On cardinal sequences of scattered spaces

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Abstract

It was proved by Dow and Simon that there are $2^{\omega_1}$ (as many as possible) pairwise nonhomeomorphic compact, $T_2$, scattered spaces of height $\omega_1$ and width $\omega$. In this paper, we prove that if $\alpha$ is an ordinal with $\omega_1 \leq \alpha < \omega_2$ and $\theta = (\kappa_\xi; \xi < \alpha)$ is a sequence of cardinals such that either $\kappa_\xi = \omega$ or $\kappa_\xi = \omega_1$ for every $\xi < \alpha$, then there are $2^{\omega_1}$ pairwise nonhomeomorphic compact, $T_2$, scattered spaces whose cardinal sequence is $\theta$. © 1998 Elsevier Science B.V. All rights reserved.

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A topological space $X$ is called scattered, if every closed subspace of $X$ has an isolated point. A useful tool in the study of scattered spaces is the Cantor–Bendixson process for topological spaces. If $X$ is a topological space and $\alpha$ is an ordinal, we define the $\alpha$-derivative of $X$ by induction on $\alpha$ as follows: $X^0 = X$; if $\alpha = \beta + 1$, $X^\alpha = \{x \in X: x$ is an accumulation point of $X^\beta\}$; and if $\alpha$ is limit, $X^\alpha = \bigcap\{X^\beta: \beta < \alpha\}$. For every ordinal $\beta$, we define the $\beta$-level of $X$ by $I_\beta(X) = X^\beta \setminus X^{\beta+1}$. It is well known that a space $X$ is scattered if and only if there is an ordinal $\alpha$ such that $X^\alpha = \emptyset$.

Suppose that $X$ is a scattered space. Then we define the height of $X$ by $ht(X) = \beta$ the least ordinal $\beta$ such that $X^\beta$ is finite, and we define the cardinal sequence of $X$ by $CS(X) = (|I_\beta(X)|: \beta < ht(X))$. All the spaces we consider are Hausdorff. By an LCS-space we mean a locally compact, Hausdorff, scattered space. Note that if $X$ is an LCS-space with cardinal sequence $\theta$ and $X$ is not compact, then the one-point compactification of $X$ has also cardinal sequence $\theta$. If $\alpha > 0$ is an ordinal and $X$ is an LCS-space, we say that $X$ is an $(\omega, \alpha)$-space if $CS(X) = \theta$ where $\theta$ is the sequence

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An LCS-space $X$ is called thin-tall, if $X$ is an $(\omega, \omega_1)$-space. It was proved by Rajagopalan and, independently, by Juhász and Weiss that there exists a thin-tall space. In [3], it was even proved by Juhász and Weiss that for every ordinal $\alpha$ such that $0 < \alpha < \omega_2$, there exists an $(\omega, \alpha)$-space. However, it is known that the existence of an $(\omega, \omega_2)$-space is independent of the axioms of Set Theory (see [1]). On the other hand, it was proved by Dow and Simon in [2] that there are $2^{\omega_1}$ (as many as possible) pairwise nonhomeomorphic thin-tall spaces. From the proof of this result we can infer by using a standard argument that for every ordinal $\alpha$ such that $\omega_1 \leq \alpha < \omega_2$, there are also $2^\omega$ pairwise nonhomeomorphic $(\omega, \alpha)$-spaces. The aim of this paper is then to prove that if $\alpha$ is an ordinal with $\omega_1 \leq \alpha < \omega_2$ and $\theta = \langle \kappa_\xi : \xi < \alpha \rangle$ is a sequence of cardinals such that either $\kappa_\xi = \omega$ or $\kappa_\xi = \omega_1$ for every $\xi < \alpha$, then there are $2^\omega$ pairwise nonhomeomorphic LCS-spaces whose cardinal sequence is $\theta$.

This paper is divided in two sections. In the first one, we consider the case of cardinal sequences of length $\omega_1$. In the second section, we first prove that for every ordinal $\alpha < \omega_2$ and every cardinal sequence $\theta = \langle \kappa_\xi : \xi < \alpha \rangle$ where $\kappa_\xi \in \{\omega, \omega_1\}$ for each $\xi < \alpha$, there is an LCS-space with cardinal sequence $\theta$, and then we prove that the construction given in Section 1 can be generalized to any uncountable ordinal $< \omega_2$.

We want to remark that results on cardinal sequences for LCS-spaces have a direct translation to the context of superatomic Boolean algebras (i.e., Boolean algebras in which every subalgebra is atomic), since it is known that the notion of a compact, Hausdorff, scattered space is the dual notion of a superatomic Boolean algebra.

1. Cardinal sequences of length $\omega_1$

We fix a cardinal sequence $\theta = \langle \kappa_\xi : \xi < \omega_1 \rangle$ where $\kappa_\xi \in \{\omega, \omega_1\}$ for every $\xi < \omega_1$. Then, by using a refinement of the argument carried out in [2, Section 2], we shall construct $2^{\omega_1}$ pairwise nonhomeomorphic LCS-spaces with cardinal sequence $\theta$. The underlying set of the $2^{\omega_1}$ spaces we want to construct will be the set

$$D = \bigcup \{\{\xi\} \times \kappa_\xi : \xi < \omega_1\}.$$ 

For every $n < \omega$, we define the column $C_n$ by $\omega_1 \times \{n\}$. Now suppose that $X$ is an LCS-space of underlying set $D$ such that $I_\xi(X) = \{\xi\} \times \kappa_\xi$ for any $\xi < \omega_1$. Let $S$ be a stationary subset of $\omega_1$. Then, for $n < \omega$, we say that $S$ is associated to $C_n$ in $X$, if for every $x = (\xi, n) \in C_n$ where $\xi$ is a limit ordinal, the following holds:

1. If $\xi \in S$, then for every neighbourhood $U$ of $x$ there is a $\zeta < \xi$ such that $\{(\mu, n) : \zeta < \mu \leq \xi\} \subseteq U$.
2. If $\xi \notin S$, there is a neighbourhood $U$ of $x$ such that $U \cap C_n = \{x\}$.

Then we say that $X$ is an admissible $\theta$-space, if the following conditions hold:

(*) (1) For each $n < \omega$, $C_n$ is a closed subset of $X$.
(2) For each $n < \omega$, there is a stationary subset of $\omega_1$ associated to $C_n$ in $X$.
(3) For every $x \in X$ there is a neighbourhood $U$ of $x$ such that $U \setminus \{x\} \subseteq \bigcup\{C_n : n < \omega\}$.
Lemma 1. If $X$ and $Y$ are admissible $\theta$-spaces and $f : X \to Y$ is a homeomorphism, then for every $k < \omega$ there are an $n < \omega$ and a $\xi < \omega_1$ such that $f''(C_k \cap X^\xi) = C_n \cap Y^\xi$.

Proof. It is clear that for every $x \in X$, if $x \in I_\beta(X)$ then $f(x) \in I_\beta(Y)$. We consider $\omega_1$ with the order topology. Then, if $N \subseteq \omega_1$ we write $N' = \{\xi < \omega_1 : \xi$ is an accumulation point of $N\}$. Let $S$ be the stationary subset associated to $C_k$ in $X$. We have that $f''(C_k) \cup \{C_n : n < \omega\}$ is countable. To check this point, note that otherwise if we put $N = \{\zeta < \omega_1 : (\zeta, \mu) \in f''(C_k) \setminus \{C_n : n < \omega\} \text{ for some } \mu < \omega_1\}$, then there is a $\rho \in S \cap N'$. Now, by using $(*)$(3), we infer that no point of $Y$ can be the image under $f$ of the point $(\rho, k)$. On the other hand, if for $k < \omega$ there are $m, n < \omega$ with $m \neq n$ such that $C_n \cap C_m$ are uncountable, then if we put $M = \{\zeta < \omega_1 : (\zeta, m) \in f''(C_k)\}$ and $N = \{\zeta < \omega_1 : (\zeta, n) \in f''(C_k)\}$, we have that there is a $\rho \in S \cap N' \cap N'$. Now, we would infer from $(*)$(1) that no point of $Y$ can be the image under $f$ of $(\rho, k)$. □

In what follows, if $x$ is a point of an LCS-space $X$, when we consider a neighbourhood $U$ of $x$, we shall tacitly assume that if $\beta$ is the ordinal such that $x \in I_\beta(X)$, then $U \cap X^\beta = \{x\}$.

By a decomposition of an infinite set $a$, we mean a partition of $a$ in infinite subsets.

Theorem 1. Let $S$ be a stationary subset of $\omega_1$. Then, there is an admissible $\theta$-space $X$ such that for each $n < \omega$, $S$ is the stationary subset associated to $C_n$ in $X$.

Proof. We construct by transfinite induction on $\xi < \omega_1$ a space $X_\xi$ satisfying the following conditions:

1. The underlying set of $X_\xi$ is $\bigcup\{X_\xi^{(\mu)} : \mu \leq \xi\}$ where $X_\xi^{(\mu)} = \{\mu\} \times \omega$ if $\kappa_\mu = \omega$ or $\xi \leq \omega$, $X_\xi^{(\mu)} = \{\mu\} \times \xi$ if $\kappa_\mu = \omega_1$ and $\xi > \omega$.
2. $X_\xi$ is an LCS-space such that $I_\mu(X_\xi) = X_\xi^{(\mu)}$ for every $\mu \leq \xi$.
3. For every $n < \omega$, $C_n \cap X_\xi$ is a closed subset of $X_\xi$.
4. If $\xi$ is limit and $\xi \in S$, then for every $n < \omega$ and every neighbourhood $U$ of $(\xi, n)$ there is a $\zeta < \xi$ such that $\{(\mu, n) : \zeta < \mu \leq \xi\} \subseteq U$.
5. If $\xi$ is limit and $\xi \notin S$, then for each $n < \omega$ there is a neighbourhood $U$ of $(\xi, n)$ such that $U \cap C_n = \{(\xi, n)\}$.
6. For every $x \in X_\xi$ there is a neighbourhood $U$ of $x$ such that $U \setminus \{x\} \subseteq \bigcup\{C_n : n < \omega\}$.
7. If $\xi < \eta$ and $x \in X_\xi$, then a neighbourhood basis of $x$ in $X_\xi$ is also a neighbourhood basis of $x$ in $X_\eta$.
8. If $\xi < \eta$, then every compact subset of $X_\xi$ is a compact subset of $X_\eta$.

We define $X_0$ as the ordinal $\omega$ with the order topology. Then, assume $\xi > 0$. Without loss of generality, we may assume that $\xi \geq \omega$ and $\kappa_\xi = \omega_1$. First, we suppose $\xi = \mu + 1$. To construct $X_\xi$ we previously define for each $\alpha \leq \mu$ an LCS-space $Y_\alpha$ such that $ht(Y_\alpha) = \xi, I_\beta(Y_\alpha) = \{\beta\} \times \xi$ if $\beta \leq \alpha$ and $\kappa_\beta = \omega_1$, and $I_\beta(Y_\alpha) = I_\beta(X_\mu)$ otherwise.
In addition, we shall have that if $\beta < \alpha \leq \mu$ and $x \in Y_\beta$, then a neighbourhood basis of $x$ in $Y_\beta$ is also a neighbourhood basis of $x$ in $Y_\alpha$. The construction of $Y_0$ is immediate. Then, assume that $\alpha$ is limit. Let $Y$ be the direct union of $\{Y_\beta: \beta < \alpha\}$. If $\kappa_\alpha = \omega$, we put $Y_\alpha = Y$. Then, suppose $\kappa_\alpha = \omega_1$. We have to define a neighbourhood basis of the point $(\alpha, \mu)$. Let $\{x_n: n < \omega\}$ be an enumeration of $Y$. For each $n < \omega$, we construct an open compact neighbourhood $U_n$ of some $y_n$ in $Y$ as follows. We take $U_0$ as an open compact neighbourhood of $x_0$ such that $U_0 \setminus \{x_0\} \subseteq \bigcup\{C_n: n < \omega\}$. If $n > 0$, let $y_n$ be the first element in the enumeration $\{x_n: n < \omega\}$ such that $y_n \not\in U_0 \cup \cdots \cup U_{n-1}$.

Then we choose $U_n$ as an open compact neighbourhood of $y_n$ such that:

1. $U_n \setminus \{y_n\} \subseteq \bigcup\{C_k: k < \omega\}$.
2. For all $m < n$, if $y_n \not\in C_m$ then $U_n \cap C_m = \emptyset$.
3. $U_n \cap (U_0 \cup \cdots \cup U_{n-1}) = \emptyset$.

Let $\{z_n: n < \omega\}$ be an enumeration of $X_\mu(\mu)$. Note that for every $n < \omega$ there is a $k_n < \omega$ such that $z_n = y_{k_n}$. Then, we define $W_n = U_{k_n}$. Let $\{\beta_n: n < \omega\}$ be a sequence of ordinals converging to $\alpha$ in a strictly increasing way. Now, for each $n < \omega$ we choose an element $v_n \in I_{\beta_n}(X_\mu) \cap W_n$ and an open compact neighbourhood $V_n$ of $v_n$ with $V_n \subseteq W_n$. Put $v = (\alpha, \mu)$. Then we define a basic neighbourhood of $v$ as a set of the form $\{v\} \cup \bigcup\{V_n: n > k\}$ where $k < \omega$. If $\alpha$ is a successor ordinal, we would proceed in a similar way. Now, put $Z = Y_\mu$. The underlying set of $X_\xi$ is $Z \cup \{\xi\} \times \xi$. If $x \in Z$, a basic neighbourhood of $x$ in $X_\xi$ is a basic neighbourhood of $x$ in $Z$. Proceeding as above, we construct for each $n < \omega$ an open compact neighbourhood $U_n$ of some $y_n$ in $Z$ satisfying $(+)$(1)–(3) in such a way that $\{U_n: n < \omega\}$ is a partition of $Z$. For each $n < \omega$, put $v_n = (\mu, n)$ and then consider the neighbourhood $V_n$ chosen for $v_n$.

Let $\{t_n: n < \omega\}$ be an enumeration of $\{\xi\} \times \xi$. Let $\{a_n: n < \omega\}$ be a decomposition of $\omega$. For $n < \omega$, we define a basic neighbourhood of $t_n$ in $X_\xi$ as a set of the form $\{t_n\} \cup \bigcup\{V_k: k \in a_n \setminus m\}$ where $m < \omega$.

Now suppose that $\xi$ is a limit ordinal. If $\xi \not\in S$, we can construct $X_\xi$ by means of an argument similar to the one given in the successor case. So, we assume that $\xi \in S$. Let $Z$ be the direct union of $\{X_\mu: \mu < \xi\}$. The underlying set of $X_\xi$ is $Z \cup (\{\xi\} \times \xi)$. If $x \in Z$, a basic neighbourhood of $x$ in $X_\xi$ is a basic neighbourhood of $x$ in $Z$. As above, for every $n < \omega$ we choose a neighbourhood $U_n$ of some $y_n$ in $Z$ verifying $(+)$(1)–(3) in such a way that $\{U_n: n < \omega\}$ is a partition of $Z$. Put $Y = \{y_n: n < \omega\}$. For every $n < \omega$, put $t_n = (\xi, n)$. Let $\{U'_n: n < \omega\}$ be an enumeration of the set $\{(\xi, \zeta): \omega < \zeta < \xi\}$. Fix $n < \omega$. Our purpose is to define a neighbourhood basis of $t_n$.

By using $(+)$(2), it is easy to check that for every $\zeta < \xi$, $Y \cap \{(\mu, n): \zeta < \mu < \xi\}$ is infinite. Set $Y \cap C_n = \{v_m: m < \omega\}$. For each $m < \omega$, put $V_m = \bigcup\{V_m': m < \omega\}$. Note that there is a $\zeta < \xi$ such that $\{(\mu, n): \zeta < \mu < \xi\} \subseteq W_n$. Then, we define a basic neighbourhood of $t_n$ as a set of the form $\{t_n\} \cup \{V_m': m > k\}$ where $k < \omega$. Note that $\{W_n: n < \omega\}$ is pairwise disjoint. To define a neighbourhood basis of a point $t'_n$, we consider a sequence of ordinals $\langle \xi_n: n < \omega\rangle$ converging to $\xi$ in a strictly increasing way and then, for each $k < \omega$, we choose $u_k \in Y \cap C_k \cap Z_\xi$. Now, for $k < \omega$, consider the neighbourhood $V'_k$ chosen for $u_k$ (as an element of $Y$). Note that $V'_k \subseteq W_k$ for each $k < \omega$. Let $\{a_n: n < \omega\}$ be a
decomposition of \( \omega \). Fix \( n < \omega \). Then, we define a basic neighbourhood of \( t'_n \) as a set of the form \( \{ t'_n \} \cup \{ V'_m : n \in \alpha_n \setminus k \} \) where \( k < \omega \).

Now we define the desired space \( X \) as the direct union of the spaces \( X_\xi \) for \( \xi < \omega_1 \).

**Theorem 2.** Let \( \theta = \langle \kappa_\alpha : \alpha < \omega_1 \rangle \) where \( \kappa_\alpha \in \{ \omega, \omega_1 \} \) for each \( \alpha < \omega_1 \). Then, there are \( 2^{\omega_1} \) pairwise nonhomeomorphic LCS-spaces with cardinal sequence \( \theta \).

**Proof.** Let \( \langle S_\xi : \xi < 2^{\omega_1} \rangle \) be a sequence of stationary subsets of \( \omega_1 \) such that if \( \mu < \xi < 2^{\omega_1} \), \( S_\xi \setminus S_\mu \) is stationary. By using Theorem 1, for every \( \xi < 2^{\omega_1} \) there is an admissible \( \theta \)-space \( X_\xi \) such that \( S_\xi \) is associated to each column in \( X_\xi \). Now, we infer from Lemma 1 that if \( \mu < \xi < 2^{\omega_1} \), then \( X_\mu \) and \( X_\xi \) are not homeomorphic.

2. Cardinal sequences of length greater than \( \omega_1 \)

Our aim here is to extend the construction given in Section 1 to any uncountable ordinal \( \eta < \omega_2 \). First, we need to prove the following result:

**Theorem 3.** Let \( \alpha \) be an ordinal such that \( 0 < \alpha < \omega_2 \). Let \( \theta = \langle \kappa_\xi : \xi < \alpha \rangle \) be a sequence of cardinals such that either \( \kappa_\xi = \omega \) or \( \kappa_\xi = \omega_1 \) for every \( \xi < \alpha \). Then, there is an LCS-space \( X \) such that \( CS(X) = \theta \).

In the proof of Theorem 3 we will extend the argument given by Juhász and Weiss in [3]. If \( \beta \) is an ordinal and \( \tau = \langle \lambda_\xi : \xi < \beta \rangle \) is a sequence of cardinals with \( \lambda_\xi \in \{ \omega, \omega_1 \} \) for every \( \xi < \beta \), we denote by \( K_\tau \) the class of all the LCS spaces \( X \) such that \( CS(X) = \tau \).

Suppose that \( \tau_1 = \langle \lambda_\xi : \xi \leq \alpha_1 \rangle \), \( \tau_2 = \langle \lambda_\xi : \xi \leq \alpha_2 \rangle \) are sequences of cardinals such that \( \lambda_\xi \in \{ \omega, \omega_1 \} \) for every \( \xi < \alpha_1 \), \( \lambda_\alpha_1 = \omega \), \( \lambda'_\omega = \omega \) and \( \lambda'_\xi \in \{ \omega, \omega_1 \} \) for every \( \xi \) such that \( 0 < \xi < \alpha_2 \). Assume that \( X \in K_{\tau_1} \) is a \( \sigma \)-compact space such that \( I_{\alpha_1 + 1}(X) = \emptyset \) and \( Y \in K_{\tau_2} \) is a space such that \( X \cap Y = \emptyset \). Then we define the LCS-space \( X \otimes Y \) as follows. The underlying set of \( X \otimes Y \) is \( X \cup (Y \setminus I_0(Y)) \). Let us consider an enumeration \( \{ u_n : n < \omega \} \) of \( I_{\alpha_1}(X) \) and an enumeration \( \{ v_n : n < \omega \} \) of \( I_0(Y) \). Since \( X \) is a paracompact space, for every \( n < \omega \) we can choose a compact open neighbourhood \( U_n \) of \( u_n \) in such a way that \( \{ U_n : n < \omega \} \) is a discrete family. Then, if \( x \in X \) we define a basic neighbourhood of \( x \) as a neighbourhood of \( x \) in \( X \), and if \( x \in Y \setminus I_0(Y) \) we define a basic neighbourhood of \( x \) as a set of the form \( \{ V \setminus I_0(Y) \} \cup \{ U_n : v_n \in V \} \), where \( V \) is a basic neighbourhood of \( x \) in \( Y \). Consider \( \tau = \langle \kappa_\xi : \xi \leq \alpha_1 + \alpha_2 \rangle \) where \( \kappa_\xi = \lambda_\xi \) for \( \xi < \alpha_1 \) and \( \kappa_\xi = \lambda'_\mu \) if \( \xi = \alpha_1 + \mu \) where \( 0 < \mu \leq \alpha_2 \). Then, it can be proved that \( X \otimes Y \in K_\tau \). Note that if in addition \( Y \) is \( \sigma \)-compact, then \( X \otimes Y \) is also \( \sigma \)-compact.

Let \( \beta \) be an ordinal such that \( \text{cf}(\beta) \leq \omega \). Let \( \tau = \langle \lambda_\xi : \xi < \beta \rangle \) be a sequence of cardinals such that \( \lambda_\xi \in \{ \omega, \omega_1 \} \) for every \( \xi < \beta \). Suppose that \( X \in K_\tau \) is a \( \sigma \)-compact space with \( I_\beta(X) = \emptyset \) and \( T = \{ t_\xi : \xi < \omega_1 \} \) is a set of different elements which do not occur in \( X \). Then we define a space \( H(X, T) \) of underlying set \( X \cup T \) such that
$H(X,T)$ is an LCS-space with $ht(H(X,T)) = \beta + 1$, $I_\xi(H(X,T)) = I_\xi(X)$ for $\xi < \beta$, $I_\beta(H(X,T)) = T$ and $I_{\beta+1}(H(X,T)) = \emptyset$. First we assume that $\beta = \gamma + 1$ is a successor ordinal. Then, if $x \in X$ we define a basic neighbourhood of $x$ as a neighbourhood of $x$ in $X$. Since $X$ is $\sigma$-compact, we infer that $I_\gamma(X)$ is a countable set. Let $\{y_n: n < \omega\}$ be an enumeration of $I_\gamma(X)$. For every $n < \omega$ we consider a compact open neighbourhood $U_n$ of $y_n$ in such a way that $\{U_n: n < \omega\}$ is a discrete family. Let $\{a_\xi: \xi < \omega_1\}$ be an almost disjoint family of $\omega$. Then, for every $\xi < \omega_1$, a basic neighbourhood of $t_\xi$ is a set of the form $\{t_\xi\} \cup \bigcup\{U_m: m \in a_\xi, m > k\}$ where $k < \omega$. Analogously, if $\text{cf}(\beta) = \omega$ we consider a sequence of ordinals $\langle \beta_n: n < \omega \rangle$ converging to $\beta$ in a strictly increasing way, and then for each $n < \omega$ we choose a point $z_n \in I_{\beta_n}(X)$ and a compact open neighbourhood $U_n$ of $z_n$ in such a way that $\{U_n: n < \omega\}$ is a discrete family. As above we consider an almost disjoint family $\{a_\xi: \xi < \omega_1\}$ of $\omega$, and then we define as a basic neighbourhood of $t_\xi$ a set of the form $\{t_\xi\} \cup \bigcup\{U_m: m \in a_\xi, m > k\}$ where $k < \omega$. Proceeding in a similar way, we can define a space $H(X,T)$ if $T$ is an infinite countable set of elements not occurring in $X$. Note that in this case $H(X,T)$ is $\sigma$-compact.

**Proof of Theorem 3.** We show that for every ordinal $\alpha < \omega_2$ and every sequence of cardinals $\theta = \langle \kappa_\xi: \xi \leq \alpha \rangle$ where $\kappa_\xi \in \{\omega, \omega_1\}$ for each $\xi \leq \alpha$, we can construct a space $X \in K_\theta$ with $I_\xi(X) = \{\xi\} \times \kappa_\xi$ for every $\xi \leq \alpha$ and $I_{\alpha+1}(X) = \emptyset$. We construct the space $X$ by transfinite induction on $\alpha$. Without loss of generality we may assume that $\kappa_\alpha = \omega_1$. The case $\alpha = 0$ is immediate. Then suppose $\alpha = \beta + 1$. Let $\theta_\beta = \langle \kappa_\xi: \xi \leq \beta \rangle$. By the induction hypothesis, $K_{\theta_\beta} \neq \emptyset$. Let $\theta'_\beta = \langle \kappa_\xi: \xi < \beta \rangle$. Since $K_{\theta_\beta} \neq \emptyset$, it follows that there is a compact space $Z_0 \in K_{\theta'_\beta}$. Let $Z_1$ be the topological sum of a family of $\omega$ disjoint copies of $Z_0$. Then we define $Z = H(Z_1, \{\alpha\} \times \omega_1)$. Now let us consider a $Y \in K_{\theta_\beta}$ such that $Y \cap Z = \emptyset$. Let $X$ be the topological sum of $Y$ and $Z$. Then, it follows that $X \in K_\theta$.

Next assume that $\alpha$ is a limit ordinal such that $\text{cf}(\alpha) = \omega$. Let $\langle \alpha_n: n < \omega \rangle$ be a sequence of ordinals converging to $\alpha$ in a strictly increasing way. For each $n < \omega$, we put $\theta_n = \langle \kappa_\xi: \xi \leq \alpha_n \rangle$. By the induction hypothesis, for each $n < \omega$ there is a compact space $Y_n \in K_{\theta_n}$. We may assume that the $Y_n$ are pairwise disjoint. Let $Y$ be the topological sum of the $Y_n$ for $n < \omega$. Then we define $X = H(Y, \{\alpha\} \times \omega_1)$. We have $X \in K_\theta$.

Now assume that $\alpha$ is a limit ordinal such that $\text{cf}(\alpha) = \omega_1$. Let $\langle \gamma_\mu: \mu < \omega_1 \rangle$ be a closed sequence of ordinals converging to $\alpha$ in a strictly increasing way such that $\text{cf}(\gamma_\mu) \leq \omega$ for each $\mu < \omega_1$. Let $\langle \alpha_\xi: \xi < \nu \rangle$ be the order-preserving enumeration of the $\gamma_\mu$ such that $\kappa_{\gamma_\mu} = \omega_1$. Without loss of generality we may suppose that $\nu = \omega_1$. In order to find a space $X \in K_\theta$, we construct by transfinite induction on $\xi \in [\omega, \omega_1]$ an “approximation” $X_\xi$ such that the following conditions hold:

1. The underlying set of $X_\xi$ is $\bigcup\{X_\xi^\beta: \beta \leq \alpha_\xi\} \cup X_\xi^{(\alpha)}$ where $X_\xi^\beta = \{\beta\} \times \kappa_\beta$ if $\beta \notin \{\alpha_\mu: \mu \leq \xi\} \cup \{\alpha\}$ and $X_\xi^{(\beta)} = \{\beta\} \times \xi$ if $\beta \in \{\alpha_\mu: \mu \leq \xi\} \cup \{\alpha\}$.
2. $X_\xi$ is a $\sigma$-compact LCS-space such that $X_\xi^{(\beta)} = I_\beta(X_\xi)$ for each $\beta \leq \alpha_\xi$ and $X_\xi^{(\alpha)} = I_{\alpha_\xi+1}(X_\xi)$. 


(3) $X_{\xi} \setminus X_{\xi}^{(\alpha)}$ with the relative topology of $X_{\xi}$ is a $\sigma$-compact LCS-space.

(4) If $\omega \leq \mu < \xi$ and $x \in X_{\mu}^{(\beta)}$ for some $\beta \leq \alpha_{\mu}$, then a neighbourhood basis of $x$ in $X_{\mu}$ is also a neighbourhood basis of $x$ in $X_{\xi}$.

(5) If $\omega \leq \mu < \xi$ and $C \subseteq X_{\mu} \setminus X_{\xi}^{(\alpha)}$ is a compact subset of $X_{\mu}$, then $C$ is a compact subset of $X_{\xi}$.

Moreover if $\omega \leq \xi < \omega_1$, we will define for each $x \in X_{\xi}^{(\alpha)}$ a canonical neighbourhood $W_{x}^{(\xi)}$ of $x$ in $X_{\xi}$ in such a way that the following two conditions hold:

(1) If $\omega \leq \mu < \xi < \omega_1$ and $x \in X_{\mu}^{(\alpha)}$, then $W_{x}^{(\mu)} \subseteq W_{x}^{(\xi)}$.

(2) If $\omega \leq \mu < \xi < \omega_1$ and $x, y \in X_{\mu}^{(\alpha)}$ with $x \neq y$, then $W_{x}^{(\mu)} \cap W_{y}^{(\mu)} = W_{x}^{(\xi)} \cap W_{y}^{(\xi)}$.

For each $x \in X_{\xi}^{(\alpha)}$, we will define a clopen neighbourhood basis of $x$ in $X_{\xi}$ from the canonical neighbourhood $W_{x}^{(\xi)}$. Furthermore, we shall have that $W_{x}^{(\xi)}$ is a compact neighbourhood of $x$.

In order to construct $X_{\omega}$, we define by induction on $n < \omega$ a $\sigma$-compact LCS-space $Y_{n}$ with $ht(Y_{n}) = \alpha_{n} + 1$, $I_{\alpha_{n}+1}(Y_{n}) = \emptyset$ and such that if $m < n < \omega$, $Y_{m}$ is an open subspace of $Y_{n}$ and for any $\zeta \leq \alpha_{n}$, $I_{\zeta}(Y_{n}) = I_{\zeta}(Y_{m})$. We assume $\alpha_{0} > 0$. Let $\tau_{0} = \langle \kappa_{\beta} : \beta < \alpha_{0} \rangle$. By the induction hypothesis, there is a compact space $Z_{0} \in K_{\tau_{0}}$. Then we define $Y_{0}$ as the topological sum of $\omega$ disjoint copies of $Z_{0}$. Next assume $n = m + 1$. Let $\delta = o.t.(\alpha_{m} \setminus \alpha_{n})$. Let $\tau = \langle \lambda_{\zeta} : \zeta < \delta \rangle$ where $\lambda_{0} = \omega$ and $\lambda_{\zeta} = \kappa_{\alpha_{n}+\zeta}$ if $0 < \zeta < \delta$. Again by the induction hypothesis, there is a compact space $Z_{0} \in K_{\tau}$. Let $Z_{1}$ be the topological sum of $\omega$ disjoint copies of $Z_{0}$. Then we define $Y_{n} = Y_{m} \otimes Z_{1}$. Let $Y'$ be the direct union of the spaces $Y_{n}$ for $n < \omega$. Without loss of generality we may suppose that $\alpha_{\omega}$ is the limit of $\{\alpha_{n} : n < \omega\}$. Then we put $Y = H(Y', \{\alpha_{\omega} \times \omega\})$. We define the underlying set of $X_{\omega}$ as $Y \cup \{\omega\} \times \omega)$. If $x \in X_{\omega}$, a basic neighbourhood of $x$ in $X_{\omega}$ is a neighbourhood of $x$ in $Y$. For each $n < \omega$, we put $y_{n} = (\omega, n)$ and $y_{n}^{*} = (\omega, n)$. For each $n < \omega$ we can choose a compact open neighbourhood $U_{n}$ of $y_{n}$ in $Y$ in such a way that $\{U_{n} : n < \omega\}$ is a discrete family. Let $\{a_{n} : n < \omega\}$ be a decomposition of $\omega$. Then we define for each $n < \omega$, the canonical neighbourhood of $x_{n}$ in $X_{\omega}$ by $W_{x_{n}}^{(n)} = \{x_{n}\} \cup \bigcup_{k \in a_{n}} \{U_{k} : k \in a_{n}\}$. Now, for every $n < \omega$, we define a basic neighbourhood of $x_{n}$ in $X_{\omega}$ as a set of the form $W_{x_{n}}^{(n)} \setminus C$ where $C \subseteq W_{x_{n}}^{(n)} \setminus \{x_{n}\}$ is a compact open subset of $Y$.

Now we assume $\xi = \mu + 1$ with $\omega \leq \mu < \omega_{1}$. In order to construct $X_{\xi}$ we define for each $\zeta \leq \mu$ a $\sigma$-compact LCS-space $Y_{\zeta}$ such that $ht(Y_{\zeta}) = \alpha_{\mu} + 2$, $I_{\beta}(Y_{\zeta}) = \{\beta\} \times \xi$ if $\beta \in \{\alpha_{\rho} : \rho \leq \zeta\}$, $I_{\beta}(Y_{\zeta}) = I_{\beta}(X_{\mu})$ otherwise. First we fix an enumeration $\{x_{n} : n < \omega\}$ of $\{a\} \times \mu$. In order to define $Y_{0}$, we assume that $\alpha_{0}$ is a successor ordinal, say $\alpha_{0} = \beta_{\delta} + 1$. If $\alpha_{0}$ is a limit ordinal, we would use a similar argument by using the fact that $cf(\alpha_{0}) = \omega$. For every $x \in X_{\mu}$, we define a basic neighbourhood of $x$ in $Y_{0}$ as a neighbourhood of $x$ in $X_{\mu}$. Now we consider a discrete family $\{V_{n} : n < \omega\}$ of compact open neighbourhoods of the points $x_{n}$ in $X_{\mu}$. For each $n < \omega$ we consider a $z_{n} \in V_{n} \cap I_{\beta_{n}}(X_{\mu})$ and a compact open neighbourhood $U_{n}$ of $z_{n}$ with $U_{n} \subseteq V_{n}$. We put $y = (\alpha_{0}, \mu)$. Then we define a basic neighbourhood of $y$ as a set of the form $\{y\} \cup \bigcup_{k \in \omega} \{U_{k} : k > m\}$ where $m < \omega$. Proceeding in a similar way, we can construct $Y_{\xi+1}$ from $Y_{\xi}$, and $Y_{\xi}$ from the union of the $Y_{\eta}$ for $\eta < \zeta$ if $\zeta$ is limit. Now we put
$Y = Y_\mu$. Again since $Y$ is a paracompact space, we can choose a discrete collection \( \{V_n : n < \omega\} \) of compact open neighbourhoods of the points $x_n$ in $Y$. For each $n < \omega$, we consider $V_n$ with the relative topology of $Y$. Then, for every $n < \omega$ we define a $\sigma$-compact LCS-space $Z_n$ such that $ht(Z_n) = \alpha_\xi + 1$, $I_\beta(Z_n) = I_\beta(V_n)$ for each $\beta \leq \alpha_\mu$, and in such a way that the $Z_n$ are pairwise disjoint. Let $\delta = o.t. (\alpha_\xi \setminus \alpha_\mu)$. Let $\tau = (\lambda_\rho : \rho < \delta)$ where $\lambda_0 = \omega$ and $\lambda_\rho = \kappa_{\alpha_\mu + \rho}$ if $0 < \rho < \delta$. Let $\{a_n : n < \omega\}$ be a decomposition of $\{\alpha_\xi\} \times \xi$.

Let us fix a natural number $n$. We put $a_n = \{y_m : m < \omega\}$. For each $m < \omega$, we consider a compact space $Z_{ym} \in K_\tau$ such that $I_\delta(Z_{ym}) = \{ym\}$. We suppose that the $Z_{ym}$ are pairwise disjoint. Then we define $Z'$ as the topological sum of the family $\{Z_{ym} : m < \omega\}$, and we put $Z_n = (V_n \setminus \{x_n\}) \otimes Z'$. Now we define $Z$ as the topological sum of the family $\{Z_n : n < \omega\}$. We then define $X_\xi$ as follows. The underlying set of $X_\xi$ is $Y \cup Z \cup \{(a, \mu)\}$. If $x \in Y \setminus \{(a) \times \xi\}$, a basic neighbourhood of $x$ is the set $V_\mu \setminus \{x\}$. If $x \in \{a\} \times \xi$, a basic neighbourhood of $x$ in $X_\xi$ is the set $U_\mu \setminus \{x\}$. Let $\{\alpha_n : n < \omega\}$ be an enumeration of $\{a\} \times \xi$. We choose a discrete collection $\{V_n : n < \omega\}$ of compact open neighbourhoods of the points $x_n$ in $Y$. Let us consider a decomposition $\{a_n : n < \omega\}$ of $\{\alpha_n\} \times \xi$. Let $\{\beta_m : m < \omega\}$ be a sequence of ordinals converging to $\alpha_\xi$ in a strictly increasing way. We fix a natural number $n$. We consider $V_n$ with the relative topology of $Y$. For each $m < \omega$, we consider a $z_m \in I_{\beta_m}(V_n)$ and a compact open neighbourhood $U_m$ of $z_m$ in $V_n$ such that $\{U_m : m < \omega\}$ is a discrete family in $V_n \setminus \{x_n\}$. We set $a_n = \{y_k : k < \omega\}$. We fix a decomposition $\{b_k : k < \omega\}$ of $\omega$. Then we define a basic neighbourhood of a point $y_k$ in $X_\xi$ as the set $\{y_k\} \cup \{U_m : m \in b_k, m > l\}$ where $l < \omega$. Now we define the canonical neighbourhood of a point $x_n$ in $X_\xi$ by $W_{\xi_n}^{(x)} = W_{x_n} \cup a_n$. Then, a basic
neighbourhood of $x_n$ in $X_\xi$ is a set of the form $W_{x_n}^{(\xi)} \setminus C$ where $C$ is a compact open subset of $W_{x_n}^{\xi} \setminus \{x_n\}$.

Finally we define the space $X$ as follows. The underlying set of $X$ is $\bigcup\{X_\xi: \omega \leq \xi < \omega_1\}$. If $x \in X_\xi \setminus \{\alpha\} \times \omega_1$ for some $\xi < \omega_1$, a basic neighbourhood of $x$ in $X$ is a basic neighbourhood of $x$ in $X_\xi$. If $x \in \{\alpha\} \times \omega_1$, we put $W_x = \bigcup\{W_x^{(\xi)}: \omega \leq \xi < \omega_1\}$. Then we define a basic neighbourhood of $x$ in $X$ as a set of the form $W_x \setminus C$ where $C \subseteq W_x \setminus \{x\}$ is a compact open subset of $X_\xi$ for some $\xi < \omega_1$. It can be verified that $X \in K_\theta$. □

Theorem 3 is in a sense best possible, since under CH we have that if $\theta = (\kappa_\xi: \xi < \eta)$ is such that $\kappa_\alpha = \omega$ and $\kappa_\beta = \omega_2$ for some $\alpha < \beta < \eta$, then there is no LCS-space $X$ such that $\text{CS}(X) = \theta$. To check this point, assume on the contrary that there is an LCS-space $X$ with $\text{CS}(X) = \emptyset$. For every $x \in X^\alpha$ consider a clopen neighbourhood $U_x$ of $x$. Now, we put $a_x = U_x \cap I_\alpha(X)$. Since we are assuming that if $\gamma$ is the ordinal such that $x \in I_\gamma(X)$ then $U_x \cap X^\gamma = \{x\}$, we have that $x \neq y$ implies $a_x \neq a_y$. Hence, we can identify every point of $X^\alpha$ with a subset of $I_\alpha(X)$. Also, it was proved by Baumgartner in [1] that if it is consistent that there exists an inaccessible cardinal, then it is consistent with $2^\omega = \omega_2$ that there is no LCS-space with cardinal sequence $\theta = (\kappa_\xi: \xi \leq \omega_1)$ where $\kappa_\xi = \omega_1$ for each $\xi < \omega_1$ and $\kappa_{\omega_1} = \omega_2$. On the other hand, Juhász has pointed out that in a collaboration with Weiss, they have proved that if $\theta = (\kappa_\xi: \xi < \omega_1)$ is a sequence of cardinals such that $\kappa_\xi < 2^\omega$ for each $\xi < \omega_1$, then there is an LCS-space $X$ such that $\text{CS}(X) = \emptyset$.

Next, combining the arguments given in the proofs of Theorems 1 and 3 we can show the following result, whose proof is left to the reader. As above, we write $C_n = \omega_1 \times \{n\}$ for $n < \omega.$

**Lemma 2.** Suppose that $\theta = (\kappa_\xi: \xi < \omega_1)$ is a sequence of cardinals such that $\kappa_\xi \in \{\omega, \omega_1\}$ for every $\xi < \omega_1$ and $\kappa_{\omega_1} = \omega_1$. Then, there is an LCS-space $X$ with $I_\xi(X) = \{\xi\} \times \kappa_\xi$ for $\xi < \omega_1$ and $I_{\omega_1+1}(X) = \emptyset$ such that the following two conditions are satisfied:

1. For every $x \in X \setminus I_{\omega_1}(X)$ and every $n < \omega$ there is a neighbourhood $U$ of $x$ such that $(U \setminus \{x\}) \cap C_n = \emptyset$.
2. For every $x \in X$ there is a neighbourhood $U$ of $x$ such that $U \setminus \{x\} \subseteq \bigcup\{C_n: n < \omega\}$.

Now, we can prove the main result.

**Theorem 4.** Let $\alpha$ be an ordinal such that $\omega_1 < \alpha < \omega_2$. Let $\theta = (\kappa_\xi: \xi < \alpha)$ be a sequence of cardinals such that either $\kappa_\xi = \omega$ or $\kappa_\xi = \omega_1$ for every $\xi < \alpha$. Then, there are $2^{\omega_1}$ pairwise nonhomeomorphic LCS-spaces with cardinal sequence $\theta$.

**Proof.** Let $\tau = (\kappa_\xi: \xi < \omega_1)$. Consider $\langle X_\xi: \xi < 2^{\omega_1} \rangle$ a sequence of pairwise nonhomeomorphic admissible $\tau$-spaces constructed as in Theorem 2. Let $X_\xi'$ be the one-point
compactification of $X_\xi$. Then, let $Y_\xi$ be the topological sum of $\omega$ disjoint copies of $X'_\xi$. Let $\beta = o.t.(\alpha \setminus \omega_1)$. Now let $\tau' = (\kappa'_\xi: \xi < \beta)$ where $\kappa'_0 = \omega$, $\kappa'_\xi = \kappa_{\omega_1+\xi}$ if $0 < \xi < \beta$. By Theorem 3, there is an LCS-space $Y$ such that $CS(Y') = \tau'$. For $\xi < 2^{\omega_1}$, we may assume that the underlying sets of $Y$ and $Y_\xi$ are disjoint. Then, we define $Z_\xi = Y_\xi \otimes Y$ for every $\xi < 2^{\omega_1}$. Note that if $\kappa_{\omega_1} = \omega$, we infer from the proof of Lemma 1 that the spaces $Z_\xi$ are pairwise nonhomeomorphic LCS-spaces with cardinal sequence $\theta$. So, assume that $\kappa_{\omega_1} = \omega_1$. Let $\tau^* = (\kappa_\xi: \xi \leq \omega_1)$. Let $Z$ be an LCS-space of cardinal sequence $\tau^*$ which verifies the conditions of Lemma 2. We may assume that for every $\xi < 2^{\omega_1}$, the underlying sets of $Z$ and $Z_\xi$ are disjoint. Then, we define $Z'_\xi$ as the topological sum of $Z$ and $Z_\xi$. By using the argument given in Lemma 1, it is now easy to check that the spaces $Z'_\xi$ are pairwise nonhomeomorphic LCS-spaces with cardinal sequence $\theta$. 

References