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Groups with a Supersoluble Triple Factorization

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1. INTRODUCTION

In the investigation of factorized groups very often one has to study groups with a triple factorization

$$G = AB = AK = BK,$$

where A and B are subgroups and K is a normal subgroup of G (see, for instance, [2, 4, 7, 15, 22]). In [3] it was shown that under certain finiteness conditions the triple factorized group G satisfies some nilpotency requirement if A, B, and K satisfy the same nilpotency requirement. In the following, similar statements are proved for some supersolubility conditions.

THEOREM A. *If the group $G = AB = AK = BK$ is the product of two supersoluble subgroups A and B and a hypercentral normal subgroup K of G, then G is supersoluble.*

Here and in the following two theorems, the condition that the normal subgroup K of G is hypercentral cannot be weakened to the condition that K is hypercyclic or even supersoluble. In fact, in [8] an example is given of

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a finite group $G = AB = AK = BK$, where A, B, and K are supersoluble and K is normal in G, but G is not supersoluble.

The following theorem is analogous to Theorem A of [3].

THEOREM B. *Let the group $G = AB = AK = BK$ be the product of two subgroups A and B and a normal hypercentral minimax subgroup K of G.*

(a) *If A and B are locally supersoluble, then G is locally supersoluble.*

(b) *If A and B are hypercyclic, then G is hypercyclic.*

Note that Theorem B even holds if the hypercentral normal subgroup K of G has finite abelian section rank and $K/T(K)$ is a minimax group, where $T(K)$ is the torsion subgroup of K. However, this cannot be weakened to the condition that K has finite Prüfer rank. In fact, an example is given by Sysak in [16] of a torsion-free nonlocally-supersoluble group $G = AB = AK = BK$, where A and B are abelian subgroups of infinite Prüfer rank and K is an abelian normal subgroup of G with Prüfer rank 1.

Our next theorem corresponds to Theorem B of [3].

THEOREM C. *Let the group $G = AB = AK = BK$ with finite abelian section rank be the product of two locally supersoluble subgroups A and B and a hypercentral normal subgroup K of G. Then G is locally supersoluble and hence hypercyclic.*

The last result is an application of Theorem C.

THEOREM D. *Let π be a set of primes. If the radical group $G = AB$ with finite abelian section rank is the product of two π-minimax subgroups A and B, one of which is hypercyclic, then G is a π-minimax group.*

Theorem D should be seen in relation with results in [4] on soluble products of two nilpotent minimax groups.

As in [3] some cohomological arguments play an important role in the proofs in this paper.

Notation. The notation is standard and can, for instance, be found in [12]. We note in particular:

A group G has *finite abelian section rank* if it has no infinite elementary abelian p-sections for every prime p;

G has *finite Prüfer rank* if there exists a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements.

A soluble group G is a *minimax group* if it has a finite series whose factors are finite or infinite cyclic or quasicyclic of p^∞-type; the number

$m(G)$ of infinite factors in such a series is called the *minimax rank* of G. In particular, if π is the set of all primes p for which there exists a section of G of p^∞-type, the minimax group G is called a *π-minimax group.*

A normal subgroup K of G is *hypercyclically embedded* in G if there exists in K an ascending G-invariant series with cyclic factors.

$\pi(G)$ is the set of primes p for which there exists an element of order p in G.

A subgroup S of a factorized group $G = AB$ is called *factorized* if $S = (A \cap S)(B \cap S)$ and $A \cap B \leqslant S$;

the *factorizer* of the normal subgroup N of $G = AB$ is the subgroup $X(N) = AN \cap BN$.

2. Auxiliary Results

The following main result of [3] is basic for our considerations.

LEMMA 2.1. *Let the group $G = AB = BK$ be the product of three hypercentral subgroups A, B, and K, where K is normal in G. Then G is hypercentral if one of the following conditions holds:*

(i) *K is a minimax group,*

(ii) *G has finite abelian section rank.*

Our proofs will depend heavily on the following cohomological result of Robinson (see [13, Lemma 10; 14, Theorem C]).

LEMMA 2.2. *Let Q be a group and let M be a Q-module which is a radicable abelian p-group of finite rank. Suppose that there exists a nilpotent normal subgroup L of Q such that $H_0(L, M) = 0$. Then for every extension*

$$M \rightarrowtail G \twoheadrightarrow Q$$

there exists a subgroup X of G such that $G = MX$ and $M \cap X$ is finite.

In the proofs of the theorems in the introduction we also need the following two lemmas.

LEMMA 2.3. *Let Q be a locally supersoluble group, and let the Q-module M have no non-trivial $\mathbb{Z}$-cyclic Q-submodules. If N is a finite Q-submodule of M, then M/N has no nontrivial $\mathbb{Z}$-cyclic Q-submodules.*

Proof. Assume that the lemma is false, and let M be a counterexample such that the Q-submodule N of M has minimal order. If P is a proper

Q-submodule of N, then M/P has no non-trivial $\mathbb{Z}$-cyclic Q-submodules. Hence N is a simple Q-module. Let C/N be a non-trivial $\mathbb{Z}$-cyclic Q-submodule of M/N, and consider the semidirect product $G = Q \ltimes C$. Then G/N is an extension of a cyclic group by a locally supersoluble group, and hence it is locally supersoluble. Since N is a minimal normal subgroup of G which is not cyclic, application of [10, Theorem 1] yields that $G = L \ltimes N$ for some subgroup L. Then $C = (C \cap L) \times N$, where $C \cap L$ is normal in $G = CL$. Therefore $C \cap L$ is a non-trivial $\mathbb{Z}$-cyclic Q-submodule of M. This contradiction proves the lemma.

LEMMA 2.4. *Let K be a hypercentral normal subgroup of the locally supersoluble group G. If K has finite abelian section rank, then K is hypercyclically embedded in G.*

Proof. Each primary component of K is a Černikov group, and hence it is hypercyclically embedded in G. Therefore we may assume that K is torsion-free, so that K is a torsion-free nilpotent group with finite Prüfer rank. The subgroup $K/(K \cap G')$ is contained in the centre of $G/(K \cap G')$, and hence it is enough to prove that $K \cap G'$ is hypercyclically embedded in G. Thus it may be assumed that K is contained in G'. Since G' is locally nilpotent, it follows by a result of Čarin (see [12, Part 2, p. 35]) that K is contained in some term with finite ordinal type of the upper central series of G'. There exists an ascending G-invariant series in K whose factors are either finite elementary abelian p-groups or torsion-free abelian groups of finite rank. Moreover, this series can be chosen in such a way that G' acts trivially on the factors and G acts irreducibly on finite factors and rationally irreducibly on infinite factors. Now the proof can be completed as the proof of Theorem 8.18 of [12].

Our last lemma in this section is a well-known statement about minimax groups.

LEMMA 2.5. *The subgroup H of the soluble minimax group G has finite index in G if and only if $m(H) = m(G)$.*

3. PROOF OF THEOREM A

It is well known that the class of finite supersoluble groups is a saturated formation. Hence the following general lemma on finite products of groups in a saturated formation contains as a special case the statement of Theorem A for finite groups.

LEMMA 3.1. *Let $\mathfrak{F}$ be a saturated formation of finite soluble groups, and* *let the group $G = AB = AK = BK$ be the product of two $\mathfrak{F}$-subgroups A and B and a nilpotent normal subgroup K of G. Then G is an $\mathfrak{F}$-group.*

Proof. Assume that the lemma is false, and choose a counterexample $G = AB = AK = BK$ of minimal order. If K is not abelian, the factor group G/K' is an $\mathfrak{F}$-group. Since K' is contained in the Frattini subgroup of G, also G is an $\mathfrak{F}$-group. This contradiction proves that K is abelian.

The $\mathfrak{F}$-residual R of G is contained in K, and hence it is also abelian. By Theorem 5.15 of [6] it follows that G splits over R and that the complements of R in G are conjugate. Thus $G = L \ltimes R$ and $K = (K \cap L) \times R$, where $K \cap L$ is a normal subgroup of $G = KL$. Since $G/(K \cap L)$ is not an $\mathfrak{F}$-group, it follows that $K \cap L = 1$ and thus $K = R$. Every proper factor group of G is an $\mathfrak{F}$-group, and hence K is a minimal normal subgroup of G. Therefore $A \cap K = B \cap K = 1$. Since the complements of K in G are conjugate, we obtain that $G = A = B$ is an $\mathfrak{F}$-group. This contradiction proves the lemma.

Proof of Theorem A.

Since G is the product of two subgroups with the maximal condition on subgroups, it has the maximal condition on normal subgroups (see [1, Corollary 3.3]). This implies that K is nilpotent and G is soluble. Now the application of the theorem of Lennox–Roseblade–Zaicev yields that G is polycyclic (see [9] or [21]). By Lemma 3.1 every finite factor of G is supersoluble, and hence also the polycyclic group G is supersoluble by a well-known result of Baer [5].

4. Proof of Theorems B and C

The proofs of Theorems B and C are similar and will be accomplished in a series of lemmas.

LEMMA 4.1. *If the group $G = AB = AK = BK$ is the product of two locally supersoluble subgroups A and B and a nilpotent Černikov normal subgroup K of G, then G is locally supersoluble.*

Proof. We may assume that K is an abelian p-group (see [11]). For each positive integer n let K_n be the subgroup of all elements of K with order $\leqslant p^n$. For every finite subset F of G there exist finite subsets F_1 of A and F_2 of B such that $F \subseteq \langle F_1, F_2 \rangle$. Since

$$G = \bigcup_{n \in \mathbb{N}} AK_n = \bigcup_{n \in \mathbb{N}} BK_n$$

there exists a positive integer h such that $F_1 \subseteq BK_h$ and $F_2 \subseteq AK_h$. The factorizer $X = X(K_h)$ of K_h in $G = AB$ has a triple factorization

$$X = A^*B^* = A^*K_h = B^*K_h, \qquad \text{where } A^* = A \cap BK_h \text{ and } B^* = B \cap AK_h.$$

Clearly $F \subseteq \langle F_1, F_2 \rangle \leqslant A^*B^* = X$. Therefore it is enough to prove that X is locally supersoluble. Thus we may suppose that K is a finite group. Then $A \cap K$ is a finite normal subgroup of G which is hypercyclically embedded in G. Therefore it can be assumed that $A \cap K = 1$. The subgroups $C_A(K)$ is normal in G, and since $G/C_A(K)$ is finite, it is supersoluble by Theorem A. It follows that G is locally supersoluble.

LEMMA 4.2. *Let the group $G = AB = AK = BK$ be the product of two locally supersoluble subgroups A and B and a hypercentral normal subgroup K of G. If there exists a finite normal subgroup N of G with locally supersoluble factor group G/N, then G is locally supersoluble.*

Proof. Assume that the lemma is false, and let G be a counterexample. Since $G/(K \cap N)$ is locally supersoluble, we may assume that N is contained in K. We may also suppose that N is a minimal normal subgroup of G. In particular N is not cyclic. By Theorem 1 of [10] we have that $G = L \ltimes N$ for some subgroup L of G. Then $C_K(N) = (C_K(N) \cap L) \times N$ and $C_K(N) \cap L$ is normal in $G = LN$. Moreover, the groups $K/C_K(N)$ and $C_K(N)/(C_K(N) \cap L)$ are finite, so that also $K/(C_K(N) \cap L)$ is finite, and hence it is nilpotent. By Lemma 4.1 it follows that $G/(C_K(N) \cap L)$ is locally supersoluble, so that also G is locally supersoluble. This contradiction proves the lemma.

The following two lemmas are special cases of Theorems B and C.

LEMMA 4.3. *Let the metanilpotent group $G = AB = AK = BK$ be the product of two locally supersoluble subgroups A and B and a nilpotent normal subgroup K of G. Then G is locally supersoluble if at least one of the following conditions holds:*

(i) *K is a minimax group,*

(ii) *G has finite abelian section rank.*

Proof. Let L be a nilpotent normal subgroup of G with nilpotent factor group G/L. Then $M = KL$ is also a nilpotent normal subgroup of G with nilpotent factor group G/M. We may assume that M is abelian (see [11]). Clearly the subgroup $A \cap K$ is normal in $G = AK$. Since $A \cap K$ is hypercyclically embedded in A by Lemma 2.4, we have that $A \cap K$ is hypercyclically embedded also in G. Therefore it is enough to prove that $G/(A \cap K)$ is locally supersoluble, and hence we may assume that

$A \cap K = 1$. Since G/K is locally supersoluble and $C_A(K)$ is a normal subgroup of G, it is sufficient to prove that $G/C_A(K)$ is locally supersoluble. We have that $C_{A/C_A(K)}(KC_A(K)/C_A(K)) = 1$, and thus it can be assumed that $C_A(K) = 1$.

The subgroup $A \cap M$ is normal in $G = AM$, so that

$$[A \cap M, K] \leqslant (A \cap M) \cap K = A \cap K = 1 \qquad \text{and} \qquad A \cap M \leqslant C_A(K) = 1.$$

It follows that A is nilpotent. As before $B \cap K$ is also a hypercyclically embedded normal subgroup of G, and hence we may assume that $B \cap K = 1$. Therefore the isomorphic subgroups A and B are both nilpotent. Application of Lemma 2.1 completes the proof.

LEMMA 4.4. *Let the group $G = AB = AK = BK$ be the product of two locally supersoluble subgroups A and B and a torsion-free abelian normal subgroup K of G with $A \cap K = 1$. Then G is locally supersoluble if at least one of the following conditions holds*:

(i) *K is a minimax group,*

(ii) *G has finite abelian section rank.*

Proof. Since G/K is locally supersoluble and $C_A(K)$ is a normal subgroup of G, it is enough to prove that the factor group $G/C_A(K)$ is locally supersoluble. Moreover, $C_{A/C_A(K)}(KC_A(K)/C_A(K)) = 1$, and hence we may assume that $C_A(K) = 1$. Therefore A is isomorphic with a group of automorphisms of the torsion-free abelian group of finite rank K, and hence the periodic subgroups of A are finite (see [12, Part 1, p. 85]). Since A is locally supersoluble, it follows that the set of primes $\pi(A)$ is finite and thus also $\pi(G)$ is finite.

If K is a torsion-free minimax group, the abelian subgroups of A are also minimax groups and hence the radical group A is a soluble minimax group (see [12, Part 2, pp. 171–173]). Hence in this case also G is a soluble minimax group. If G has finite abelian section rank, the subgroups A and B are hypercyclic by a theorem of Baer (see [12, Part 2, p. 89]). Then G is an $\mathfrak{S}_1$-group in the sense of [12, Part 2, p. 137]. We have shown that in both cases (i) and (ii) G is an $\mathfrak{S}_1$-group. In particular A and B are hypercyclic.

Since K is torsion-free and $A \cap K = 1$, the periodic subgroups of G are finite. Then there exists a nilpotent normal subgroup N of G with polycyclic factor group G/N (see [12, Part 2, p. 169]). The finite epimorphic images of G are supersoluble by Theorem A, so that the polycyclic group G/N is supersoluble (see [5]). Then $M = KN$ is a nilpotent normal subgroup of G with supersoluble factor group G/M. Let L be any normal subgroup of G with $M' \leqslant L < M$, and write $\bar{G} = G/L$. If

$\bar{A} \cap \bar{M} = \bar{B} \cap \bar{M} = 1$, then $\bar{A}$ and $\bar{B}$ both are supersoluble and hence $\bar{G}$ is supersoluble by Theorem A. Suppose now that $\bar{A} \cap \bar{M} \neq 1$. Then $\bar{A} \cap \bar{M}$ contains a non-trivial cyclic subgroup $\bar{H}$ which is normal in $\bar{A}$. Since $\bar{M}$ is abelian, it follows that $\bar{H}$ is normal in $\bar{G} = \bar{A}\bar{M}$. Similarly if $\bar{B} \cap \bar{M} \neq 1$, it contains a non-trivial cyclic $\bar{G}$-invariant subgroup. Hence M/M' is hypercyclically embedded in G/M', so that G/M' is hypercyclic. It follows that G is hypercyclic (see [11]). The lemma is proved.

Our last lemma deals with a special situation.

LEMMA 4.5. *Let the group $G = AB = AK = BK$ be the product of two locally supersoluble subgroups A and B and a proper normal subgroup K of G with $A \cap K = B \cap K = 1$. If K is a radicable abelian p-group of finite rank, then K is properly contained in its centralizer $C_G(K)$.*

Proof. Assume that $C_G(K) = K$, so that $C_A(K) = 1$. Then the locally supersoluble group A is linear over the field of p-adic numbers. Hence it is nilpotent-by-finite and its periodic subgroups are finite (see [17, p. 168; 12, Part 1, p. 84]). Let A_0 be a nilpotent normal subgroup of finite index of A and let C be the centre of A_0. If C is finite, then A_0 has finite exponent, and hence A_0 is finite. In this case the isomorphic subgroups A and B are finite. This is impossible since G contains the non-trivial radicable subgroup K. Therefore C is infinite and it contains an element x of infinite order.

The normal closure $A_1 = x^A$ of x in A is a finitely generated A-invariant subgroup of C. Let $A_1 = \langle a_1, ..., a_t \rangle$, and write $a_i = b_i k_i$, where $b_i \in B$ and $k_i \in K$. Then there exists a finite characteristic subgroup K_1 of K which contains the set $\{k_1, ..., k_t\}$. Since $A_1 \leqslant BK_1$, the factor group $A_1/(A_1 \cap B)$ is finite. Then for some positive integer n we have that $A_2 = A_1^n \leqslant A_1 \cap B$, and A_1/A_2 is finite. The subgroup A_2K is normal in $G = AK$, and hence $A_2K \cap B$ is normal in B. Since $A_2K \cap B = A_2(K \cap B) = A_2$, it follows that A_2 is normal in $G = AB$. Then $A_2 \leqslant C_G(K) = K$ and hence $A_2 = 1$. This shows that A_1 is finite, which is a contradiction since $x \in A_1$. The lemma is proved.

The proofs of Theorem C and of statement (a) of Theorem B are now the same. Therefore it is enough to prove Theorem B.

Proof of Statement (a)

It is easy to see that we may suppose that the torsion subgroup $T(K)$ of K is a p-group for some prime p. In particular $T(K)$ is a Černikov group. Assume now that statement (a) is false, and among the counterexamples for which the torsion-free rank of K is minimal choose one $G = AB = AK = BK$ such that the finite residual R of $T = T(K)$ has minimal Prüfer rank.

(i) *The case: K nilpotent.* In this case we may assume that K is abelian (see [11]). The subgroup $A \cap K$ is normal in $G = AK$ and it is hypercyclically embedded in A by Lemma 2.4. Then $A \cap K$ is hypercyclically embedded also in G and hence $G/(A \cap K)$ is not locally supersoluble. Therefore $A \cap K$ is periodic and $G/(A \cap K)$ is also a minimal counterexample. So we may suppose that $A \cap K = 1$. Clearly the subgroup $C_A(K)$ is normal in G and $G/C_A(K)$ is not locally supersoluble. Since $C_{A/C_A(K)}(KC_A(K)/C_A(K)) = 1$, it can be assumed that $C_A(K) = 1$, and hence we have that $C_G(K) = K$. By Lemma 4.4 the factor group G/T is locally supersoluble, and hence $G'T/T$ is locally nilpotent.

The subgroup $C = C_{G'}(T) \cap C_{G'}(K/T)$ is normal in G and $C \cap K = C_C(K)$. It follows that $C/(C \cap K)$ is isomorphic with a group of automorphisms of K which stabilizes the series $K \geqslant T \geqslant 1$, and hence it is abelian. Therefore

$$[C', C, C] \leqslant [C \cap K, C, C] \leqslant [T, C] = 1.$$

Thus C and also $L = KC$ are nilpotent normal subgroups of G. Write $K_0 = [K, C]$. If G/K_0 is locally supersoluble, then so is G/L'. Hence G is locally supersoluble (see [11]). This contradiction proves that G/K_0 is not locally supersoluble, which implies that K_0 is finite, of order e, say.

For all elements $k \in K$ and $c \in C$ we have that $1 = [k, c]^e = [k^e, c]$, so that C is contained in the centralizer $C_G(K^e)$. The factor group K/K^e is finite, and hence $C_1 = C_C(K/K^e)$ is a normal subgroup of G which has finite index in C. Since $C_1 \cap K = C_{C_1}(K)$, the group $C_1/(C_1 \cap K)$ is isomorphic with a group of automorphisms of K which stabilizes the series $K \geqslant K^e \geqslant 1$. Therefore $C_1/(C_1 \cap K)$ is isomorphic with a subgroup of $\mathrm{Hom}(K/K^e, K^e)$, and hence it is finite. Then $C/(C \cap K)$ and also $C(G' \cap K)/(G' \cap K)$ are finite.

The locally nilpotent group $G'/C_{G'}(T)$ is isomorphic with a group of automorphisms of the abelian p-group of finite rank T, and hence it is nilpotent (see [12, Part 2, p. 31]). Moreover, the locally nilpotent group $G'/C_{G'}(K/T)$ is nilpotent as a linear group over the field of rational numbers (see [12, Part 2, p. 31]). It follows that G'/C is nilpotent and $G'/(G' \cap K)$ is finite-by-nilpotent. Therefore $G'/(G' \cap K) \simeq G'K/K$ is nilpotent. The group $G'K/T$ is locally nilpotent, and by a result of Čarin (see [12, Part 2, p. 35]) it follows that $K/T \leqslant Z_n(G'K/T)$ for some non-negative integer n. This implies that $G'K/T$ is nilpotent, so that in particular $G'T/T$ is nilpotent.

If Γ is the last term of the lower central series of G', we have that $\Gamma \leqslant G' \cap T$ and G'/Γ is nilpotent, since T satisfies the minimal condition on subgroups. Then G/Γ is nilpotent-by-abelian, and it is locally supersoluble by Lemma 4.3. For every positive integer n the group G/Γ^n is finite-by-locally supersoluble, and hence it is locally supersoluble by Lemma 4.2.

Therefore G'/Γ^n is locally nilpotent and so is nilpotent. It follows that $\Gamma^n = \Gamma$ for every positive integer n, so that Γ is radicable. Since $[\Gamma, G'] = \Gamma$, we have that $H_0(G'/\Gamma, \Gamma) = 0$ and by Lemma 2.2 there exists a finite G-invariant subgroup F of Γ such that $\bar{G} = \bar{M} \ltimes \bar{\Gamma}$ for some subgroup $\bar{M}$, where $\bar{G} = G/F$. Then $\bar{K} = (\bar{K} \cap \bar{M}) \times \bar{\Gamma}$ and $\bar{K} \cap \bar{M}$ is normal in $\bar{G} = \bar{K}\bar{M}$. The group $G^* = \bar{G}/(\bar{K} \cap \bar{M})$ has a triple factorization

$$G^* = A^*B^* = A^*K^* = B^*K^*, \qquad \text{where} \quad K^* = \bar{K}/(\bar{K} \cap \bar{M}) \simeq \bar{\Gamma}.$$

Therefore G^* is locally supersoluble by Lemma 4.1. As $\bar{G}/\bar{\Gamma} \simeq G/\Gamma$ is locally supersoluble, we have that also $\bar{G}$ is locally supersoluble. Since F is finite, it follows by Lemma 4.2 that G is locally supersoluble. This contradiction proves statement (a) of Theorem B when K is nilpotent.

(ii) *The general case.* Since the factor group K/R is nilpotent (see [12, Part 2, p. 35]), the last term Γ_1 of the lower central series of K is contained in R, and K/Γ_1 is nilpotent. By case (i) it follows that G/Γ_1 is locally supersoluble. For every positive integer n the group K/Γ_1^n is finite-by-nilpotent and hence nilpotent. Therefore $\Gamma_1^n = \Gamma_1$ and Γ_1 is radicable. Since $[\Gamma_1, K] = \Gamma_1$ we have that $H_0(K/\Gamma_1, \Gamma_1) = 0$. By Lemma 2.2 there exists a subgroup L of G such that $G = L\Gamma_1$ and $L \cap \Gamma_1$ is finite. Then $L \cap \Gamma_1$ is normal in G and $G/(L \cap \Gamma_1)$ is not locally supersoluble by Lemma 4.2. We may assume that $G = L \ltimes \Gamma_1$, and hence $R = (R \cap L) \times \Gamma_1$, where $R \cap L$ is a normal subgroup of $G = LR$. The group $G/(R \cap L)$ is not locally supersoluble and this forces the radicable subgroup $R \cap L$ to be trivial. Therefore $R = \Gamma_1$ and $G = L \ltimes R$.

Let R_0 be the maximal G-invariant subgroup of R which is hypercyclically embedded in G. The factor group G/R_0 is not locally supersoluble, and we may assume that R does not contain non-trivial cyclic G-invariant subgroups. The subgroup $C_L(R)$ is normal in $G = LR$ and the factor group $G/C_L(R)$ is not locally supersoluble. Since $C_{L/C_L(R)}(RC_L(R)/C_L(R)) = 1$, we may assume that $C_L(R) = 1$. Hence we have that $C_G(R) = R$. The homology group $H_0(K/R, R)$ is trivial, and so by Lemma 5.11 of [14] it follows that $H^0(K/R, R)$ is finite. Therefore the centralizer $C_R(K)$ is finite.

The subgroup $M = RZ_n(K)$ is a nilpotent normal subgroup of G, so that G/M' is not locally supersoluble (see [11]). This implies that the factor group $G/[R, Z_n(K)]$ is not locally supersoluble. It follows that the subgroup $[R, Z_n(K)]$ is finite and so $[R, Z_n(K)] = 1$ since R is radicable. This shows that $Z_n(K) \leqslant C_G(R) = R$ and $Z_\omega(K) = \bigcup_n Z_n(K) \leqslant R$. Therefore $R = Z_\omega(K)$. In particular $Z(K) = C_R(K)$ is finite and so every $Z_n(K)$ is finite.

Assume that $A \cap R \neq 1$. Then $A \cap R$ contains a non-trivial cyclic A-invariant subgroup $\langle a \rangle$. Let n be the least positive integer such that $a \in Z_n(K)$, and write $\bar{G} = G/Z_{n-1}(K)$. Then $\langle \bar{a} \rangle$ is a non-trivial cyclic $\bar{G}$-invariant subgroup of $\bar{R}$. This contradicts Lemma 2.3. Therefore $A \cap R = 1$ and similarly it follows that $B \cap R = 1$. Let $X = X(R)$ be the factorizer of R in $G = AB$. Then X has a triple factorization

$$X = A^*B^* = A^*R = B^*R, \qquad \text{where } A^* = A \cap BR \text{ and } B^* = B \cap AR.$$

We have also that $A^* \cap R = B^* \cap R = 1$. Lemma 4.5 shows that $R = X$ since $C_X(R) = R$. It follows that R is factorized and $R = (A \cap R)(B \cap R) = 1$. This contradiction proves statement (a) of Theorem B.

Proof of Statement (b)

The group in question is locally supersoluble by statement (a). Application of Lemma 2.4 yields that K is hypercyclically embedded in G. Therefore G is hypercyclic. This completes the proof of Theorem B.

5. Proof of Theorem D

Theorem D is an easy consequence of Theorem C once we have proved the following lemma.

LEMMA 5.1. *If the hypercyclic group $G = AB$ with finite abelian section rank is the product of two minimax subgroups A and B, then G is a minimax group.*

Proof. Assume that the lemma is false, and let G be a counterexample such that the minimax rank $m(A)$ of A is minimal.

The commutator subgroup G' of G is hypercentral. Suppose that the torsion subgroup T of G' is not a Černikov group. Then the set of primes $\pi(T)$ is infinite and $T = L \times M$, where L and M are G-invariant subgroups of T such that the sets of primes $\pi(L)$ and $\pi(M)$ are infinite. The factorizer $X(L)$ of L in G is not a minimax group, and hence $m(A \cap BL) = m(A)$. By Lemma 2.5 it follows that $|A : A \cap BL| < \infty$ and so also $|G : BL| < \infty$. This implies that $|T : T \cap BL| < \infty$. Since $T \cap BL = L(T \cap B)$ and $\pi(T/L) = \pi(M)$ is infinite, we have that the set of primes $\pi(T \cap B)$ is also infinite. But B is a minimax group. This contradiction proves that T is Černikov group. Hence we may assume that G' is torsion-free, so that G' is nilpotent (see [12, Part 2, p. 35]). Therefore G'/G'' is not a minimax group, and it may be assumed that G' is an abelian group.

The factorizer $X = X(G')$ of G' in G has the triple factorization

$$X = A^*B^* = A^*G' = B^*G', \qquad \text{where } A^* = A \cap BG' \text{ and } B^* = B \cap AG'.$$

The abelian minimax subgroup $C=(A^* \cap G')(B^* \cap G')$ is normal in X, and the factor group X/C is the product of two abelian minimax subgroups. Application of Theorem 3.9 of [20] yields that X/C is a minimax group. Hence also X and G' are minimax groups. This contradiction proves the lemma.

Proof of Theorem D

Since a radical minimax group is soluble, it follows that it is sufficient to prove that G is a minimax group (see [19, Corollary 3, p. 340]).

The factorizer $X=X(H)$ of the abelian normal H of G has the triple factorization

$$X=A^*B^*=A^*H=B^*H, \qquad \text{where } A^*=A \cap BH \text{ and } B^*=B \cap AH.$$

Clearly the abelian minimax subgroup $C=(A^* \cap H)(B^* \cap H)$ is normal in X. Since the isomorphic subgroups A^*C/C and B^*C/C are hypercyclic, it follows from Theorem C that the factor group X/C is hypercyclic. Hence X/C and X are minimax groups by Lemma 5.1. Then also H is a minimax group. We have shown that every abelian normal section of G is a minimax group.

The Hirsch–Plotkin radical R of G is hypercental. If T is the torsion subgroup of R, then the factor group R/T is nilpotent (see [12, Part 2, p. 35]). It follows that R/T is a minimax group. The socle S of T is an abelian normal subgroup of G, and hence it is finite. Then T is a Černikov group and R is a minimax group. For every positive integer n, let R_n be the nth term of the ascending Hirsch–Plotkin series of G. Since the hypotheses of Theorem D are inherited by factor groups, every R_n is a minimax group. Therefore the factorizer $X(R_n)$ of R_n in G is a soluble minimax group, and we have that

$$m(R_n) \leqslant m(X(R_n)) \leqslant m(A)+m(B)$$

(see [19, Theorem 1]). This implies that there exists a positive integer t such that $m(R_t)=m(R_{t+1})$. Then R_{t+1}/R_t is finite by Lemma 2.5, and hence the radical group G/R_t is finite. This proves Theorem D.

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