Topology Vol. 23, No. 1, pp. 117-125, 1984
Printed in Great Britain.
0040-9383/84 $3.00 + .00

INCOMPRESSIBLE SURFACES VIA BRANCHED SURFACES

W. FLOYD† and U. OERTEL

(*Received* 25 *September* 1981)

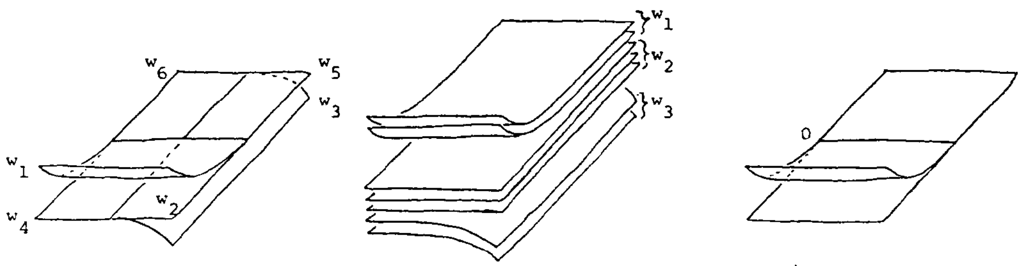
A COMPACT, irreducible, orientable 3-manifold which contains a 2-sided incompressible surface is called a *Haken* manifold. While a Haken manifold may contain an infinite number of nonisotopic, incompressible surfaces, Haken showed that one can generate all isotopy classes of incompressible, ∂-incompressible surfaces from a finite set of surfaces by certain cut-and-paste operations (See [3] or [7].)

However, these operations also yield surfaces which are not incompressible. In this paper we use branched surfaces to produce exactly the 2-sided, incompressible, ∂-incompressible surfaces (up to isotopy) in a Haken 3-manifold. Our approach parallels Haken's to some extent, because we obtain a finiteness statement by putting incompressible, ∂-incompressible surfaces in Haken's normal form relative to a fixed handle decomposition of the 3-manifold.

A branched surface B in a 3-manifold M^3 is a subspace locally modelled on the space $\mathscr{U}$ shown in Fig. 1(a) (locally modelled on the space $\mathscr{U}^+$ of Fig. 1(c) near ∂M). A surface S properly embedded in M is carried by B if S can be isotoped so that it runs nearly parallel to B, as indicated in Fig. 1. (A precise definition will be given in §1.) If S is carried by B, then the number of sheets of S near a point in B determines an integer weight for each component of the complement of the branch locus in B. The weights satisfy certain obvious conditions: in Fig. 1(a), $w_2 + w_3 = w_4$, $w_1 + w_2 = w_5$, etc. Conversely, given a set of weights on B satisfying these additive conditions, one can construct a corresponding surface carried by B by glueing parallel copies of the components of B-(branch locus) along the branch locus.

The motivation for using branched surfaces comes from Thurston's use of train tracks (branched 1-manifolds) to study simple closed curves on a compact, orientable surface S. In [9] he defines what it means for a curve system to be "carried" by a train track and gives conditions on a train track which imply that no curve it carries is null-homotopic. He then shows there are two such train tracks which carry all (up to isotopy) of the non-trivial simple closed curves on S.

THEOREM 1. *Let M be a Haken 3-manifold with incompressible boundary. There are a*



(a) $\mathscr{U}$ (b) S (c) $\mathscr{U}^+$

†Supported in part by an NSF grant.

finite number of branched surfaces, $B_1, \ldots, B_k$, properly embedded in M such that (a) each surface carried with positive weights by one of the B_i's is ∂-injective and injective, and (b) every two-sided, incompressible, ∂-incompressible surface in M is isotopic to a surface carried by one of the B_i's with positive weights.

In §1 we give definitions and an example. Section 2 gives conditions on a branched surface so that all of the surfaces it carries with positive weights are ∂-injective and injective. In §3 we use Haken's normal surface theory to find a finite number of branched surfaces satisfying the appropriate conditions, thereby proving the theorem. In §4 we briefly mention some applications and make further remarks.

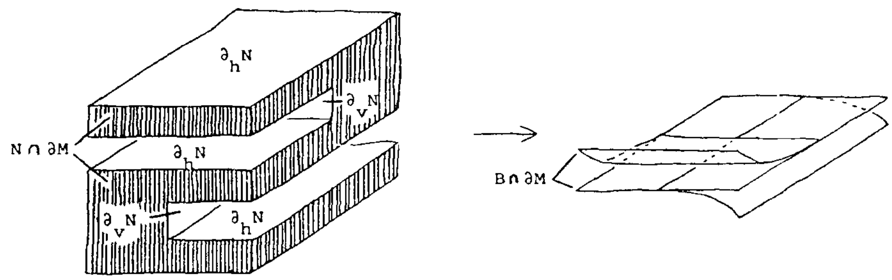
We would like to thank Allen Hatcher and William Thurston for conversations about branched surfaces. Allen Hatcher also helped in the simplification of statements and proofs. Finally, we would like to thank William Jaco for reading our manuscript and pointing out flaws and errors.

§1. DEFINITIONS

Unless stated otherwise, all manifolds and maps are smooth, and a surface $S \subset M^3$ is properly embedded. Let $S \subset M^3$ be a surface which is not S^2, $\mathbb{P}^2$, or a disk which can be pushed, rel boundary, into ∂M. A compressing disk for S is an embedded 2-disk D in M such that $D \cap S = \partial D$ and ∂D does not bound a disk in S. S is *incompressible* if it has no compressing disks. S is *injective* if the map $i_* : \pi_1(S) \to \pi_1(M)$ is an injection. A surface S is injective if and only if $\partial N(S)$ (the boundary of a regular neighborhood of S) is incompressible. S is *∂-incompressible* if for every disk $D \subset M$ with $\partial D = \alpha \cup \beta$, where $D \cap S = \alpha$ is a properly embedded arc and $D \cap \partial M = \beta$, there is a disk $D' \subset S$ with $\partial D' = \alpha \cup \beta'$ and $\beta' = D' \cap \partial S$. S is *∂-injective* if $\partial N(S)$ is ∂-incompressible. Clearly injective, ∂-injective surfaces are incompressible and ∂-incompressible.

A *branched surface fibered neighborhood* in M is a codimension zero, compact submanifold $N = N(B) \subset M^3$ foliated by intervals (fibers) in a way locally isomorphic to the model, foliated by vertical arcs, as shown in Fig. 2(a). The *branched surface B* itself is the quotient space of $N(B)$ obtained by identifying the interval fibers to points. B inherits naturally a smooth structure from $N(B)$, and with this structure B can be embedded in $N(B)$ transverse to the fibers so that the composition $B \to N(B) \to B$ is near the identity. The *branch locus* is $\{b \in B : b$ does not have a neighborhood homeomorphic to $\mathbb{R}^2$ or $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}\}$, $\partial B = B \cap \partial M$. With our definition, the branch locus of B is generic; i.e. it is self-transverse and intersects itself in double points only. (See Fig. 2b.) Any branched surface with non-generic branch locus can be replaced by another, equivalent for our purposes, having generic branch locus. A more general definition of a branched manifold is given in [10].

The boundary of the branched surface fibered neighborhood is the union of three



(a) N (b) B

Fig. 2.

compact subsurfaces $\partial_h N$, $\partial_v N$ and $N \cap \partial M$ which meet only in their common boundary points; a fiber of N meets $\partial_h N$ transversely at its endpoints, while a fiber of N intersects $\partial_v N$ (if at all) in a closed interval in the interior of the fiber. A surface $S \subset M$ is *carried by* B if S can be isotoped into $\mathring{N}$ so that it intersects the fibers transversely. S is carried by B *with positive weights* if S intersects every fiber.

Here is an example to illustrate how branched surfaces arise in studying incompressible surfaces. Let $M = S_g \times S^1$, where we are identifying S^1 with the unit circle in the complex plane and $S_g = S_g \times 1$ is a closed, orientable surface of genus $g \geq 1$. Let $\alpha \subset S_g$ be a nonseparating, simple closed curve, and $U \subset S_g$ a regular neighborhood of α with boundary components α_1 and α_2. Given $n \in \mathbb{N}$ we will construct a surface $S \subset M$ as follows. Let $W = S_g - \mathring{U}$, $V = \bigcup_{i=1,\ldots,n} W \times e^{2\pi i/n}$. S is obtained from V by adding n annuli in $U \times S^1$, so that the ith annulus $A_i \subset U \times e^{[2\pi i/n,\, 2\pi(i+1)/n]}$ with $\partial A_i = (\alpha_1 \times e^{2\pi i/n}) \cup (\alpha_2 \times e^{2\pi(i+1)/n})$. S has genus $n(g-1)+1$, and S is an incompressible surface in M(S is actually the fiber of a fibration of M over S^1). Although this construction yields a countably infinite family of incompressible surfaces in M, they are all carried by a single branched surface B. B is the union of S_g and an annulus $A \subset U \times S^1$, and the branch locus of B is $\alpha_1 \cup \alpha_2$. S is carried by B with weight n on $S_g - U$, weight $n-1$ on $\mathring{U}$, and weight 1 on $\mathring{A}$. In Fig. 3 we use branched 1-manifolds to give a schematic drawing of how S is carried by B.

§2. BRANCHED SURFACES AND INCOMPRESSIBLE SURFACES

Let M be a Haken 3-manifold with incompressible boundary, $B \subset M$ a properly embedded branched surface, and N a fibered neighborhood of B. B is *incompressible* if the following three conditions are satisfied:

(i) There is no disk $D \subset N$ such that D is transverse to the fibers and $\partial D \subset \operatorname{int}(\partial_v N)$; there is no disk $D \subset N$ such that D is transverse to the fibers with $\partial D = \alpha \cup \beta$, where $\mathring{\alpha} \subset \operatorname{int}(\partial_v N)$ and $\beta \subset \partial M$ are arcs and $\alpha \cap \beta = S^0$.

(ii) Each component of $\partial_h N$ is incompressible and ∂-incompressible in $M - \mathring{N}$.

(iii) There is no disk $D \subset M - \mathring{N}$ with $\partial D = D \cap N = \alpha \cup \beta$, where $\alpha \subset \partial_v N$ is a fiber (vertical arc) and $\beta \subset \partial_h N$.

Conditions (i)–(iii) are sufficient to ensure that the surfaces carried by B with positive weights are ∂-injective and injective, as will be shown below. Of the 3 conditions, (i) is for technical convenience and rules out disks of contact and half-disks of contact in B. Condition (iii) rules out monogons in $M - B$ (see Fig. 4). In condition (ii), notice that $\partial_h N$ is not properly embedded in $M - \mathring{N}D$ – by ∂-incompressible we mean the following: if there is a disk D in $M - \mathring{N}$ with $\mathring{D} \subset M - N$ and $\partial = \alpha \cup \beta$, where $\alpha \subset \partial_h N$ is a properly

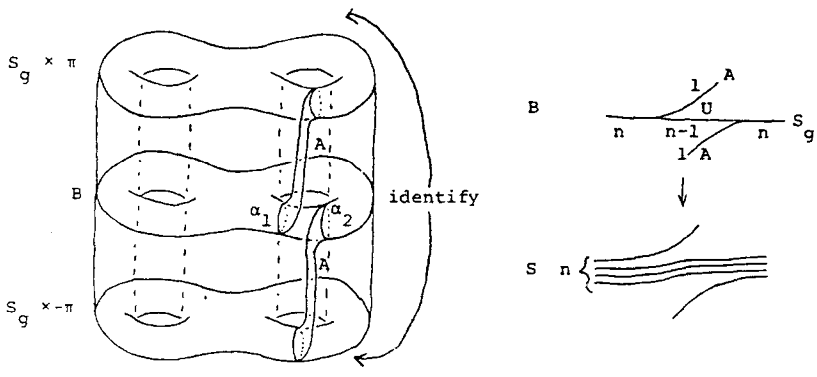


Fig. 3. Schematic view of $M = S_g \times S^1 = S_g \times [-\pi, \pi]/(x, -\pi) \sim (x, \pi)$.

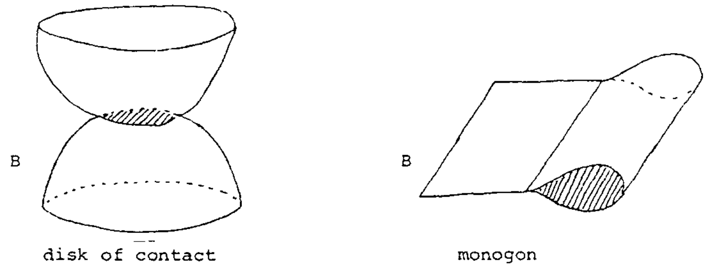


Fig. 4.

embedded arc and $\beta = D \cap \partial M$, then there is a disk $D' \subset \partial_h N$ with $\partial D' = \alpha \cup \beta'$, where $\beta' = D' \cup \partial M$.

THEOREM 2. *Let M be a Haken 3-manifold with incompressible boundary, and $B \subset M$ a properly embedded branched surface. If B is incompressible, then every surface carried by B with positive weights is ∂-injective and injective.*

Proof. Let N' be a fibered neighborhood of B, and let F be a surface carried by B with positive weights. To prove that F is injective, it suffices to show that $S = \partial N(F)$, the boundary of a regular neighborhood of F, is incompressible. Since S is carried by B, we can isotope S so that $S \subset \mathring{N}'$ and S is transverse to the fibers of N'. S intersects each fiber at least twice, so we can isotope S so that $S \subset N'$ and $\partial_h N' \subset S$ by pushing the sheets of S outermost in N' to $\partial_h N'$. Note that $N' - S$ is an interval bundle L, with fibers coming from the fibers of N'.

If the core of an annulus A of $\partial_v N'$ bounds a disk D in $M - \mathring{N}$, then by (ii) the two boundary components of A bound disks D_0 and D_1 in $\partial_h N'$. The 2-sphere $D_0 \cup A \cup D_1$ bounds a 3-ball in M (on the side containing D by (i)), so we extend N' and the interval bundle L to that 3-ball. Similarly, if there is a disk D with $\partial D = \alpha \cup \beta$, where $\beta \subset \partial M$ and α is the core (transverse to the fibers) of a rectangle $R \subset \partial_v N'$ corresponding to a branch arc of B, then we extend N' and the interval bundle L to the component of $M - N'$ containing D (this uses the ∂-incompressibility of $\partial_h N'$, the incompressibility of ∂M, and the irreducibility of M). If we denote the extended N' by N, then conditions (i)–(iii) hold for N.

Suppose that S is not incompressible, and let D be a compressing disk for S, with $\partial D = \alpha$. (It follows from (i) and (ii) that no component of S is S^2 or a boundary-parallel D^2.) We can assume that $D \cap \partial_v N$ consists of arcs and simple closed curves after isotoping D to make it transverse to $\partial_v N$. Suppose γ is an innermost circle of $D \cap \partial_n N$ bounding a disk E in D. γ cannot be isotopic to the core of an annulus of $\partial_v N$, since otherwise either $E \subset M - \mathring{N}$ (contradicting the construction of N from N') or $E \subset N$ (contradicting (i)). Thus γ bounds a disk in $\partial_v N$, and we can isotope D across this disk to eliminate γ from $D \cap \partial_v N$. We eliminate all circles of $D \cap \partial_v N$ in this way.

Suppose γ is an arc of $D \cap \partial_v N$ which is edgemost in D, cutting a disk E from D. Either γ is isotopic to a fiber of $\partial_v N$, or it is isotopic to an arc in S. γ cannot be isotopic to a fiber of $\partial_v N$, since otherwise either $E \subset M - \mathring{N}$ (contraidcting (iii)) or $E \subset N$ (this cannot occur since the pullback of the I-bundle L to ∂E would be non-trivial). If γ is not edgemost in $\partial_v N$, replace it by an arc of $D \cap \partial_v N$ which is edgemost in $\partial_v N$. When γ is isotoped to S, one of the two disks obtained by cutting D along γ is a compressing disk which intersects $\partial_v N$ in fewer arcs. Replace D by this disk.

When all arcs of $D \cap \partial_v N$ have been eliminated, we obtain a compressing disk D which does not intersect $\partial_v N$. If $D \subset M - \mathring{N}$, then D is a compressing disk for $\partial_h N$, contrary to (ii). D cannot be contained in N, since otherwise D would be a compressing disk for $S \cap \partial L$ in the I-bundle L. This completes the proof of incompressibility.

It remains to show that S is ∂-incompressible. Note that $d(B)$, the double of B, is a branched surface in $d(M)$, the double of M, which satisfies the hypotheses of Theorem 2. The above argument implies that $d(S)$, the double of S, is an incompressible surface in $d(M)$, and hence $S \subset M$ is ∂-incompressible. □

§3. FINITENESS

In this section we complete the proof of the main theorem that in a Haken 3-manifold M with incompressible boundary there are finitely many branched surfaces so that the two-sided surfaces they carry with positive weights are exactly the two-sided, incompressible, ∂-incompressible surfaces in M. Given a two-sided, incompressible, ∂-incompressible surface $S \subset M$, we isotope it to a normal form relative to a fixed handle decomposition of M using Haken's theory and construct a branched surface $B_S \subset M$ which carries S with positive weights. We then complete the proof by showing that B_S is incompressible and that only finitely many non-isotopic branched surfaces are produced by the construction.

Haken's results show that any incompressible, ∂-incompressible surface S in M can be isotoped to a normal form relative to a handle decomposition of $(M, \partial M)$ coming from a triangulation of M (see [7]).

$$M = \left(\bigcup_{j=1}^{r} H_j^0\right) \cup \left(\bigcup_{j=1}^{k} H_j^1\right) \cup \left(\bigcup_{j=1}^{l} H_j^2\right) \cup \left(\bigcup_{j=1}^{m} H_j^3\right),$$

and

$$\partial M = \left(\bigcup_{j=1}^{r} H_j^0 \cap \partial M\right) \cup \left(\bigcup_{j=1}^{k} H_j^1 \cap \partial M\right) \cup \left(\bigcup_{j=1}^{l} H_j^2 \cap \partial M\right).$$

S is in *normal form* if:

(1) S is disjoint from the 3-handles.

(2) ∂S is disjoint from the 2-handles of ∂M.

(3) S intersects each i-handle $H_j^i (i = 0, 1, 2)$ in a collection of disjoint disks whose boundaries are homotopically non-trivial in

$$\partial H_j^i - \left[\left(\bigcup_{j=1}^{m} H_j^3\right) \cup \left(\bigcup_{j=1}^{l} H_j^2 \cap \partial M\right)\right],$$

and the disks of $S \cap \partial H_j^1$ are isotopic to products $I \times \alpha$ in the product structure $H_j^1 = I \times D^2$ of H_j^1.

(4) ∂S intersects each $H_j^i \cap \partial M (i = 0, 1)$ in non-trivial arcs.

(5) S intersects the 0- and 1-handles of the induced handle decomposition of $\partial H_j^i - \partial M$ in non-trivial arcs disjoint from the 2-handles of $\partial H_j^i - \partial M$.

(6) Each circle of $S \cap \partial H_j^0$ intersects ∂M or any 2-handle H_k^2 in at most one arc.

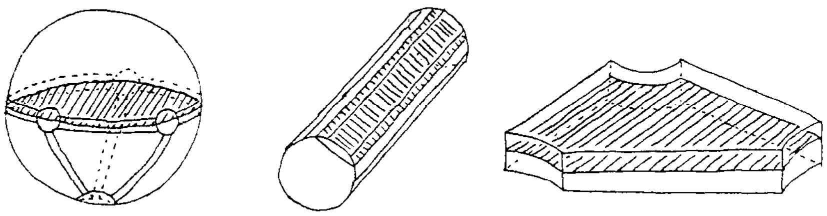
Figure 5 shows typical disks of $S \cap H_j^i$.

There are a finite number of possible *disk-types* in H_j^i, where two disks of $S \cap H_j^i$ are of the same type if their boundaries are isotopic in

$$\partial H_j^i - \left[\left(\bigcup_{n=1}^{m} H_n^3\right) \cup \left(\bigcup_{n=1}^{l} H_n^2 \cap \partial M\right)\right].$$

In particular, there is only one disk-type in each 2-handle. We define the *complexity* $\gamma(S)$ of a normal form of S to be the total number of disks in which S intersects the 2-handles.

Relative to the fixed handle decomposition of M, let S be in a normal form with



H_j^0 H_j^1 H_j^2

Fig. 5.

minimal complexity over all normal forms of S. Let $F = \partial N(S)$, where $N(S) \approx S \times I$ is a regular neighborhood of S. F is just two copies of S, and we can assume that F is also in a normal form of minimal complexity, with twice as many of each disk-type as S. We construct a branched surface B_S carrying S with positive weights. Roughly speaking, B_S will be the branched surface obtained by pushing disks of the same type toward each other and identifying them. The result is a branched surface with non-generic branch locus, but a slight restriction of the area of identification yields a branched surface with generic branch locus which carries the same surfaces. More precisely, we construct the branched surface fibered neighborhood $\hat{N}_S$ as follows. Given two adjacent disks E_0 and E_1 of $F \cap H_j^i$ ($i = 0, 1, 2$) of the same disk-type, the 3-ball between them in H_j^i can be given a product structure $E \times I$ with $E \times 0 = E_0$, $E \times 1 = E_1$, and $\partial E \times I \subset \partial H_j^i$. We can choose these product structures consistently, so that two adjacent $E \times I$'s intersects in a union of fibers of either product structure. If the 3-ball is contained in $N(S)$, we may assume that the fibers are the fibers of the product structure on $N(S)$. Let $\hat{N}_S$ be the union of these fibered balls over all pairs of adjacent disks of the same disk-type. Clearly $\hat{N}_S$ satisfies $\partial_h \hat{N}_S \subset F$.

In general $\hat{N}_S$ fails to satisfy condition (i), so a further modification is necessary. Since we want to produce a finite number of branched surfaces for Theorem 1, we need to make this modification relatively independent of F.

Claim 1. If $\hat{B}_S$ carries a 2-sphere, then there is a disk $G \subset F$ yielding a disk of contact for $\hat{B}_S$. By this we mean that $\partial G \subset \partial_v \hat{N}_S$ and there is a neighborhood U of ∂G in G satisfying $(U - \partial G) \subset \operatorname{int}(\hat{N}_S)$. G is not itself a disk of contact, but it becomes one if ∂G is pushed into $\operatorname{int}(\partial_v \hat{N}_S)$. We will abuse notation and call G a "disk of contact in F".

To produce this disk G, we first isotope F along fibers of $\hat{N}_S$ into $\operatorname{int}(\hat{N}_S)$. Then we choose a 2-sphere P in $\hat{N}_S$ which is transverse to the fibers of $\hat{N}_S$ and intersects F transversely in the least possible number of curves. If $P \cap F \neq \emptyset$, let α be a curve of $P \cap F$ which is innermost in F and hence bounds a disk D in F with $\mathring{D} \cap P = \emptyset$. Then one of the spheres obtained by surgering P on D is carried by $\hat{B}_S$ and has fewer curves of intersection with F than P. This contradicts our assumption on P, so $P \cap F = \emptyset$.

P bounds a 3-ball B in M and $B \cap F = \emptyset$ since $P \cap F = \emptyset$ and no component of F can lie in a ball. Thus one can isotope P and B so that $(\partial_h \hat{N}_S \cap B) \subset P$, $P \subset \hat{N}_S$, $B \cap F = \emptyset$, and $P - \partial_h \hat{N}_S$ is transverse to the fibers of $\hat{N}_S$. Then $B \cap \hat{N}_S$ is an I-bundle and $B \cap \partial_v \hat{N}_S$ is a collection $\{A_1, \ldots, A_w\}$ of annuli. By an easy argument, $B \cap \hat{N}_S$ is a product. If $A_j \in \{A_1, \ldots, A_w\}$, there are disks D_{1j} and D_{2j} in P with $D_{1j} \cap D_{2j} = \emptyset$ and $A_j \cup D_{1j} \cup D_{2j}$ a topological 2-sphere. Since B is irreducible, $A_j \cup D_{1j} \cup D_{2j}$ bounds a ball B_j in B. We call A_j outermost if $B_j \not\subset B_s$ for any $s \in \{1, \ldots, w\}$, $s \neq j$. Let $A_t \in \{A_1, \ldots, A_w\}$ be an outermost annulus. Then there is a component W of $P \cap \partial_h \hat{N}_S$ with $\partial A_t \subset \partial W$. Isotope F along fibers in $\hat{N}_S$ so that, once again, $\partial_h \hat{N}_S \subset F$, hence $W \subset F$, and $F - \partial_h \hat{N}_S$ is transverse to the fibers of $\hat{N}_S$. Let β_1 and β_2 be the two components of ∂A_t. Since F is incompressible, β_1 bounds a disk D in F. If D is not a disk of contact in F, then $W \subset D$ and there is a disk $G \subset D$ with $\partial G = \beta_2$. G is a disk of contact in F. This completes the proof of Claim 1.

Claim 2. If there is a disk of contact for $\hat{B}_S$ but $\hat{B}_S$ carries no 2-sphere, then there is a disk of contact G in F. To prove this, assume that $\hat{B}_S$ carries no 2-sphere and that D

is a disk of contact for $\hat{B}_S$ as in (i). Isotope F along fibers in $\hat{N}_S$ so that $F \subset \operatorname{int}(\hat{N}_S)$ and F is transverse to the fibers of $\hat{N}_S$. We can assume that D intersects F transversely. If $D \cap F \neq \emptyset$, let α be a component of $D \cap F$ which bounds a disk D' in F with $\operatorname{int}(D') \cap D = \emptyset$, and let D'' be the disk in D bounded by α. $D' \cup D''$ is a 2-sphere in $\hat{N}_S$. Since there are no 2-spheres carried by $\hat{B}_S$, one can isotope D'' to D' (the isotopy may not stay in $\hat{N}_S$) and then push it off F to get a new disk of contact, which we still call D, with the same boundary and with fewer components of intersection with F. Repeating as necessary, we can isotope D (rel ∂D) to a disk of contact D with $D \cap F = \emptyset$. Now isotope F along fibers in $\hat{N}_S$ so that once again $\partial_h \hat{N}_S \subset F$ and $F - \partial_h \hat{N}_S$ is transverse to the fibers of $\hat{N}_S$. Let α be a circle of $\partial_v \hat{N}_S \cap F$ such that α and ∂D are in the same component of $\partial_v \hat{N}_S$. Since F is incompressible, α bounds a disk G in F. Since there are no 2-spheres carried by $\hat{N}_S$, G is a disk of contact in F. This establishes Claim 2.

Claim 3. If G is a disk of contact in F, then G has minimal complexity among all disks E in normal form with $\partial E = \partial G$. (Since

$$\partial G \subset \left(\left(\bigcup_{j=1}^{r} \partial H_j^0\right) \cup \left(\bigcup_{j=1}^{k} \partial H_j^1\right)\right),$$

the definitions of normal form and complexity make sense for E.) If not, let E be a disk in normal form with $\partial G = \partial E$ and $\gamma(E) < \gamma(G)$. Isotope E (rel ∂E) among normal surfaces so that $\operatorname{int}(E)$ intersects F transversely and

$$(E \cap F) \subset \left(\left(\bigcup_{j=1}^{r} H_j^0\right) \cup \left(\bigcup_{j=1}^{k} H_j^1\right)\right).$$

If $E \cap F = \partial E$, then F does not have minimal complexity since $\gamma(E) < \gamma(G)$. If $E \cap F \neq \partial E$, let α be a circle of $E \cap F$ which is innermost in E. α bounds a disk E' in E and a disk D' in F. Since F has minimal complexity, $\gamma(E') \geq \gamma(D')$. Then homotoping E' to D' and beyond it extends to a homotopy (rel ∂E) of E to an immersed disk E_1 with only double curves of self-intersections and $\gamma(E_1) \leq \gamma(E)$. Also, $E_1 \cap F$ has fewer components than $E \cap F$. If E_1 is not embedded, let β be a double curve in E_1 which bounds an embedded disk J in E_1 and an immersed disk J' in E_1. J is homotopic to J' since $\pi_2(M) = 0$. If $J \subset J'$, homotoping J' to J extends to a homotopy of E_1(rel ∂E_1) to an immersed disk E_2 such that $\gamma(E_2) \leq \gamma(E_1)$ and E_2 has fewer double curves than E_1. If $J \not\subset J'$ let E_2 be the disk obtained from E_1 by cutting and pasting along α (interchanging J and J'). Then E_2 is homotopic (rel ∂E_2) to E_1, $\gamma(E_2) = \gamma(E_1)$, and E_2 has fewer double curves than E_1. Repeating as necessary, one can homotope E_1 (rel ∂E_1) to an embedded disk E_3 such that $\gamma(E_3) \leq \gamma(E_1)$ and $E_3 \cap F$ has no more components than $E_1 \cap F$. One can then isotope E_3 to a normal disk E_4 such that $\gamma(E_4) \leq \gamma(E)$ and $E_4 \cap F$ has fewer components than $E \cap F$. Repeating as necessary, one can homotope E_4 (rel ∂E_4) to a normal disk E_5 with $E_5 \cap F = \partial E_5$ and $\gamma(E_5) \leqslant \gamma(E)$. This implies that $E_5 \cap F = \partial G$ and $\gamma(E_5) < \gamma(G)$, which is impossible since F has minimal complexity. Thus G has minimal complexity among all disks E in normal form with $\partial E = \partial G$. This completes the proof of Claim 3.

If there are any disks of contact for $\hat{B}_S$, it follows from Claims 1 and 2 that there is a disk of contact G in F. After isotoping F along fibers in $\hat{N}_S$ so that $F \subset \operatorname{int}(\hat{N}_S)$, we push G off F to a disk of contact for $\hat{B}_S$. We then eliminate all fibers of $\hat{N}_S - F$ which intersect G. This process eliminates a disk of contact for $\hat{B}_S$, and the resulting branched surface carries S with positive weights. We eliminate all disks of contact in the same way. If there are any half-disks of contact, we eliminate one by using the above method to eliminate a disk of contact in the double. We eliminate all half-disks of contact in this way. We then eliminate some fibers of $\hat{N}_S - F$ so that the branch locus is transverse to itself. We denote the branched surface by B_S and the fibered neighborhood by N_S.

PROPOSITION 3. *Let M, S, F, B_S be as above. Then B_S is incompressible.*

Proof. Conditions (i) and (ii) follow directly from the construction. To show that B_S satisfies (iii), suppose $D \subset M - \mathring{N}$ is a monogon: $\partial D = D \cap N = \alpha \cup \beta$, where $\alpha \subset \partial_v N_S$ is a fiber and $\beta \subset \partial_h N_S$. There are two cases to consider, depending on whether the component A of $\partial_v N_S$ containing α is an annulus or a rectangle.

Suppose first that A is an annulus. Let D_1 and D_2 be two parallel copies of the monogon D on opposite sides of D with $\partial D_i = \alpha_i \cup \beta_i$ (Fig. 6a). The arcs α_1 and α_2 divide A into two rectangles R and R_0, where R_0 contains α. Then $E = R \cup D_1 \cup D_2$ is a disk with $E \cap F = \partial E$. Since F is incompressible there is a disk $E' \subset F$ with $\partial E' = \partial E$. The disk E' does not contain β, since otherwise there is a disk of contact with boundary in A. The 2-sphere $E \cup E'$ bounds a 3-cell C since M is irreducible, and C does not contain D since otherwise a component of F would be contained in the 3-ball. If we isotope $D_1 \subset \partial C$ to $D_2 \subset \partial C$ in the obvious way and identify D_1 and D_2, we obtain a solid torus T with $\partial T = A \cup A'$, where A' is an annulus in F. Isotoping $A' \subset F$ to A and a little beyond reduces the number of intersections of F with the 2-handles, although after the isotopy F is no longer in normal form. One can then isotope F to a normal form with smaller complexity, contrary to assumption.

The proof is similar if A is a rectangle. Again let D_1 and D_2 be parallel copies of D with $\partial D_i = \alpha_i \cup \beta_i$. In this case α_i and α_2 divide A into 3 rectangles R_0, R_1, and R_2 (Fig. 6b), and $E_1 = D_1 \cup R_1$ and $E_2 = D_2 \cup R_2$ are potential ∂-compressing disks for F. Thus there are half-disks E_1' and E_2' in F with $\partial E_i' - \partial M = \partial E_i - \partial M$. For $i = 1, 2$ $E_i \cup E_i'$ cuts a 3-cell C_i from M, with $C_1 \cap C_2 = \emptyset$. Again we isotope $D_1 \subset \partial C_1$ to $D_2 \subset \partial C_2$ and identify D_1 and D_2 to obtain a 3-cell C with $\partial C - \partial M = A \cup A'$, where $A' \subset F$. Isotoping A' to A again shows that the complexity of F can be reduced, contrary to assumption. □

Theorem 1 stated in the introduction, now follows easily from Theorem 2 and Proposition 3.

Proof of Theorem 1. Given any two-sided, incompressible, ∂-incompressible surface $S \subset M$, we can construct, by the method of Proposition 3, a properly embedded branched surface $B_S \subset M$ which carries S with positive weights and is incompressible. By Theorem 2 every surface carried by B_S with positive weights is injective and ∂-injective. We will finish the proof by showing that only finitely many of the B_S's are distinct, up to the modification making the branch locus generic.

In the construction of $\hat{B}_S$ from S, each disk-type occurs at most once in $\hat{B}_S$, and $\hat{B}_S$ is determined completely by which disk-types are present. Since there are only a finite number of disk-types, there are only a finite numer of possibilities for $\hat{B}_S$ and $\hat{N}_S$. N_S is obtained from $\hat{N}_S$ by cutting on disks (and half-disks) of contact of minimal complexity. For each component of $\partial_v \hat{N}_S$ whose core bounds a disk (half-disk) of contact there are only finitely many disks (half-disks) of contact having minimal complexity. Thus there are only finitely many possibilities for B_S, up to the above modification. □

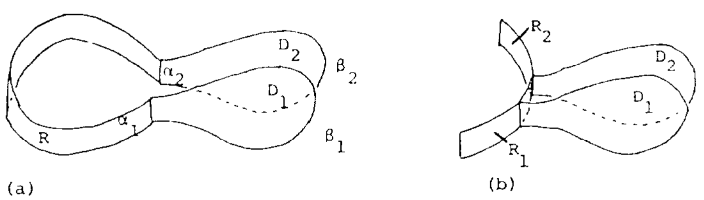


Fig. 6.

§4. FURTHER REMARKS

Branched surfaces are convenient tools in studying incompressible surfaces in 3-manifolds, enabling one to describe surfaces by a finite set of weights. It is often possible to read off information about the surfaces (e.g. genus, number of boundary components, isotopy classes of boundary components) directly from the branched surface data, and one can use branched surfaces to prove that surfaces are incompressible and ∂-incompressible. For examples of this, see [5] or [6].

Using our main result, Hatcher has given a fairly easy proof of the following theorem [4].

THEOREM (Hatcher). *If M is a Haken 3-manifold with $\partial M = T^2$, then there are only a finite number of slopes realized by boundary curves of two-sided incompressible, ∂-incompressible surfaces in M.*

Hatcher's result, which was one of our original motivations for studying branched surfaces, and the results of [1] imply that, if M is an atoroidal, Haken 3-manifold with $\partial M = T^2$ and $c \in \mathbb{N}$, then there are only a finite number of non-isotopic, orientable, incompressible, ∂-incompressible surfaces $S \subset M$ with $|\chi(S)| \leq c$. By Haken's work this is known in general for an atoroidal and anannular Haken 3-manifold.

One of Thurston's powerful tools in his work on surface diffeomorphisms [8] and on hyperbolic structures on 3-manifolds [9] is his construction of the projective lamination spaces $\mathscr{PL}(S)$ and $\mathscr{PL}_0(S)$ for a surface S. $\mathscr{PL}_0(S)$ is topologically a sphere, and contains the (non-peripheral) simple closed curves on S, up to isotopy, as a dense set of "rational" points. $\mathscr{PL}_0(S)$ can be used to give a natural compactification of Teichmüller space. One approach to the projective lamination spaces uses "train tracks", or branched 1-manifolds [9]. One can construct a projective lamination γ on S by assigning weights to a transversely recurrent train track; rational weights give systems of simple closed curves. Similarly, one can construct injective, ∂-injective surfaces in a Haken 3-manifold by assigning rational weights (we are measuring weights projectively here) to suitable branched surfaces. Our main result, together with the results of [2] and [5], suggest that there may be a "projective lamination" space $\mathscr{PL}(M)$, with the injective, ∂-injective surfaces, up to isotopy, as a dense set of "rational" points.

REFERENCES

1. J. EISNER: *Finiteness Theorems for Incompressible Surfaces in 3-Manifolds*. Preprint. Oklahoma State University (1977).
2. W. FLOYD and A HATCHER: *Incompressible Surfaces in 2-Bridge Link Complements*. Preprint. University of Michigan (1981).
3. W. HAKEN: Theorie der Normalflächen. *Acta Math.* **105** (1961), 245–375.
4. A. HATCHER: On the boundary curves of incompressible surfaces. *Pacific J. Math.* **99** (1982), 373–377.
5. A. HATCHER and W. THURSTON: *Incompressible Surfaces in 2-Bridge Knot Complements*. Preprint. UCLA (1979).
6. U. OERTEL: *Incompressible Surfaces in Complements of Star Links*. Dissertation, UCLA (1980).
7. H. SCHUBERT: Bestimmung der Primfaktorzerlegung von Verkettungen. *Math. Z.* **76** (1961), 116–148.
8. W. THURSTON: *On the Geometry and Dynamics of Diffeomorphisms of Surfaces, I*. Preprint. Princeton University (1977).
9. W. THURSTON: *The Geometry and Topology of 3-Manifolds*. Mimeographed Notes. Princeton University (1978).
10. R. WILLIAMS: Expanding Attractors. *Pub. Math. No.* 43. I.H.E.S., 169–203 (1974).

University of Michigan
Ann Arbor, MI 48109, *U.S.A.*

Michigan State University
East Lansing, MI 48824, *U.S.A.*