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A Characteristic Property of Labelings and Linear Extensions of Posets of Dimension 2*

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1. INTRODUCTION

The notion of dimension of a partially ordered set was introduced by Dushnik and Miller in 1941 [2]. Every partial order is the intersection of a family of linear orders; the dimension of the partial order is defined as the minimum number of linear orders in such a representation.

In particular, finite posets of dimension 2 are isomorphic to suborders of $N \times N$ or $N \times N^*$, where N denotes the set of nonnegative integer numbers with the natural linear order and N^* denotes the dual order. Furthermore, without loss of generality, one can say [10] that a poset of dimension 2 over n points is isomorphic to a suborder of the product of $\{1, 2, \dots, n\}$ and its dual. In general, such an isomorphism is not unique for a given poset of dimension 2. There are, in fact, several ways to assign a pair of "coordinates" to each element of the poset, so that for p and q elements with coordinates (a, b) and (a', b'), one has $p \leq q \Leftrightarrow a \leq a'$ and $b \geq b'$. We will call such a correspondence a *labeling* of the poset.

The main result of this work, Theorem 2, consists of proving that in every poset of dimension 2 the difference between the coordinates of each element does not depend on the chosen labeling. In other words, the

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theorem states that, given a linear extension of the poset P, there is at most one way to extend this order to a labeling of P.

This result yields an efficient algorithm for computing all labelings of a finite poset. If the algorithm does not find any labeling, then the poset has dimension strictly greater than 2.

Note that a labeling of a poset of dimension 2 can be seen as a permutation; thus it corresponds to a pair of standard Young tableaux [6, 8, 9], which are strictly related to the structure of the poset (refer to [3, 4]). Our algorithm produces all permutations and, hence, all pairs of Young tableaux associated with a given finite poset of dimension 2. For this reason the algorithm seems to be a useful tool for the study of the relationships between these pairs of tableaux.

2. LABELINGS OF FINITE POSETS OF DIMENSION 2

Recall that a finite partially ordered set $(P, \leq)$ has dimension 2 whenever the order relation "$\leq$" is the intersection of two different linear orders on P (refer to [2]). For our purposes it is convenient to express this fact in terms of labels, as in the following proposition, in which $[n]$ denotes the set $\{1, 2, \dots, n\}$.

PROPOSITION 1. *Let $(P, \leq)$ be a finite poset and $n = |P|$. Then $(P, \leq)$ has dimension 2 if and only if there exists a map φ: $P \to [n]^2$ such that for every $p, q \in P$ with, say, $\varphi(p) = (a, b)$ and $\varphi(q) = (a', b')$, we have*

$$p \leq q \quad \Leftrightarrow \quad a \leq a' \text{ and } b \geq b'.$$

Without loss of generality [10], we can assume that $p \neq q$ implies $a \neq a'$ and $b \neq b'$.

We call such a map φ a *labeling* of P. Note that if φ is a labeling of P, then the maps

$$\varphi_l: P \to [n], \varphi_l(p) := a \qquad \text{and} \qquad \varphi_r: P \to [n], \varphi_r(p) := b,$$

where $\varphi(p) = (a, b)$, are linear extensions of P and of its dual P^*, respectively. Conversely, a linear extension f: $P \to [n]$ of P will be called a *semilabeling* of P if there exists a linear extension g of P^* such that the map

$$\varphi: P \to [n]^2, \varphi(p) = (f(p), g(p))$$

is a labeling of P.

Remark. It is easy to check that for a labeling φ of P, the map $\tilde{\varphi}$ defined by

$$\tilde{\varphi}: P \to [n]^2, \tilde{\varphi}(p) := (n + 1 - \varphi_r(p), n + 1 - \varphi_l(p))$$

is a labeling of P as well. We call $\tilde{\varphi}$ the *complementary labeling* of φ.

As an immediate consequence of the previous remark, we get that every finite poset of dimension 2 admits at least two different labelings and that a finite poset has dimension 1 if and only if it admits exactly one labeling φ that coincides with $\tilde{\varphi}$.

3. THE VARIANCE

We now introduce the concept of variance of a point p in P, which will be used in Theorem 2 below.

DEFINITION. Let $(P, \leq)$ be a finite poset. For every $p \in P$, set

$$\delta^+(p) := |\{q \in P; q \geq p\}|, \quad \delta^-(p) := |\{q \in P; q \leq p\}|.$$

We define the variance of p as the integer

$$\delta(p) := \delta^+(p) - \delta^-(p).$$

THEOREM 2. *Let $(P, \leq)$ be a finite poset of dimension 2, with $n = |P|$ and $\varphi: P \to [n]^2$ a labeling of P. Then we have, for all $p \in P$,*

$$\delta(p) = \varphi_r(p) - \varphi_l(p).$$

Proof. Let p be an arbitrary element in P with $\varphi(p) = (a, b)$. Consider the sets

$$A := \{(x, y) \in \varphi(P); x > a \text{ and } y > b\},$$

$$B := \{(x, y) \in \varphi(P); x < a \text{ and } y < b\},$$

$$C := \{(x, y) \in \varphi(P); x < a \text{ and } y > b\},$$

Then we have

$$n - \delta^+(p) = |\{q \in P; q \ngeq p\}| = |A \cup B \cup C|$$

$$= |A \cup C| + |B \cup C| - |C|.$$

Since p is the only element in P with $\varphi_l(p) = a$ and $\varphi_r(p) = b$, and since $\varphi_l(P) = \varphi_r(P) = [n]$, we obtain

$$|A \cup C| = |\{y \in [n]; y > b\}| = n - b$$

$$|B \cup C| = |\{x \in [n]; x < a\}| = a - 1,$$

while $|C| = |\{q \in P;\ q < p\}| = \delta^-(p) - 1$. Hence

$$n - \delta^+(p) = n - b + a - \delta^-(p),$$

which gives the assertion.

4. THE ALGORITHM

Theorem 2 implies that a labeling φ of P is completely determined by the variance and by the semilabeling φ_l of P or, equivalently, by the semilabeling φ_r of P^*.

This suggests using the notion of variance to implement an algorithm that checks whether a given finite poset P has dimension 2 and, in such a case, constructs all possible labelings of P. We need to build all linear extensions φ_l of P and test whether $\varphi(p) = (\varphi_l(p), \varphi_l(p) + \delta(p))$ defines a labeling of P.

Consider the classical algorithm for the construction of all linear extensions of a partial order (see, for instance, [7]), where $\min(A)$ denotes the set of minimal points of $A \subseteq P$ with respect to the given partial order and $n = |P|$:

> Let $P_0 = P$. Choose $p_0 \in \min(P_0)$ and set $\omega(p_0) := 1$, $P_1 = P_0 \setminus \{p_0\}$. Suppose now that $P_1, P_2, \dots, P_i$ have already been constructed for some i with $1 \le i < n$. Choose $p_i \in \min(P_i)$ and set $\omega(p_i) = i + 1$. Then let $P_{i+1} = P_i \setminus \{p_i\}$ and repeat the procedure for $i + 1$ until $i = n$.

We eventually get $P_n = \emptyset$ and $\omega\colon P \to [n]$ is a linear extension of P, having defined $p <_\omega q \Leftrightarrow \omega(p) < \omega(q)$.

By executing all possible choices of $p \in \min(P_i)$, the algorithm produces all possible linear extensions of P by depth-first search. By Theorem 2, we only need to check if ω is a semilabeling, i.e., if the map $\varphi(p) := (\omega(p), \omega(p) + \delta(p))$ is a labeling of P.

A partial check can be done during the construction itself, as soon as a new element $p \in P_i$ is chosen. Assume that $\omega(p)$ is initially set to zero for all $p \in P$ and that the variance $\delta(p)$ has been computed; then the algorithm takes the following form:

> Let $P_0 = P$. Choose $p_0 \in \min(P_0)$ and set $\omega(p_0) := 1$, $P_1 = P_0 \setminus \{p_0\}$. Suppose now that $P_1, P_2, \dots, P_i$ have already been constructed for some i with $1 \le i < n$. Choose $p_i \in \min(P_i)$ with the property that there is no $j < i$ such that $\omega(p_j) + \delta(p_j) = (i + 1) + \delta(p_i)$. If such an admissible p_i does not exist, then set $\omega(p_{i-1}) = 0$ and go back to step $i - 1$ to select a different p_{i-1}. Otherwise set $\omega(p_i) = i + 1$ and let $P_{i+1} = P_i \setminus \{p_i\}$. Repeat recursively the procedure for $i + 1$ until $i = n$ or no feasible choice is possible.

By considering all possible elements p_i, one gets all possible candidates for labelings of P. Note that we still have to test whether $(\omega, \omega + \delta)$ is indeed a labeling. The intermediate tests on the admissibility of the p_i's, though, cut a considerable number of branches from the search tree and shorten the execution time.

Terminating the algorithm whenever the first labeling is found gives a procedure for checking whether P has dimension 2.

5. SUPER-GREEDY DIMENSION

In [5] the super-greedy dimension of a poset P is defined to be the least integer t such that P is the intersection of t super-greedy linear extensions. A super-greedy linear extension of the finite poset P is defined as a linear extension of P obtained by the following linearization algorithm, called the Super-Greedy algorithm:

Set $P_0 = \min(P)$ and choose $p_1 \in P_0$.
Suppose $p_1, p_2, \dots, p_i$ have been chosen for some i with $1 \leq i < |P|$. Let $M_i = \min(P \setminus \{p_1, \dots, p_i\})$
and $J_i = \{j;\ 1 \leq j \leq i$, and there exists $p \in M_i$ such that $p_j < p$ in $P\}$. If $J_i \neq \emptyset$, let k be the largest integer in J_i and set $P_i = \{p \in M_i;\ p_k < p\}$. Else, set $P_i = M_i$. Choose $p_{i+1} \in P_i$.

As in the previous section, we can add to the choosing step in this algorithm the intermediate check of admissibility of the semilabeling. Thus, we can speak of super-greedy labelings of P.

Furthermore, if P has dimension 2, then the super-greedy dimension of P is 2 as well (see Theorem 3 in [5]).

Therefore we can use the modified super-greedy algorithm to check if the dimension of P is 2. This last method is more efficient than the algorithm in the previous section, since more branches of the search tree are not involved, as a consequence of a more restrictive choice of the p_i's.

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