots, c_{2n})
\]
\[
+ \text{pf}(c_1, c_j)(-1)^j \frac{\partial}{\partial x} \text{pf}(c_2, \ldots, \hat{c}_j, \ldots, c_{2n}) \}. \tag{A.29}
\]
We now employ an induction. If \( n = 1 \) the differential formula is equal to the formula (i). Under the assumption that the differential formula holds if \( n = n - 1 \), we have
\[ \frac{\partial}{\partial x} f_0 = \sum_{j=2}^{2n} \left\{ \left[ -pf(d'_0, d_1, c_1, \ldots, c_j) \right] (-1)^j pf(c_2, \ldots, \hat{c}_j, \ldots, c_{2n}) - pf(c_1, c_j) (-1)^j pf(d'_0, d_1, c_2, \ldots, \hat{c}_j, \ldots, c_{2n}) \right\}. \] (A.30)

(3) The r.h.s is reduced, due to the expansion formula (ii), to
\[ = - pf(d'_0, d_1, c_1, c_2, \ldots, c_{2n}). \] (A.31)

Therefore the differential formula holds for an arbitrary \( n \). Thus we have shown the differential formula
\[ \frac{\partial}{\partial x} f_0 = - pf(d'_0, d_1, c_1, c_2, \ldots, c_{2n}). \] (A.32)

Next, we take a differential formula of \( f' = pf(d_0, \beta'_0, c_1, c_2, \ldots, c_{2n}) \) with respect to \( x \) as another example.

(1) We expand \( f' \) using the expansion formula (i),
\[ f' = \sum_{1 \leq j, k \leq 2n} pf(d_0, \beta'_0, c_j, c_k) (-1)^{j+k-1} pf(c_1, c_2, \ldots, \hat{c}_j, \ldots, \hat{c}_k, \ldots, c_{2n}). \] (A.33)

(2) We obtain, using the differential formula (iii) and Eq. (A.32), a derivative of \( f' \),
\[ \frac{\partial}{\partial x} f' = \sum_{1 \leq j, k \leq 2n} (-1)^{j+k-1} \left\{ pf(d_1, \beta'_0, c_j, c_k) pf(c_1, c_2, \ldots, \hat{c}_j, \ldots, \hat{c}_k, \ldots, c_{2n}) - pf(d_0, \beta'_0, c_j, c_k) pf(d'_0, d_1, c_2, \ldots, \hat{c}_j, \ldots, \hat{c}_k, \ldots, c_{2n}) \right\}. \] (A.34)

(3) The r.h.s is reduced, due to the expansion formula (ii) and Eq. (A.25), to
\[ = pf(d_1, \beta'_0, c_1, c_2, \ldots, c_{2n}). \] (A.35)

Hence we have shown the differential formula
\[ \frac{\partial}{\partial x} f' = pf(d_1, \beta'_0, c_1, c_2, \ldots, c_{2n}). \] (A.36)

The differential formulae (A.32) and (A.35) give the differential formula of \( f \) (\( = f_0 + f' \)),
\[ \frac{\partial}{\partial x} f = - pf(d_0, d_1, c_1, c_2, \ldots, c_{2n}), \] (A.37)

which is one of the differential formulae used in Section 3.

The other differential formulae listed in Section 3 are obtained by using the same procedure.

References
