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Study of Green's Tensor in Magneto-Visco-Elastic Media

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Abstract—The Green's tensor for an infinite medium with interacting magnetic and viscoelastic fields has been studied in terms of Fourier integrals. The exact evaluation of the integrals has been carried out in the case of an electrically nonconducting medium as a particular case for verification with the existing result.

Keywords—Green's function, Viscoelastic medium, Fourier integral.

1. INTRODUCTION

The Green's tensor has direct applications to the solutions of boundary value problems and to the study of wave propagation in interacting magnetic and elastic (or viscoelastic) fields in random media, as can be found in the discussions of many authors like Karal and Keller [1], Chow [2] and Bhattacharyya [3]. Among many other applications, it is seen that a knowledge of Green's function is also essential for the one body scattering problem, and for the problem of multiple scattering by randomly distributed scatterers [4], and for exact solution of wave propagation in a medium with random refractive index [5].

In the present analysis, the components of the Green's tensor for interacting magnetic and viscoelastic fields in an infinite homogeneous medium is expressed in the form of Fourier integrals by the use of Fourier Transforms. In general, the integrands are found to be extremely complicated and to evaluate the integrals exactly, a particular case of a nonconducting medium has been considered. The results obtained here could also be used in the study of magnetoelastic wave propagation in an infinite random medium.

2. FORMULATION OF THE PROBLEM

In the study of the effect of magnetic fields on wave propagation in random viscoelastic media, we have obtained the following (linearised) unperturbed equations:

$$Lv = \begin{bmatrix} M & P \\ N & Q \end{bmatrix} \begin{bmatrix} u \\ h \end{bmatrix} = 0, \tag{2.1}$$

where u is the displacement vector, h is the perturbation in the magnetic field and

$$\begin{aligned} M &= (\lambda+\mu)(\nabla.) + \mu\nabla^2 + (\nabla\lambda)(\nabla.) + (\nabla\mu)\times(\nabla\times) + 2(\nabla\mu.\nabla) + \rho\omega^2, \\ P &= -\nu H_0 \times (\nabla\times), \\ N &= -i\omega\sigma\nu[\{(H_0.\nabla) - H_0(\nabla.)\} - H_0(\nabla\nu.)], \\ Q &= +\left[\nabla\left\{\frac{1}{\nu}(\nabla\nu.)\right\} + \nabla^2 + i\omega\nu\sigma\right]. \end{aligned} \tag{2.2}$$

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Here, λ, μ are the viscoelastic constants involving ω, ν the magnetic permeability, σ the conductivity and $H_0(H_{01}, H_{02}, H_{03})$ is the constant unperturbed magnetic field. In the case of elastic medium, λ, μ reduce to Lamè constants and (2.2) reduces to

$$\begin{aligned} M_0 &= (\lambda+\mu)\nabla(\nabla.) + \mu\nabla^2 + \rho\omega^2, \\ P_0 &= -\nu H_0 \times (\nabla\times), \\ N_0 &= -i\omega\sigma\nu[(H_0.\nabla) - H_0(\nabla.)], \\ Q_0 &= +\left[\nabla^2 + i\omega\nu\sigma\right]. \end{aligned} \tag{2.3}$$

The time dependence of both u and h is given by a factor $e^{-i\omega t}$. Our main object is to calculate an appropriate Green's tensor $G(\mathbf{x}, \mathbf{x}')$ such that

$$L_0 G(\mathbf{x}, \mathbf{x}') = -I\delta(\mathbf{x} - \mathbf{x}'), \tag{2.4}$$

where I is the identity matrix and $\delta(\mathbf{x}-\mathbf{x}')$ is the Dirac delta function, $\mathbf{x}, \mathbf{x}'$ being the field and source points, respectively, and L_0 corresponds to the set (2.3). We shall first solve (2.4) for the elastic medium and for the viscoelastic medium the results will automatically follow.

3. SOLUTION OF THE PROBLEM

We write the equation (2.1) for the elastic medium with a delta function singularity in the form

$$\begin{aligned} M_0 u + P_0 h &= -C\delta(\mathbf{x} - \mathbf{x}'), \\ N_0 u + Q_0 h &= -D\delta(\mathbf{x} - \mathbf{x}'), \end{aligned} \tag{3.1}$$

where C and D are constant vectors to be chosen suitably. Setting

$$[u_j, h_j] = \int [A_j(\mathbf{k}), iB_j(\mathbf{k})]\, e^{i\mathbf{k}.\mathbf{x}}\, d\mathbf{k}, \tag{3.2}$$

and using

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}')}\, d\mathbf{k}, \tag{3.3}$$

we obtain the following equations:

$$-(\lambda+\mu)\left[k_j k_p A_p + \rho\omega^2 - \mu k_p k_p\right] A_j + \nu\left[k_j H_p B_p - B_j H_p k_p\right] = -\frac{C_j}{(2\pi)^3} e^{-i\mathbf{k}.\mathbf{x}'}, \tag{3.4}$$

and

$$\omega\nu\sigma\left[A_p k_p H_j - H_p k_p A_j\right] + \left[ik_p k_p + \omega\nu\sigma\right] B_j = -\frac{D_j}{(2\pi)^3} e^{-i\mathbf{k}.\mathbf{x}'}. \tag{3.5}$$

The set of equations (3.4) and (3.5) may be written as

$$R\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = -\frac{1}{(2\pi)^3}\begin{bmatrix} C \\ D \end{bmatrix} e^{-i\mathbf{k}.\mathbf{x}'}, \tag{3.6}$$

where

$$\begin{aligned} (A_{11})_{lj} &= -[A_0\delta_{lj} + B_0 k_l k_j], \\ (A_{12})_{lj} &= -\nu[k_l H_j - (k.H)\delta_{lj}], \\ (A_{21})_{lj} &= -\omega\nu\sigma[(k.H)\delta_{lj} - H_l k_j], \\ (A_{22})_{lj} &= -(ik^2 + \omega\nu\sigma)\delta_{lj}, \end{aligned} \tag{3.7}$$

with $k^2 = k_p k_p$, $A_0 = \mu k^2 - \rho\omega^2$ and $B_0 = \lambda + \mu$. From equation (3.6), we obtain

$$R^{-1} = \begin{bmatrix} A_{11}^{-1} + \left(A_{11}^{-1}A_{12}\right)E^{-1}\left(A_{21}A_{11}^{-1}\right) & -\left(A_{11}^{-1}A_{12}\right)E^{-1} \\ -E^{-1}\left(A_{21}A_{11}^{-1}\right) & E^{-1} \end{bmatrix}, \tag{3.8}$$

assuming that E^{-1}, A_{11}^{-1} exist and

$$E = A_{22} - A_{21}\left(A_{11}^{-1}A_{12}\right). \tag{3.9}$$

It is to be noted that E, E^{-1}, A_{11}, A_{11}^{-1} and A_{22} are all symmetric matrices.

Then the solution of the set of equations can be obtained as

$$\begin{aligned} A &= \begin{bmatrix} -\left\{A_{11}^{-1} + \left(A_{11}^{-1}A_{12}\right)E^{-1}\left(A_{21}A_{11}^{-1}\right)\right\} \\ C + \left(-A_{11}^{-1}A_{12}\right)E^{-1}D \end{bmatrix} \frac{1}{(2\pi)^3} e^{-i\mathbf{k}.\mathbf{x}'}, \\ B &= \left[\left(-E^{-1}\right) \quad \left(A_{21}A_{11}^{-1}\right)C - E^{-1}D\right] \frac{1}{(2\pi)^3} e^{-i\mathbf{k}.\mathbf{x}'}, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \left(A_{11}^{-1}\right)_{lj} &= \frac{1}{A_0\left(A_0 + B_0k^2\right)}\left[\left(A_0 + B_0k^2\right)\delta_{lj} - B_0k_lk_j\right], \\ \left(E^{-1}\right)_{lj} &= \Big[E_0^2\delta_{lj} + E_0\alpha\Big\{\left(H_0^2\delta_{lj} - H_{ol}\right)A_0k^2 + A_0\left(k_lH_{oj} + k_jH_{ol}\right)(k.H_0) + B_0(k^2\delta_{lj} \\ &\quad + k_lk_j\,(k.H_0)\Big\} + \left(A_0 + B_0k^2\right)k_lk_j\,(k.H_0)^2\left\{A_0H_0^2 + B_0\,(k.H_0)^2\right\}\alpha^2\Big] \Big/ \Delta E, \end{aligned}$$

where

$$E_0 = ik^2 + \omega\nu\sigma, \qquad F_0 = \frac{\omega\nu^2\sigma}{A_0}, \tag{3.11}$$

and

$$\begin{aligned} \Delta E = E_0\Big[&E_0^2 + E_0\alpha\left\{A_0H_0^2 + B_0k^2\,(k.H_0)^2 + (k.H_0)^2\left(A_0 + B_0k^2\right)\right\} \\ &+ k^2\alpha^2\,(k.H_0)^2\left\{A_0\left(A_0H_{01}^2 + B_0k^2H_0^2\right) + B_0\,(k.H_0)^2\left(A_0 + B_0k^2\right)\right\}\Big], \end{aligned}$$

with

$$\alpha = \frac{F_0}{\left(A_0 + B_0k^2\right)}, \qquad H_0^2 = H_{op}H_{op}. \tag{3.12}$$

If $\sigma = 0$, then $\Delta E = E_0^3$, $F_0 = 0$ and

$$E_0 = ik^2. \tag{3.13}$$

4. COMPUTATION OF GREEN'S FUNCTION

It is clear that the expressions obtained in (3.11) and (3.12) are extremely cumbrous. For this reason we confine ourselves to the evaluation of Green's function of Green's tensor (which has 36 components) for the particular case of nonconducting medium ($\sigma = 0$). In this case, the expression becomes remarkably simple since F_0 vanishes.

The Green's function corresponding to R^{-1} for the assumption of $\sigma = 0$, is given by

$$G_{rs} = \int \left(R^{-1}\right)_{rs} e^{-i\mathbf{k}.(\mathbf{x}-\mathbf{x}')}\, d\mathbf{k}, \tag{4.1}$$

where the integration variables (k_1, k_2, k_3), the components of $\mathbf{k}$, are to be transformed by $k_1 = k\sin\theta\cos\phi$, $k_2 = k\sin\theta\sin\phi$, $k_3 = k\cos\theta$, where $k = |\mathbf{k}|$.

Let us choose C and D suitably as unit vectors. Then the denominator of $(R^{-1})_{rs}$ has zeroes at

$$k = \pm\sqrt{\frac{\rho}{\mu}}\,\omega, \ \pm\sqrt{\frac{\rho}{(\lambda + 2\mu)}}\,\omega,$$

on the real axis.

In evaluating the integrals, we choose k_3-axis along r so that

$$\mathbf{k}.\mathbf{r} = kr\cos\theta. \tag{4.2}$$

The integration is now carried out by the method of contour integration [6]. Let us now set

$$G = \int (R^{-1}) e^{-i\mathbf{k}.(\mathbf{x}-\mathbf{x}')}\, d\mathbf{k} = \begin{bmatrix} G_0 & G_1 \\ 0 & G_2 \end{bmatrix}. \tag{4.3}$$

We obtain

$$G_0 = \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & G_{22} & 0 \\ 0 & 0 & G_{33} \end{bmatrix}, \tag{4.4}$$

where

$$G_{11} = \frac{1}{(2\pi)^3} \begin{bmatrix} -\dfrac{2\pi^2}{\mu r} e^{ira} - (ira - 1)\dfrac{2\pi^2 e^{ira}}{\mu a^2 r^3} \\ +(irb - 1)\dfrac{2\pi^2 e^{irb}}{(\lambda + 2\mu) b^2 r^3} \end{bmatrix}, \qquad G_{22} = G_{11},$$

$$G_{33} = \frac{1}{(2\pi)^3} \begin{bmatrix} \dfrac{2\pi^2}{\mu r} e^{ira} + (a^2r^2 + 2ira - 1)\dfrac{2\pi^2 e^{ira}}{\mu a^2 r^3} \\ -(b^2r^2 + 2irb - 2)\dfrac{2\pi^2 e^{irb}}{(\lambda + 2\mu) b^2 r^3} \end{bmatrix}, \qquad \text{with} \tag{4.5}$$

$$a = \sqrt{\frac{\rho}{\mu}}\,\omega, \qquad b = \left[\frac{\rho}{(\lambda + 2\mu)}\right]^{1/2} \omega.$$

This Green's tensor G_0 coincides, as it is expected with the result derived by Karal and Keller [1, equation (37)], in the absence of the magnetic field when transformed to dyadic form. Also,

$$G_1 = \begin{bmatrix} \bar{G}_{11} & 0 & \bar{G}_{13} \\ 0 & \bar{G}_{22} & \left(\dfrac{H_{02}}{H_{01}}\right) \bar{G}_{12} \\ \bar{G}_{31} & \left(\dfrac{H_{02}}{H_{01}}\right) \bar{G}_{31} & \bar{G}_{33} \end{bmatrix}, \tag{4.6}$$

where

$$\bar{G}_{11} = \frac{\nu}{8\pi^2} \begin{bmatrix} \dfrac{ira}{\mu} \dfrac{e^{ira}}{(ar)^2} - \left\{(ar)^2 + 3iar - 3\right\} \dfrac{e^{iar}}{2\pi (ar)^4} \\ + \left\{(br)^2 + 3ibr - 3\right\} \dfrac{e^{irb}}{2(\lambda + 2\mu)(br)^4} \end{bmatrix}, \qquad \bar{G}_{22} = \bar{G}_{11},$$

$$\bar{G}_{13} = \frac{\nu H_{01}}{8\pi^2} \begin{bmatrix} - \left\{(ar)^2 + 3iar - 3\right\} \dfrac{e^{ira}}{2\mu (ar)^4} \\ + \left\{(br)^2 + 3ibr - 3\right\} \dfrac{e^{ibr}}{2(\lambda + 2\mu)(br)^4} \end{bmatrix},$$

$$\bar{G}_{31} = \frac{\nu H_{01}}{8\pi^2} \begin{bmatrix} -(ibr - 1)\dfrac{e^{ira}}{(\lambda + 2\mu)(br)^2} - \left\{(ar)^2 + 3ibr - 3\right\} \\ \times \dfrac{e^{iar}}{2\mu (ar)^4} + \left\{(br)^2 + 3ibr - 3\right\} \dfrac{e^{ibr}}{2(\lambda + 2\mu)(br)^4} \end{bmatrix}, \tag{4.7}$$

$$\bar{G}_{33} = \frac{\nu H_{03}}{8\pi^2} \begin{bmatrix} (ira - 1)\dfrac{e^{ira}}{\mu (ar)^2} - (ibr - 1)\dfrac{e^{ira}}{(\lambda - 2\mu)(br)^2} \\ - \left\{i(ar)^3 - 3(ar)^2 - 6iar + 6\right\} \dfrac{e^{ira}}{2\mu (ar)^4} \\ + \left\{i(br)^3 - 3(br)^2 - 6ibr + 6\right\} \dfrac{e^{ibr}}{2(\lambda + 2\mu)(br)^4} \end{bmatrix}.$$

Furthermore,

$$G_2 = \frac{2\pi^2 i}{r} I, \tag{4.8}$$

where I is the unit matrix of order 3.

Proceeding in a similar way, we can easily solve the equation (2.1) for the viscoelastic medium replacing M_0, P_0, N_0, Q_0 by their corresponding values M, P, N, Q given by (2.2).

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