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On an integral operator

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ABSTRACT

In this paper we derive some criteria for univalence of a general integral operator for analytic functions in the open unit disk.

1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in $\mathcal{U}$.

Miller and Mocanu [1] have considered the integral operator M_α given by

$$M_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z (f(u))^{\frac{1}{\alpha}} u^{-1} \mathrm{d}u \right\}^\alpha, \quad z \in \mathcal{U} \tag{1.1}$$

for functions $f(z)$ belonging to the class $\mathcal{A}$ and for some complex numbers α, $\alpha \neq 0$. It is well known that $M_\alpha(z) \in \mathcal{S}$ for $f(z) \in \mathcal{S}^*$ and $\alpha > 0$, where $\mathcal{S}^*$ denotes the subclass of $\mathcal{S}$ consisting of all starlike functions $f(z)$ in $\mathcal{U}$.

In this paper we consider the integral operator denoted by $J_{\gamma,\beta}$, for $\gamma, \beta \in \mathbb{C}$ defined by

$$J_{\gamma,\beta}(z) = \left\{ \frac{1}{\gamma} \int_0^z u^{-\beta} (f(u))^{\frac{1}{\gamma}+\beta-1} \mathrm{d}u \right\}^\gamma, \quad z \in \mathcal{U}. \tag{1.2}$$

From (1.2), for $\beta = 1$ and $\gamma = \alpha$ we obtain the integral operator $M_\alpha(z)$.

If $\frac{1}{\gamma} = 1$ and $\beta \in \mathbb{C} - \{0, 1\}$, from (1.2) we obtain the integral operator

$$K_\beta(z) = \int_0^z \left(\frac{f(u)}{u} \right)^\beta \mathrm{d}u, \quad z \in \mathcal{U} \tag{1.3}$$

which is the integral operator Kim–Merkes [2].

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From (1.2), for $\frac{1}{\gamma} = 1$ and $\beta = 1$ we obtain the integral operator Alexander define by

$$J_{1,1}(z) = \int_0^z \frac{f(u)}{u} \mathrm{d}u. \tag{1.4}$$

If $\beta = 0$, from (1.2) we obtain the integral operator define by

$$J_{\gamma,0}(z) = \left\{ \frac{1}{\gamma} \int_0^z (f(u))^{\frac{1}{\gamma}-1} \mathrm{d}u \right\}^{\gamma}. \tag{1.5}$$

2. Preliminary results

To discuss our problems for univalence of integral operator $J_{\gamma,\beta}$ we need the following lemmas.

Lemma 2.1 ([3]). *Let α be a complex number with $\operatorname{Re}\alpha > 0$ and $f(z) \in \mathcal{A}$. If $f(z)$ satisfies*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \le 1 \tag{2.1}$$

for all $z \in \mathcal{U}$, then the function

$$F_{\alpha}(z) = \left\{ \alpha \int_0^z u^{\alpha-1} f'(u) \mathrm{d}u \right\}^{\frac{1}{\alpha}} \tag{2.2}$$

is in the class $\mathcal{S}$.

Lemma 2.2 (Schwarz [4]). *Let $f(z)$ be the function regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiplicity greater then m, then*

$$|f(z)| \le \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R \tag{2.3}$$

the equality (in the inequality (2.3) for $z \neq 0$) can hold only if

$$f(z) = \mathrm{e}^{\mathrm{i}\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

Lemma 2.3 (Caratheodory [5,6]). *Let f be analytic function in $\mathcal{U}$, with $f(0) = 0$.*
If f satisfies

$$\operatorname{Re} f(z) \le M \tag{2.4}$$

for some $M > 0$, then

$$(1 - |z|) |f(z)| \le 2M |z|, \quad z \in \mathcal{U}. \tag{2.5}$$

3. Main results

Theorem 3.1. *Let γ be a complex number, $\operatorname{Re}\frac{1}{\gamma} > 0$, $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \cdots$. If*

$$\operatorname{Re}\left\{ \mathrm{e}^{\mathrm{i}\theta} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} \le \frac{|\gamma| \operatorname{Re}\frac{1}{\gamma}}{4\,(1 + |\gamma|\,|\beta - 1|)}, \quad 0 < \operatorname{Re}\frac{1}{\gamma} < 1 \tag{3.1}$$

or

$$\operatorname{Re}\left\{ \mathrm{e}^{\mathrm{i}\theta} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} \le \frac{|\gamma|}{4\,(1 + |\gamma|\,|\beta - 1|)}, \quad \operatorname{Re}\frac{1}{\gamma} \ge 1 \tag{3.2}$$

for all $z \in \mathcal{U}$, $\theta \in [0, 2\pi]$ and $\beta \in \mathbb{C}$, then the integral operator $J_{\gamma,\beta}$ is in the class $\mathcal{S}$.

Proof. The integral operator $J_{\gamma,\beta}$ has the form

$$J_{\gamma,\beta}(z) = \left\{ \frac{1}{\gamma} \int_0^z u^{\frac{1}{\gamma}-1} \left(\frac{f(u)}{u} \right)^{\frac{1}{\gamma}+\beta-1} \mathrm{d}u \right\}^{\gamma}. \tag{3.3}$$

We consider the function

$$g(z) = \int_0^z \left(\frac{f(u)}{u} \right)^{\frac{1}{\gamma}+\beta-1} \mathrm{d}u. \tag{3.4}$$

regular in $\mathcal{U}$.

We have

$$\frac{zg''(z)}{g'(z)} = \left(\frac{1}{\gamma} + \beta - 1 \right) \left(\frac{zf'(z)}{f(z)} - 1 \right). \tag{3.5}$$

Let us consider the function

$$\psi(z) = \mathrm{e}^{\mathrm{i}\theta} \left(\frac{zf'(z)}{f(z)} - 1 \right), \quad z \in U, \ \theta \in [0, 2\pi] \tag{3.6}$$

and we observe that $\psi(0) = 0$.

By (3.1) and Lemma 2.3. for $\mathrm{Re}\,\frac{1}{\gamma} \in (0, 1)$ we obtain

$$|\psi(z)| \leq \frac{|z|\,|\gamma|\,\mathrm{Re}\,\frac{1}{\gamma}}{2\,(1-|z|)\,(1+|\gamma|\,|\beta-1|)}, \quad z \in \mathcal{U}, \ \beta \in \mathbb{C}. \tag{3.7}$$

From (3.2) and Lemma 2.3, for $\mathrm{Re}\,\frac{1}{\gamma} \in [1, \infty)$ we have

$$|\psi(z)| \leq \frac{|z|\,|\gamma|}{2\,(1-|z|)\,(1+|\gamma|\,|\beta-1|)}, \quad z \in \mathcal{U}, \ \beta \in \mathbb{C}. \tag{3.8}$$

From (3.5) and (3.7) we get

$$\frac{1-|z|^{2\mathrm{Re}\,\frac{1}{\gamma}}}{\mathrm{Re}\,\frac{1}{\gamma}} \left| \frac{zg''(z)}{g'(z)} \right| \leq \frac{\left(1-|z|^{2\mathrm{Re}\,\frac{1}{\gamma}}\right)|z|}{2\,(1-|z|)}, \quad z \in \mathcal{U}, \ \mathrm{Re}\,\frac{1}{\gamma} \in (0,1)\,. \tag{3.9}$$

Because $1-|z|^{2\mathrm{Re}\,\frac{1}{\gamma}} \leq 1-|z|^2$ for $\mathrm{Re}\,\frac{1}{\gamma} \in (0,1)\,,\ z \in \mathcal{U}$, from (3.9) we have

$$\frac{1-|z|^{2\mathrm{Re}\,\frac{1}{\gamma}}}{\mathrm{Re}\,\frac{1}{\gamma}} \left| \frac{zg''(z)}{g'(z)} \right| \leq 1 \tag{3.10}$$

for all $z \in \mathcal{U},\ \mathrm{Re}\,\frac{1}{\gamma} \in (0, 1)$.

For $\mathrm{Re}\,\frac{1}{\gamma} \in [1, \infty)$ we have $\frac{1-|z|^{2\mathrm{Re}\,\frac{1}{\gamma}}}{\mathrm{Re}\,\frac{1}{\gamma}} \leq 1-|z|^2$, $z \in \mathcal{U}$ and from (3.5) and (3.8) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\,\frac{1}{\gamma}}}{\mathrm{Re}\,\frac{1}{\gamma}} \left| \frac{zg''(z)}{g'(z)} \right| \leq 1 \tag{3.11}$$

for all $z \in \mathcal{U}$, $\mathrm{Re}\,\frac{1}{\gamma} \in [1, \infty)$.

Using (3.10) and (3.11) by Lemma 2.1. it results that $J_{\gamma,\beta}$ given by (1.2) is in the class $\mathcal{S}$. □

Corollary 3.2. *Let γ be a complex number,* $\mathrm{Re}\,\frac{1}{\gamma} > 0$, $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \cdots$. *If*

$$\mathrm{Re}\left\{ \mathrm{e}^{\mathrm{i}\theta} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} \leq \frac{|\gamma|\,\mathrm{Re}\,\frac{1}{\gamma}}{4}, \quad \mathrm{Re}\,\frac{1}{\gamma} \in (0,1)$$

or

$$\mathrm{Re}\left\{ \mathrm{e}^{\mathrm{i}\theta} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} \leq \frac{|\gamma|}{4}, \quad \mathrm{Re}\,\frac{1}{\gamma} \in [1,\infty)$$

for all $z \in \mathcal{U}$ and $\theta \in [0, 2\pi]$, then the integral operator M_γ define by (1.1) is in the class $\mathcal{S}$.

Proof. In Theorem 3.1. we take $\beta = 1$. □

Corollary 3.3. *Let* $\beta \in \mathbb{C} - \{0\}$ *and* $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \cdots$. *If*

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} \le \frac{1}{4(1 + |\beta - 1|)}$$

for all $z \in \mathcal{U}$ *and* $\theta \in [0, 2\pi]$, *then the integral operator* K_β *given by* (1.3) *is in the class* $\mathcal{S}$.

Proof. For $\gamma = 1$, from Theorem 3.1. we obtain Corollary 3.3. □

Corollary 3.4. *Let* γ *be a complex number* $\operatorname{Re}\frac{1}{\gamma} > 0$ *and* $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \cdots$. *If*

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} \le \frac{|\gamma| \operatorname{Re}\frac{1}{\gamma}}{4(1 + |\gamma|)}, \quad \operatorname{Re}\frac{1}{\gamma} \in (0, 1)$$

or

$$\operatorname{Re}\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} \le \frac{|\gamma|}{4(1 + |\gamma|)}, \quad \operatorname{Re}\frac{1}{\gamma} \in [1, \infty)$$

then the integral operator $J_{\gamma,0}$ *given by* (1.5) *is in the class* $\mathcal{S}$.

Proof. In Theorem 3.1. we take $\beta = 0$. □

Theorem 3.5. *Let* γ *be a complex number* $a = \operatorname{Re}\frac{1}{\gamma} > 0$, $\beta \in \mathbb{C}$ *and* $f(z) \in \mathcal{A}$ *with the form* $f(z) = z + a_2z^2 + \cdots$. *If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{(2a+1)^{\frac{2a+1}{2a}} |\gamma|}{2(1 + |\gamma| \, |\beta - 1|)} \tag{3.12}$$

for all $z \in \mathcal{U}$, *then the integral operator* $J_{\gamma,\beta}$ *define by* (1.2) *is in the class* $\mathcal{S}$.

Proof. We observe that $J_{\gamma,\beta}(z)$ has the form (3.3).

Consider the function $g(z)$ defined in formula (3.4), regular in $\mathcal{U}$.

We define the function $p(z) = \frac{zg''(z)}{g'(z)}$, $z \in \mathcal{U}$ and we obtain

$$p(z) = \frac{zg''(z)}{g'(z)} = \left(\frac{1}{\gamma} + \beta - 1\right)\left(\frac{zf'(z)}{f(z)} - 1\right). \tag{3.13}$$

From (3.12) and (3.13) we have

$$|p(z)| \le \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}$$

for all $z \in \mathcal{U}$.

The function p satisfies the condition $p(0) = 0$ and applying Lemma 2.2 we obtain

$$|p(z)| \le \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} |z|, \quad z \in \mathcal{U}. \tag{3.14}$$

From (3.14) we get

$$\frac{1 - |z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \le \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \frac{\left(1 - |z|^{2a}\right)}{a} |z| \tag{3.15}$$

for all $z \in \mathcal{U}$.

Because

$$\max_{|z| \le 1}\left\{\frac{1 - |z|^{2a}}{a} |z|\right\} = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}$$

from (3.15) we obtain

$$\frac{1 - |z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \le 1 \tag{3.16}$$

for all $z \in \mathcal{U}$.

From (3.16) and because $g'(z) = \left(\frac{f(z)}{z}\right)^{\frac{1}{\gamma}+\beta-1}$, by Lemma 2.1. we obtain that the integral operator $J_{\gamma,\beta}$ is in the class $\mathcal{S}$. □

Corollary 3.6. *Let γ be a complex number, $a = \operatorname{Re}\frac{1}{\gamma} > 0$ and $f(z) \in \mathcal{A}$, $f(z) = z + a_2z^2 + \cdots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|\gamma|$$

for all $z \in \mathcal{U}$, then the integral operator M_γ given by (1.1) is in the class $\mathcal{S}$.

Proof. For $\beta = 1$, from Theorem 3.5 we obtain that $M_\gamma(z)$ is in the class $\mathcal{S}$. □

Corollary 3.7. *Let $\beta \in \mathbb{C} - \{0, 1\}$ and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \cdots$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2\,(1 + |\beta - 1|)}$$

for all $z \in \mathcal{U}$, then the integral operator K_β define by (1.3) belongs to class $\mathcal{S}$.

Proof. We take $\frac{1}{\gamma} = 1$ in Theorem 3.5 and we get $K_\beta \in \mathcal{S}$. □

Corollary 3.8. *Let the function $f(z) \in \mathcal{A}$, $f(z) = z + a_2z^2 + \cdots$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2}, \quad z \in \mathcal{U}$$

then, the integral operator $J_{1,1}$ define by (1.4) is in the class $\mathcal{S}$.

Proof. In Theorem 3.5. we take $\frac{1}{\gamma} = 1$ and $\beta = 1$. □

Corollary 3.9. *Let γ be a complex number $a = \operatorname{Re}\frac{1}{\gamma} > 0$ and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(2a+1)}{2}^{\frac{2a+1}{2a}} \cdot \frac{|\gamma|}{1 + |\gamma|}$$

for all $z \in \mathcal{U}$, then the integral operator $J_{\gamma,0}$ given by (1.5) is in the class $\mathcal{S}$.

Proof. For $\beta = 0$, from Theorem 3.5 we have $J_{\gamma,0} \in \mathcal{S}$. □

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