Existence of three positive pseudo-symmetric solutions for a one dimensional $p$-Laplacian

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Abstract

We apply the Five Functionals Fixed Point Theorem to verify the existence of at least three positive pseudo-symmetric solutions for the three point boundary value problem,

$$(g(u'))'+a(t)f(u) = 0,$$  \(u(0) = 0, \) and \(u(\nu) = u(1), \) where \(g(v) = |v|^{p-2}v, \) with \(p > 1 \) and \(\nu \in (0, 1). \)

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1. Introduction

There is much current interest in questions of multiple positive solutions of boundary value problems for ordinary differential equations. A good deal of this attention is due to the applicability of fixed point theorems like the Leggett–Williams multiple fixed point theorem [5,6,9] and [10]. The book by Agarwal et al. [1] gives a good overview of much of this work, and the works by Avery [2] and Henderson and Thompson [8] also contribute nice applications of the Leggett–Williams theorem. Avery [3] obtained a multiple fixed point theorem, now called the Five Functionals Fixed Point Theorem, which generalized the Leggett–Williams theorem in terms of functionals rather than norms, which in turn, has led to improvements for some of the results in [1]. Applications of the Avery theorem to boundary value problems for multiple solutions have been made in [4].
In a recent paper, He and Ge [7] investigated the existence of at least three positive solutions to a one-dimensional \( p \)-Laplacian by applying the Leggett–Williams theorem. In particular, they studied the three point boundary value problem

\[
\left( g(u') \right)' + a(t)f(u) = 0, \quad 0 < t < 1, \quad (1)
\]

\[
u(0) = 0 \quad \text{and} \quad u(\nu) = u(1), \quad (2)
\]

where \( g(v) = |v|^{p-2}v \), with \( p > 1 \), and \( \nu \in (0, 1) \). They obtained three positive solutions of (1), (2) by proving the existence of three positive fixed points to an operator \( A \) defined by

\[
Au(t) = w(t) = \begin{cases}
\int_0^t G \left( \int_s^\sigma u a(r)f(u(r)) dr \right) ds, & 0 \leq t \leq \sigma_u, \\
w(t) + \int_{\sigma_u}^1 G \left( \int_s^\sigma u a(r)f(u(r)) dr \right) ds, & \sigma_u \leq t \leq 1,
\end{cases}
\]

where \( G(w) = |w|^{1/(p-1)} \text{sgn}(w) \) is the inverse of \( g \), and \( \sigma_u \in [\nu, 1] \) is the unique solution of the equation

\[
\int_{\nu}^x G \left( \int_s^x a(r)f(u(r)) dr \right) ds = \int_x^1 G \left( \int_s^x a(r)f(u(r)) dr \right) ds.
\]

In this paper, we apply the Avery Five Functional Fixed Point Theorem, along with techniques developed in [4] corresponding to the study of the existence of symmetric solutions to the conjugate boundary value problem

\[
u'' + h(u) = 0, \quad 0 < t < 1, \quad (3)
\]

\[
u(0) = \nu(1), \quad (4)
\]

to obtain at least three positive pseudo-symmetric solutions of (1), (2), where we now define what we mean by a pseudo-symmetric function.

**Definition 1.** For \( \nu \in (0, 1) \) a function \( u \in C[0, 1] \) is said to be pseudo-symmetric if \( u \) is symmetric over the interval \([\nu, 1] \). That is, for \( t \in [\nu, 1] \) we have \( u(t) = u(1 - (t - \nu)) \).

In this setting we are able to verify that for all \( u \) in our cone \( \sigma_u = (1 + \nu)/2 \).

2. Preliminaries and the five functionals fixed point theorem

In this section, we provide some background material from the theory of cones in Banach spaces. We also state in this section the Avery Five Functional Fixed Point Theorem.
Definition 2. Let $E$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is called a cone, if it satisfies the following two conditions:

(i) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$;

(ii) $x \in P$, $-x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by

$x \leq y$ if and only if $y - x \in P$.

Definition 3. An operator is called completely continuous, if it is continuous and maps bounded sets into precompact sets.

Definition 4. A map $\alpha$ is said to be a nonnegative, continuous, concave functional on a cone $P$ of a real Banach space $E$, if

$\alpha : P \to [0, \infty)$

is continuous, and

$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly, we say the map $\beta$ is a nonnegative, continuous, convex functional on a cone $P$ of a real Banach space $E$, if

$\beta : P \to [0, \infty)$

is continuous, and

$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$

for all $x, y \in P$ and $t \in [0, 1]$.

Let $\gamma, \beta, \theta$ be nonnegative, continuous, convex functionals on $P$ and $\alpha, \psi$ be nonnegative, continuous, concave functionals on $P$. Then, for nonnegative real numbers $h, a, b, d$ and $c$, we define the convex sets,

$P(\gamma, c) = \{x \in P : \gamma(x) < c\}$,

$P(\gamma, \alpha, a, c) = \{x \in P : a \leq \alpha(x), \gamma(x) \leq c\}$,

$Q(\gamma, \beta, d, c) = \{x \in P : \beta(x) \leq d, \gamma(x) \leq c\}$,

$P(\gamma, \theta, a, b, c) = \{x \in P : a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\}$,

and

$Q(\gamma, \beta, \psi, h, d, c) = \{x \in P : h \leq \psi(x), \beta(x) \leq d, \gamma(x) \leq c\}$.

The following theorem is the Five Functionals Fixed Point Theorem [3], a generalization of the Leggett–Williams Fixed Point Theorem.
Theorem 5. Let $P$ be a cone in a real Banach space $E$. Suppose there exist positive numbers $c$ and $M$, nonnegative, continuous, concave functionals $\alpha$ and $\psi$ on $P$, and nonnegative, continuous, convex functionals $\gamma$, $\beta$, and $\theta$ on $P$, with

$$\alpha(x) \leq \beta(x) \quad \text{and} \quad \|x\| \leq M\gamma(x)$$

for all $x \in P(\gamma, c)$. Suppose $A : P(\gamma, c) \to P(\gamma, c)$ is completely continuous and there exist nonnegative numbers $h, a, k, b$, with $0 < a < b$ such that:

(i) $\{x \in P(\gamma, \theta, a, b, k, c) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Ax) > b$ for $x \in P(\gamma, \theta, a, b, k, c)$;
(ii) $\{x \in Q(\gamma, \beta, \psi, h, a, c) : \beta(x) < a\} \neq \emptyset$ and $\beta(Ax) < a$ for $x \in Q(\gamma, \beta, \psi, h, a, c)$;
(iii) $\alpha(Ax) > b$ for $x \in P(\gamma, a, b, c)$ with $\theta(Ax) > k$;
(iv) $\beta(Ax) < a$ for $x \in Q(\gamma, a, c)$ with $\psi(Ax) < h$.

Then $A$ has at least three fixed points $x_1, x_2, x_3 \in P(\gamma, c)$ such that

$$\beta(x_1) < a,$$
$$b < \alpha(x_2),$$

and

$$a < \beta(x_3) \quad \text{with} \quad \alpha(x_3) < b.$$
and
\[ m^* = \int_0^\delta G \left( \int_\delta^\sigma a(r) dr \right) ds, \]
satisfy the inequality
\[ \nu < \frac{m^*}{M^*}. \]
Trivially, we have
\[ \delta < \frac{m^*}{M^*}. \]
We let
\[ m = \int_0^\delta G \left( \int_\delta^\nu a(r) dr \right) ds = \delta G \left( \int_\delta^\nu a(r) dr \right) \]
and
\[ M = \int_0^\sigma G \left( \int_\delta^\sigma a(r) dr \right) ds, \]
as well as
\[ h_1 = \int_\delta^\nu G \left( \int_\delta^\sigma a(r) dr \right) ds, \]
\[ h_2 = \int_0^\delta G \left( \int_\delta^\sigma a(r) dr \right) ds, \]
and
\[ h_3 = \int_0^\delta G \left( \int_\delta^\sigma a(r) dr \right) ds. \]
Define the nonnegative, continuous, concave functionals \( \alpha, \psi \), and the nonnegative, continuous, convex functionals \( \beta, \theta, \gamma \) on the cone \( P \) by:
\[ \gamma(x) = \theta(x) := \max_{t \in [0,1]} x(t) = x(\sigma), \]
\[ \beta(x) := \max_{t \in [\delta, \nu]} x(t) = x(\nu), \]
\[ \alpha(x) = \psi(x) := \min_{t \in [\delta, \nu]} x(t) = x(\nu). \]
In our main result, we will make use of the following lemma. The lemma is easily proved using the concavity and the pseudo-symmetry of all \( u \in P \).
Lemma 6. Let \( u \in P \). Then

(D1) \( u(\delta) \geq \delta u(1) = \delta u(v) \), and
(D2) \( \sigma u(v) \geq vu(\sigma) = v\|u\| \).

We are now ready to apply the Five Functionals Fixed Point Theorem to an operator \( A \) to give sufficient conditions for the existence of at least three positive pseudo-symmetric solutions to (1), (2).

Theorem 7. Assume that \( v \in (0, 1) \), \( a : [0, 1] \to [0, \infty) \) is a pseudo-symmetric continuous function, \( \delta \in (0, v) \) such that
\[
\int_{\delta}^{v} a(t) dt > 0 \quad \text{with} \quad v < \frac{m^*}{M^*},
\]
and \( f : [0, \infty) \to [0, \infty) \) is continuous. Let \( 0 < a < b < c \), with \( ch^2 < aM \), and suppose that \( f \) satisfies the following conditions:

(i) \( f(x) > g(b/m) \) for all \( b \leq x \leq b/\delta \),
(ii) \( f(x) < g(Ma - c/a) \) for all \( a\delta \leq x \leq a\sigma/v \), and
(iii) \( f(x) \leq g(c/M) \) for all \( 0 \leq x \leq c \).

Then the three point boundary value problem (1), (2) has at least three positive pseudo-symmetric solutions \( u_1, u_2, u_3 \) such that
\[
\max_{t \in [\delta, v]} u_1(t) < a < \max_{t \in [v, 1]} u_2(t) \quad \text{and} \quad \min_{t \in [v, 1]} u_2(t) < b < \min_{t \in [\delta, v]} u_3(t).
\]

Proof. Define the completely continuous operator \( A \) on \( P \) by
\[
Au(t) = w(t) = \begin{cases} 
\int_{0}^{t} G \left( \int_{0}^{s} a(r) f(u(r)) \, dr \right) \, ds, & 0 \leq t \leq \sigma, \\
w(v) + \int_{t}^{1} G \left( \int_{\sigma}^{s} a(r) f(u(r)) \, dr \right) \, ds, & \sigma \leq t \leq 1,
\end{cases}
\]
where
\[
\sigma = \frac{v + 1}{2}.
\]
We first note that for \( u \in P \) we have \( Au(t) \geq 0 \), \( Au(0) = 0 \), and applying the Fundamental Theorem of Calculus we have that \( Au \) is concave. Furthermore, for \( t \in [v, 1] \)
\[
Au(t) = Au(1 - (t - v)).
\]
Consequently, \( Au \in P \), that is, \( A : P \to P \). Moreover, since
\[
\left( g((Au)') \right)'(t) = -a(t) f(u(t)) \leq 0 \quad \text{for all} \quad 0 < t < 1,
\]
we have that all fixed points of $A$ are solutions of (1), (2). Thus we set out to verify that the operator $A$ satisfies the Five Functionals Fixed Point Theorem which will prove the existence of three fixed points of $A$ which satisfy the conclusion of the theorem.

If $u \in \overline{P(\gamma, c)}$, then $\gamma(u) = \max_{t \in [0,1]} u(t) \leq c$. Thus

$$\gamma(Au) = Au(\sigma)$$
$$= \int_0^\sigma G \left( \int_s^\sigma a(r) f(u(r)) \, dr \right) \, ds$$
$$\leq \int_0^\sigma G \left( \int_s^\sigma a(r) g \left( \frac{c}{M} \right) \, dr \right) \, ds$$
$$= \left( \frac{c}{M} \right) \int_0^\sigma G \left( \int_s^\sigma a(r) \, dr \right) \, ds$$
$$= c.$$

Hence,

$$A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}.$$

Next, let $N = (m^* + \delta M^*)/2$. Thus $m^* > N > \delta M^*$, and if we define

$$u_P(t) = \begin{cases} 
\left( \frac{b}{N} \right) \int_0^t G \left( \int_s^\sigma a(r) \, dr \right) \, ds, & 0 \leq t \leq \sigma, \\
 u_P(\nu) + \left( \frac{b}{N} \right) \int_\nu^1 G \left( \int_\sigma^x a(r) \, dr \right) \, ds, & \sigma \leq t \leq 1, 
\end{cases}$$

and

$$u_Q(t) = \begin{cases} 
\left( \frac{a\delta}{N} \right) \int_0^t G \left( \int_s^\sigma a(r) \, dr \right) \, ds, & 0 \leq t \leq \sigma, \\
 u_Q(\nu) + \left( \frac{a\delta}{N} \right) \int_\nu^1 G \left( \int_\sigma^x a(r) \, dr \right) \, ds, & \sigma \leq t \leq 1, 
\end{cases}$$

then, clearly $u_P, u_Q \in P$. Furthermore,

$$\alpha(u_P) = u_P(\delta) = \frac{b}{N} \int_0^\delta G \left( \int_s^\sigma a(r) \, dr \right) \, ds = \frac{bm^*}{N} > b$$

and
\[\theta(u_P) = u_P(v) = \frac{b}{N} \int_0^v G \left( \int_0^\sigma a(r) \, dr \right) \, ds = \frac{b M^*}{N} < \frac{b}{\delta},\]
as well as
\[\psi(u_Q) = u_Q(\delta) = \frac{a \delta}{N} \int_0^\delta G \left( \int_0^\sigma a(r) \, dr \right) \, ds = \frac{a \delta m^*}{N} > a \delta\]
and
\[\beta(u_Q) = u_Q(v) = \frac{a \delta}{N} \int_0^v G \left( \int_0^\sigma a(r) \, dr \right) \, ds = \frac{a M^* \delta}{N} < a.\]

Therefore,
\[u_P \in \{ u \in P(\gamma, \theta, \alpha, b, b/\delta, c) : \alpha(u) > b \}\]
and
\[u_Q \in \{ u \in Q(\gamma, \beta, \psi, a \delta, a, c) : \beta(u) < a \},\]
hence, these sets are nonempty.

If \(u \in P(\gamma, \theta, \alpha, b, b/\delta, c)\), then
\[b \leq u(t) \leq \frac{b}{\delta},\]
for all \(t \in [\delta, v]\), and thus by condition (i) of this theorem,
\[\alpha(Au) = Au(\delta)\]
\[= \frac{b}{m} \int_0^\delta G \left( \int_0^\sigma a(r) \, dr \right) \, ds\]
\[\geq \int_0^\delta G \left( \int_0^\sigma a(r) \, dr \right) \, ds\]
\[> \int_0^\delta G \left( \int_0^\sigma a(r) g \left( \frac{b}{m} \right) \, dr \right) \, ds\]
\[= \frac{b}{m} \int_0^\delta G \left( \int_0^\delta a(r) \, dr \right) \, ds\]
\[= b.\]

Hence, condition (i) of the Five Functionals Fixed Point Theorem is satisfied.

If \(u \in P(\gamma, \alpha, b, c)\) with \(\theta(Au) > b/\delta\), then by Lemma 6(D1), we have
\[\alpha(Au) = Au(\delta) \geq \delta Au(1) = \delta Au(v) = \delta \theta(Au) > b.\]
Thus, condition (iii) of the Five Functionals Fixed Point Theorem is satisfied. If \( u \in Q(\gamma, \beta, \psi, a\delta, a, c) \), then
\[
a\delta \leq u(t) \leq a,
\]
for all \( t \in [\delta, \nu] \), and thus by Lemma 6(D2),
\[
a\delta \leq u(t) \leq \frac{a\sigma}{\nu},
\]
for all \( t \in [\delta, \sigma] \). Thus by condition (ii) of this theorem,
\[
\beta(Au) = Au(\nu) = \int_0^\nu G\left( \int_s^\sigma a(r)f(u(r)) \, dr \right) \, ds
\]
\[
\leq \int_0^\delta G\left( \int_s^\sigma a(r)f(u(r)) \, dr \right) \, ds + \int_\delta^\nu G\left( \int_s^\sigma a(r)f(u(r)) \, dr \right) \, ds
\]
\[
\leq \int_0^\delta \int_s^\sigma a(r)f(u(r)) \, dr \, ds + \int_\delta^\nu \int_s^\sigma a(r)f(u(r)) \, dr \, ds
\]
\[
< \frac{c}{M} \int_0^\delta \int_s^\sigma a(r) \, dr \, ds + \frac{Ma - ch_2}{M(h_1 + h_3)} \int_\delta^\nu \int_s^\sigma a(r) \, dr \, ds
\]
\[
+ \frac{Ma - ch_2}{M(h_1 + h_3)} \int_0^\nu \int_s^\sigma a(r) \, dr \, ds
\]
\[
= \frac{ch_2}{M} + \frac{(Ma - ch_2)h_3}{M(h_1 + h_3)} + \frac{(Ma - ch_2)h_1}{M(h_1 + h_3)}
\]
\[
= a.
\]
Hence, condition (ii) of the Five Functionals Fixed Point Theorem is satisfied.

If \( u \in Q(\gamma, \beta, a, c) \), with \( \psi(Au) < a\delta \), then by Lemma 6, we have
\[
\beta(Au) = Au(\nu) \leq \frac{Au(\delta)}{\delta} = \frac{\psi(Au)}{\delta} < a.
\]

Consequently, condition (iv) of the Five Functionals Fixed Point Theorem is also satisfied. Therefore, the hypotheses of the Five Functionals Fixed Point Theorem 5 are satisfied, and there exist at least three positive pseudo-symmetric solutions \( u_1, u_2, u_3 \in \overline{P(\gamma, c)} \) for the three point boundary value problem (1), (2) such that
\[
\beta(u_1) < a < \beta(u_2) \quad \text{and} \quad \alpha(u_2) < b < \alpha(u_3).
\]
References