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Interaction between nonlinear diffusion and geometry of domain ☆

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A B S T R A C T

Let Ω be a domain in $\mathbb{R}^N$, where $N \geqslant 2$ and $\partial\Omega$ is not necessarily bounded. We consider nonlinear diffusion equations of the form $\partial_t u = \Delta\phi(u)$. Let $u = u(x,t)$ be the solution of either the initial-boundary value problem over Ω, where the initial value equals zero and the boundary value equals 1, or the Cauchy problem where the initial data is the characteristic function of the set $\mathbb{R}^N \setminus \Omega$.

We consider an open ball B in Ω whose closure intersects $\partial\Omega$ only at one point, and we derive asymptotic estimates for the content of substance in B for short times in terms of geometry of Ω. Also, we obtain a characterization of the hyperplane involving a stationary level surface of u by using the sliding method due to Berestycki, Caffarelli, and Nirenberg. These results tell us about interactions between nonlinear diffusion and geometry of domain.

1. Introduction

Let Ω be a C^2 domain in $\mathbb{R}^N$, where $N \geqslant 2$ and $\partial\Omega$ is not necessarily bounded, and let $\phi : \mathbb{R} \to \mathbb{R}$ satisfy

$$\phi \in C^2(\mathbb{R}), \qquad \phi(0) = 0, \quad \text{and} \quad 0 < \delta_1 \leqslant \phi'(s) \leqslant \delta_2 \quad \text{for } s \in \mathbb{R}, \tag{1.1}$$

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where δ_1, δ_2 are positive constants. Consider the unique bounded solution $u = u(x,t)$ of either the initial-boundary value problem:

$$\partial_t u = \Delta\phi(u) \quad \text{in } \Omega \times (0,+\infty), \tag{1.2}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0,+\infty), \tag{1.3}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{1.4}$$

or the Cauchy problem:

$$\partial_t u = \Delta\phi(u) \quad \text{in } \mathbb{R}^N \times (0,+\infty) \quad \text{and} \quad u = \chi_{\Omega^c} \quad \text{on } \mathbb{R}^N \times \{0\}; \tag{1.5}$$

here χ_{Ω^c} denotes the characteristic function of the set $\Omega^c = \mathbb{R}^N \setminus \Omega$. Note that the uniqueness of the solution of either problem (1.2)–(1.4) or (1.5) follows from the comparison principle (see Theorem A.1 in the present paper). Since $\partial\Omega$ is of class C^2, we can construct barriers at any point on the boundary $\partial\Omega \times (0,+\infty)$ for problem (1.2)–(1.4). Thus, by the theory of uniformly parabolic equations (see [6]), we have the existence of a solution $u \in C^{2,1}(\Omega \times (0,+\infty)) \cap L^\infty(\Omega \times (0,+\infty)) \cap C^0(\overline{\Omega} \times (0,+\infty))$ such that $u(\cdot,t) \to 0$ in $L^1_{loc}(\Omega)$ as $t \to 0$ for problem (1.2)–(1.4). For problem (1.5), since for any bounded measurable initial data there exists a bounded solution of the Cauchy problem for $\partial_t u = \Delta\phi(u)$ by the theory of uniformly parabolic equations, we always have a solution $u \in C^{2,1}(\mathbb{R}^N \times (0,+\infty)) \cap L^\infty(\mathbb{R}^N \times (0,+\infty))$ such that $u(\cdot,t) \to \chi_{\Omega^c}(\cdot)$ in $L^1_{loc}(\mathbb{R}^N)$ as $t \to 0$ for any domain Ω, that is, in the case of problem (1.5), we only need that the set Ω is measurable.

The differential equation in (1.2) or in (1.5) has the property of *infinite* speed of propagation of disturbances from rest, since

$$\int_0^1 \frac{\phi'(\xi)}{\xi}\, d\xi = +\infty, \tag{1.6}$$

as it follows from (1.1).

By the strong comparison principle, we know that

$$0 < u < 1 \quad \text{either in } \Omega \times (0,+\infty) \text{ or in } \mathbb{R}^N \times (0,+\infty);$$

also, as $t \to 0^+$, u exhibits a *boundary layer*: while $u \to 0$ in Ω, u remains equal to 1 on $\partial\Omega$. The profile of u as $t \to 0^+$ is controlled by the function Φ defined by

$$\Phi(s) = \int_1^s \frac{\phi'(\xi)}{\xi}\, d\xi \quad \text{for } s > 0. \tag{1.7}$$

In fact, in [9, Theorems 1.1 and 4.1] we showed that, if $\partial\Omega$ is bounded and u is the solution of either problem (1.2)–(1.4) or problem (1.5), then

$$\lim_{t\to 0^+} -4t\Phi\big(u(x,t)\big) = d(x)^2 \quad \text{uniformly on every compact subset of } \Omega. \tag{1.8}$$

Here, $d = d(x)$ is the distance function:

$$d(x) = \operatorname{dist}(x, \partial\Omega) \quad \text{for } x \in \Omega. \tag{1.9}$$

Formula (1.8) generalizes one obtained by Varadhan [13] for the heat equation (and quite general linear parabolic equations); in that case, $\Phi(s) = \log s$ since $\phi(s) \equiv s$; (1.8) tells us about an interaction between nonlinear diffusion and geometry of domain, since the function $d(x)$ is deeply related to geometry of Ω.

We point out that (1.8) was proved in [9] when $\partial\Omega$ is bounded. In Theorem 2.1 in Section 2, we will show how to extend its validity to the case in which $\partial\Omega$ is unbounded. Moreover, with Theorem 2.1 in hand, in Theorem 2.3 we obtain a characterization of hyperplanes as stationary level surfaces of the solution u (i.e. surfaces where u remains constant at any given time); this result generalizes one of those obtained in [8,10] for the heat equation. As in [8, Theorem 3.4], the proof still relies on the sliding method due to Berestycki, Caffarelli, and Nirenberg [2] but, by a different argument, allows us to treat more general assumptions on Ω.

Let us now state our main theorem which shows a more intimate link between short-time nonlinear diffusion and the geometry of the domain Ω.

Theorem 1.1. *Let u be the solution of either problem* (1.2)–(1.4) *or problem* (1.5)*. Let $x_0 \in \Omega$ and assume that the open ball $B_R(x_0)$ centered at x_0 and with radius R is contained in Ω and such that $\overline{B_R(x_0)} \cap \partial\Omega = \{y_0\}$ for some $y_0 \in \partial\Omega$.*

Then we have:

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int\limits_{B_R(x_0)} u(x,t)\,dx = c(\phi, N)\left\{\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j(y_0)\right]\right\}^{-\frac{1}{2}}. \tag{1.10}$$

Here, $\kappa_1(y_0), \dots, \kappa_{N-1}(y_0)$ denote the principal curvatures of $\partial\Omega$ at y_0 with respect to the inward normal direction to $\partial\Omega$ and $c(\phi, N)$ is a positive constant depending only on ϕ and N (of course, $c(\phi, N)$ depends on the problems (1.2)–(1.4) *or* (1.5)*).*

When $\kappa_j(y_0) = \frac{1}{R}$ for some $j \in \{1, \dots, N-1\}$, the formula (1.10) *holds by setting the right-hand side to $+\infty$ (notice that $\kappa_j(y_0) \leqslant 1/R$ for every $j \in \{1, \dots, N-1\}$).*

Remark 1.2. In view of the proof given in the end of Section 3, under the existence of the solution u of problem (1.2)–(1.4), we need not assume that the entire $\partial\Omega$ is of class C^2 but only that it is of class C^2 in a neighborhood of the point y_0. Of course, in the case of problem (1.5) we only need to assume that $\partial\Omega$ is of class C^2 in a neighborhood of y_0.

A version of Theorem 1.1 was proved in [7] for problem (1.2)–(1.4), under the assumptions that $\partial\Omega$ is bounded and ϕ satisfies either $\int_0^1 \frac{\phi'(\xi)}{\xi}\,d\xi < +\infty$ or $\phi(s) \equiv s$. The reason why we could not treat cases in which $\int_0^1 \frac{\phi'(\xi)}{\xi}\,d\xi = +\infty$ and ϕ is nonlinear was merely technical. To be precise, in [7], the construction of supersolutions and subsolutions to problem (1.2)–(1.4) was eased by the property of *finite* speed of propagation of disturbances from rest that descends from the assumption $\int_0^1 \frac{\phi'(\xi)}{\xi}\,d\xi < +\infty$. In fact, such barriers were constructed in a set $\Omega_\rho \times (0, \tau]$, with

$$\Omega_\rho = \{x \in \Omega\colon\ d(x) < \rho\}, \tag{1.11}$$

where ρ and τ were chosen sufficiently small so that the solution u equals zero on the set $\Gamma_\rho \times (0, \tau]$, with

$$\Gamma_\rho = \{x \in \Omega\colon\ d(x) = \rho\}. \tag{1.12}$$

This property does not occur when (1.6) is in force. However, formula (1.10) seems general and is expected to hold for general diffusion equations. Here, we in fact overcome some of those technical difficulties and prove (1.10) for a class of nonlinear diffusion equations satisfying (1.6); moreover, the method of the proof of the present article enables us to treat also the case in which $\partial\Omega$ is unbounded. To be more specific, we construct the supersolutions and subsolutions for u without using the linearity of the heat equation and the result of Varadhan [13] as done in [7], but instead we exploit Theorem 2.1 together with a result of Atkinson and Peletier [1, Lemma 4, p. 383] concerning the asymptotic behavior of one-dimensional similarity solutions (see (3.15) in the present paper). Then, as in [7], we take advantage of their explicit form $f_\pm(t^{-\frac{1}{2}}d(x))$ (see Lemmas 3.1 and 3.2 in the present paper) to calculate their integrals over the ball $B_R(x_0)$ with the aid of the co-area formula. The proof of Theorem 1.1 is finally completed by letting $t \to 0^+$ and using a geometric lemma [7, Lemma 2.1, p. 376] (see Lemma 3.3 in the present paper). These will be done in Section 3.

In Appendix A, we give proofs of several facts used in Section 3, and prove a comparison principle (see Theorem A.1) for $\partial_t u = \Delta\phi(u)$ over general domains Ω including the case where $\partial\Omega$ is unbounded (in this case we could not find a proof of Theorem A.1 in the literature). Once the comparison principle is proved, then the strong comparison principle follows immediately.

2. Short-time asymptotic profile in the unbounded case and application

We begin with our extension of formula (1.8) to the case in which $\partial\Omega$ is unbounded.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be any domain with boundary $\partial\Omega$ of class C^2 and let u be the solution of either problem* (1.2)–(1.4) *or* (1.5).

Then (1.8) *holds true.*

Remark 2.2. In view of the proof given below, instead of assuming that $\partial\Omega$ is of class C^2, we only need to assume that $\partial\Omega = \partial(\mathbb{R}^N \setminus \overline{\Omega})$ under the existence of the solution u of problem (1.2)–(1.4). Of course, in the case of problem (1.5), we only need to assume that $\partial\Omega = \partial(\mathbb{R}^N \setminus \overline{\Omega})$.

Proof. The case where $\partial\Omega$ is bounded is treated in [9]; here, we shall assume that $\partial\Omega$ is unbounded.

Take any point $x_0 \in \Omega$. For each $\varepsilon > 0$, there exists an open ball $B_\delta(z)$, centered at z and with radius δ, contained in $\mathbb{R}^N \setminus \overline{\Omega}$, and such that $|x_0 - z| < d(x_0) + \varepsilon$.

Consider problem (1.2)–(1.4) first. Let $u^\pm = u^\pm(x,t)$ be bounded solutions of the following initial-boundary value problems:

$$
\begin{aligned}
\partial_t u^+ &= \Delta\phi\big(u^+\big) \quad \text{in } B_{d(x_0)}(x_0) \times (0, +\infty),\\
u^+ &= 1 \quad \text{on } \partial B_{d(x_0)}(x_0) \times (0, +\infty),\\
u^+ &= 0 \quad \text{on } B_{d(x_0)}(x_0) \times \{0\},
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_t u^- &= \Delta\phi\big(u^-\big) \quad \text{in } \big(\mathbb{R}^N \setminus \overline{B_\delta(z)}\big) \times (0, +\infty),\\
u^- &= 1 \quad \text{on } \partial B_\delta(z) \times (0, +\infty),\\
u^- &= 0 \quad \text{on } \big(\mathbb{R}^N \setminus \overline{B_\delta(z)}\big) \times \{0\},
\end{aligned}
$$

respectively. Then it follows from the comparison principle that

$$
u^-(x_0,t) \leqslant u(x_0,t) \leqslant u^+(x_0,t) \quad \text{for every } t > 0, \tag{2.1}
$$

which gives

$$
-4t\Phi\big(u^-(x_0,t)\big) \geqslant -4t\Phi\big(u(x_0,t)\big) \geqslant -4t\Phi\big(u^+(x_0,t)\big) \quad \text{for every } t > 0.
$$

By [9, Theorem 1.1], letting $t \to 0^+$ yields that

$$\big(d(x_0)+\varepsilon\big)^2 \geqslant \limsup_{t\to 0^+}\big(-4t\Phi\big(u(x_0,t)\big)\big) \geqslant \liminf_{t\to 0^+}\big(-4t\Phi\big(u(x_0,t)\big)\big) = d(x_0)^2.$$

This implies (1.8), since $\varepsilon > 0$ is arbitrary. Furthermore, let ρ_0 and ρ_1 be given such that $0 < \rho_0 \leqslant \rho_1 < +\infty$; then by a scaling argument, we infer that the convergence in (1.8) is uniform in every subset F of $\{x \in \Omega\colon\ \rho_0 \leqslant d(x) \leqslant \rho_1\}$ in which the number $\delta > 0$ can be chosen independently of each point $x \in F$. In particular, when F is compact, it was shown in [13, Lemma 3.11, p. 444] that $\delta > 0$ can be chosen independently of each point $x \in F$ only under the assumption that $\partial\Omega = \partial(\mathbb{R}^N \setminus \overline{\Omega})$.

It remains to consider problem (1.5). Let $u^\pm = u^\pm(x,t)$ be bounded solutions of the following initial value problems:

$$\partial_t u^+ = \Delta\phi\big(u^+\big) \quad \text{in } \mathbb{R}^N \times (0,+\infty) \quad \text{and} \quad u^+ = \chi_{B_{d(x_0)}(x_0)^c} \quad \text{on } \mathbb{R}^N \times \{0\},$$

and

$$\partial_t u^- = \Delta\phi\big(u^-\big) \quad \text{in } \mathbb{R}^N \times (0,+\infty) \quad \text{and} \quad u^- = \chi_{\overline{B_\delta(z)}} \quad \text{on } \mathbb{R}^N \times \{0\},$$

respectively. Then by the comparison principle we get (2.1). Thus, (1.8) follows similarly also in this case, with the aid of [9, Theorem 4.1]. □

We now give a simple application of the theorem just proved. Let $f \in C^2(\mathbb{R}^{N-1})$ and set

$$\Omega = \big\{x \in \mathbb{R}^N\colon\ x_N > f\big(x'\big)\big\},$$

where $x' = (x_1,\dots,x_{N-1}) \in \mathbb{R}^{N-1}$. Consider the solution $u = u(x,t)$ of either problem (1.2)–(1.4) or problem (1.5). In the sequel, it will be useful to know that

$$\frac{\partial u}{\partial x_N} < 0 \quad \text{either in } \Omega \times (0,+\infty) \text{ or in } \mathbb{R}^N \times (0,+\infty); \tag{2.2}$$

this is obtained by applying the comparison principle to $u(x', x_N + h, t)$ and $u(x,t)$ for $h > 0$ and then the strong maximum principle to the resultant nonnegative function $\frac{\partial\phi(u)}{\partial x_N}$, since $v = \phi(u)$ satisfies $\partial_t v = \phi'(u)\Delta v$.

A hypersurface Γ in Ω is said to be a *stationary level surface* of u if at each time t the solution u remains constant on Γ (a constant depending on t). The following theorem characterizes the boundary $\partial\Omega$ in such a way that u has a stationary level surface in Ω.

Theorem 2.3. *Assume that for each $y' \in \mathbb{R}^{N-1}$ there exists $h(y') \in \mathbb{R}$ such that*

$$\lim_{|x'|\to\infty}\big[f\big(x'+y'\big) - f\big(x'\big)\big] = h\big(y'\big). \tag{2.3}$$

Let u be the solution of either problem (1.2)–(1.4) *or problem* (1.5). *Suppose that u has a stationary level surface Γ in Ω.*

Then f is affine, that is, $\partial\Omega$ must be a hyperplane.

Remark 2.4. In view of the proof given below, instead of assuming that $f \in C^2(\mathbb{R}^{N-1})$, we only need to assume that $f \in C^0(\mathbb{R}^{N-1})$ under the existence of the solution u of problem (1.2)–(1.4). Of course, in the case of problem (1.5), we can replace the assumption $f \in C^2(\mathbb{R}^{N-1})$ with $f \in C^0(\mathbb{R}^{N-1})$.

Proof. We shall use the sliding method due to Berestycki, Caffarelli, and Nirenberg [2]. The condition (2.3) is a modified version of (7.2) of [2, p. 1108], in which $h(y')$ is supposed identically zero.

Since Γ is a stationary level surface of u, it follows from Theorem 2.1, (2.2) and the implicit function theorem that there exist a number $R > 0$ and a function $g \in C^2(\mathbb{R}^{N-1})$ such that

$$\Gamma = \{(x', g(x')) \in \mathbb{R}^N \colon x' \in \mathbb{R}^{N-1}\} = \{x \in \mathbb{R}^N \colon d(x) = R\}; \tag{2.4}$$

moreover, it is easy to verify that the function g satisfies

$$g(x') = \sup_{|x'-y'| \leqslant R} \left\{ f(y') + \sqrt{R^2 - |x' - y'|^2} \right\} \quad \text{for every } x' \in \mathbb{R}^{N-1}. \tag{2.5}$$

Conversely, let $\nu(y')$ denote the unit upward normal vector to Γ at $(y', g(y')) \in \Gamma$; the facts that g is smooth, $\partial\Omega$ is a graph, and $(y', g(y')) - R\nu(y') \in \partial\Omega$ for every $y' \in \mathbb{R}^{N-1}$ imply that

$$f(x') = \inf_{|x'-y'| \leqslant R} \left\{ g(y') - \sqrt{R^2 - |x' - y'|^2} \right\} \quad \text{for every } x' \in \mathbb{R}^{N-1}; \tag{2.6}$$

$$\partial\Omega = \{x \in \mathbb{R}^N \colon \operatorname{dist}(x, \{y \in \mathbb{R}^N \colon y_N \geqslant g(y')\}) = R\}. \tag{2.7}$$

Thus, it follows from (2.4) and (2.7) that for every $x \in \partial\Omega$ there exists $z \in \Gamma$ satisfying

$$B_R(z) \subset \Omega \quad \text{and} \quad \partial B_R(z) \cap \partial\Omega = \{x\}. \tag{2.8}$$

For fixed $y' \in \mathbb{R}^{N-1}$ and $h \in \mathbb{R}$, we define the translates:

$$\Omega_{y',h} = (y', h) + \Omega, \qquad \Gamma_{y',h} = (y', h) + \Gamma;$$

(2.3) guarantees that the values

$$\begin{aligned} h_+(y') &= \inf\{h \in \mathbb{R} \colon \Omega_{y',h} \subset \Omega\} \quad \text{and} \\ h_-(y') &= \sup\{h \in \mathbb{R} \colon \Omega \subset \Omega_{y',h}\} \end{aligned} \tag{2.9}$$

are finite, since in fact, $h_-(y') \leqslant h(y') \leqslant h_+(y')$ for every $y' \in \mathbb{R}^{N-1}$.

To complete our proof, it suffices to show that

$$h_-(y') = h(y') = h_+(y').$$

Indeed, this yields that $\Omega = \Omega_{y',h(y')}$ for every $y' \in \mathbb{R}^{N-1}$ and hence

$$f(x') = f(x' - y') + h(y') \quad \text{for every } x', y' \in \mathbb{R}^{N-1}. \tag{2.10}$$

Then, $\nabla f(x') = \nabla f(x' - y')$ for every $x', y' \in \mathbb{R}^{N-1}$ and hence ∇f must be constant in $\mathbb{R}^{N-1}$. Namely, f is affine and $\partial\Omega$ must be a hyperplane. When it is assumed only that $f \in C^0(\mathbb{R}^{N-1})$, without using differentiability of f, we can solve (2.10) as a functional equation with the help of continuity of f and we can also conclude that f is affine.

Thus, set $h_+ = h_+(y')$ and suppose by contradiction that $h_+ > h(y')$. Then it follows from (2.3) and (2.8) that there exist $x_0 \in \partial\Omega \cap \partial\Omega_{y',h_+}$ and $z \in \Gamma \cap \Gamma_{y',h_+}$ satisfying

$$\Omega_{y',h_+} \subsetneqq \Omega \quad \text{and} \quad \partial B_R(z) \cap \partial\Omega \cap \partial\Omega_{y',h_+} = \{x_0\}.$$

On the other hand, from the strong comparison principle we have

$$u\big(x' - y', x_N - h_+, t\big) > u(x,t) \quad \text{for every } (x,t) \in \Omega_{y',h_+} \times (0,\infty).$$

Therefore, $u(z' - y', x_N - h_+, t) > u(z,t)$ which contradicts the fact that $z \in \Gamma \cap \Gamma_{y',h_+}$ and that Γ is a stationary level surface of u.

The proof that $h_-(y') = h(y')$ runs similarly. □

3. Short-time asymptotics and curvature

This section is devoted to the proof of Theorem 1.1. We first prove two lemmas in which we construct useful barriers for problems (1.2)–(1.4) and (1.5), respectively.

In the former lemma, we use a result from Atkinson and Peletier [1]: for every $c > 0$, there exists a unique C^2 solution $f_c = f_c(\xi)$ of the problem:

$$\big(\phi'(f_c) f_c'\big)' + \frac{1}{2}\xi f_c' = 0 \quad \text{in } [0,+\infty), \tag{3.1}$$

$$f_c(0) = c, \qquad f_c(\xi) \to 0 \quad \text{as } \xi \to +\infty, \tag{3.2}$$

$$f_c' < 0 \quad \text{in } [0,+\infty). \tag{3.3}$$

Note that, if we put $w(s,t) = f_c(t^{-\frac{1}{2}} s)$ for $s > 0$ and $t > 0$, then w satisfies the one-dimensional problem:

$$\partial_t w = \partial_s^2 \phi(w) \quad \text{in } (0,+\infty)^2, \qquad w = c \quad \text{on } \{0\} \times (0,+\infty), \quad \text{and} \quad w = 0 \quad \text{on } (0,+\infty) \times \{0\}.$$

Lemma 3.1. *Let $\partial\Omega$ be bounded and of class C^2 and let $\rho_0 > 0$ be such that the distance function d belongs to $C^2(\overline{\Omega_{\rho_0}})$ (see [5]); then, set $\rho_1 = \max\{2R, \rho_0\}$. Let $u = u(x,t)$ be the solution of problem (1.2)–(1.4).*

Then, for every $\varepsilon \in (0, 1/4)$, there exist two C^2 functions $f_\pm = f_\pm(\xi) : [0,+\infty) \to \mathbb{R}$ satisfying

$$0 < f_\pm(\xi) \leqslant \alpha e^{-\beta\xi^2} \quad \textit{for every } \xi \in [0,+\infty); \tag{3.4}$$

$$f_\pm \to f_1 \quad \textit{as } \varepsilon \to 0 \textit{ uniformly on } [0,+\infty), \tag{3.5}$$

where α and β are positive constants independent of ε, and there exists a number $\tau = \tau_\varepsilon > 0$ such that the functions $w_\pm$, defined by

$$w_\pm(x,t) = f_\pm\big(t^{-\frac{1}{2}} d(x)\big) \quad \textit{for } (x,t) \in \Omega \times (0,+\infty), \tag{3.6}$$

satisfy the inequalities:

$$w_- \leqslant u \leqslant w_+ \quad \textit{in } \overline{\Omega_{\rho_1}} \times (0,\tau]. \tag{3.7}$$

Proof. We begin by deriving some properties of the solution f_c of problem (3.1)–(3.3); by writing $v_c = v_c(\xi) = \phi(f_c(\xi))$ for $\xi \in [0,+\infty)$, we see that

$$\frac{v_c''}{v_c'} = -\frac{1}{2}\xi \frac{1}{\phi'(f_c)} \quad \text{in } [0,+\infty). \tag{3.8}$$

With the aid of the last assumption in (1.1), integrating (3.8) yields that

$$v_c'(0)\exp\left\{-\frac{\xi^2}{4\delta_2}\right\} \leqslant v_c'(\xi) \leqslant v_c'(0)\exp\left\{-\frac{\xi^2}{4\delta_1}\right\} < 0 \quad \text{for every } \xi > 0, \tag{3.9}$$

and hence

$$\frac{v_c'(0)}{\delta_1}\exp\left\{-\frac{\xi^2}{4\delta_2}\right\} \leqslant f_c'(\xi) \leqslant \frac{v_c'(0)}{\delta_2}\exp\left\{-\frac{\xi^2}{4\delta_1}\right\} < 0 \quad \text{for every } \xi > 0. \tag{3.10}$$

Furthermore, by integrating (3.10) and using (3.2), we have that for every $\xi > 0$

$$-\frac{v_c'(0)}{\delta_1}\int_\xi^\infty \exp\left\{-\frac{\eta^2}{4\delta_2}\right\} d\eta \geqslant f_c(\xi) \geqslant -\frac{v_c'(0)}{\delta_2}\int_\xi^\infty \exp\left\{-\frac{\eta^2}{4\delta_1}\right\} d\eta. \tag{3.11}$$

Thus, with the aid of (3.9) and (3.11), by integrating (3.1), we have:

$$-v_c'(0) = \frac{1}{2}\int_0^\infty f_c(\xi)\,d\xi \quad \text{for } c > 0. \tag{3.12}$$

Moreover, a comparison argument will give us

$$0 < f_{c_1} < f_{c_2} \quad \text{on } [0, +\infty) \text{ if } 0 < c_1 < c_2 < +\infty; \tag{3.13}$$

$$0 > v_{c_1}'(0) > v_{c_2}'(0) \quad \text{if } 0 < c_1 < c_2 < +\infty. \tag{3.14}$$

In Appendix A, we will give a proof of (3.12)–(3.14).

Furthermore, [1, Lemma 4, p. 383] tells us that, for every compact interval I contained in $(0, +\infty)$,

$$\frac{-4\Phi(f_c(\xi))}{\xi^2} \to 1 \quad \text{as } \xi \to +\infty \text{ uniformly for } c \in I. \tag{3.15}$$

Let $0 < \varepsilon < \frac{1}{4}$. Then, by continuity we can find a sufficiently small $0 < \eta_\varepsilon \ll \varepsilon$ and two C^2 functions $f_\pm = f_\pm(\xi)$ for $\xi \geqslant 0$ satisfying:

$$f_\pm(\xi) = f_{1\pm\varepsilon}\left(\sqrt{1 \mp 2\eta_\varepsilon}\,\xi\right) \quad \text{if } \xi \geqslant \eta_\varepsilon;$$

$$f_\pm' < 0 \quad \text{in } [0, +\infty);$$

$$f_- < f_1 < f_+ \quad \text{in } [0, +\infty);$$

$$\left(\phi'(f_\pm) f_\pm'\right)' + \frac{1}{2}\xi f_\pm' = h_\pm(\xi) f_\pm' \quad \text{in } [0, +\infty),$$

where $h_\pm = h_\pm(\xi)$ are defined by

$$h_\pm(\xi) = \begin{cases} \pm\eta_\varepsilon\xi & \text{if } \xi \geqslant \eta_\varepsilon, \\ \pm\eta_\varepsilon^2 & \text{if } \xi \leqslant \eta_\varepsilon. \end{cases}$$

(Here, in order to use the functions $h_\pm$ also in Lemma 3.2 later, we defined $h_\pm(\xi)$ for all $\xi \in \mathbb{R}$.) It is important to notice that

$$h_+ = -h_- \geqslant \eta_\varepsilon^2 \quad \text{on } \mathbb{R}. \tag{3.16}$$

Moreover, (3.5) follows directly from the above construction of $f_\pm$, and (3.11) together with (3.13) yields (3.4).

Set $\Psi = \Phi^{-1}$. Then it follows from (3.15) that there exists $\xi_\varepsilon > 1$ such that

$$\Psi\left(-\frac{\xi^2}{4}\left(1-\frac{\eta_\varepsilon}{2}\right)\right) > f_c(\xi) > \Psi\left(-\frac{\xi^2}{4}\left(1+\frac{\eta_\varepsilon}{2}\right)\right) \quad \text{for } \xi \geqslant \xi_\varepsilon \text{ and } c \in I_\varepsilon, \tag{3.17}$$

where we set $I_\varepsilon = [1-2\varepsilon, 1+2\varepsilon]$.

Since $\partial\Omega$ is bounded and of class C^2, Theorem 2.1 yields that

$$-4t\Phi\big(u(x,t)\big) \to d(x)^2 \quad \text{as } t \to 0^+ \text{ uniformly on } \overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}. \tag{3.18}$$

Then there exists $\tau_{1,\varepsilon} > 0$ such that for every $t \in (0, \tau_{1,\varepsilon}]$ and every $x \in \overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}$

$$\left|-4t\Phi\big(u(x,t)\big) - d(x)^2\right| < \frac{1}{2}\eta_\varepsilon \rho_0^2 \leqslant \frac{1}{2}\eta_\varepsilon\, d(x)^2,$$

which implies that

$$\Psi\left(-\frac{(1-\frac{1}{2}\eta_\varepsilon)}{4}\frac{d(x)^2}{t}\right) > u(x,t) > \Psi\left(-\frac{(1+\frac{1}{2}\eta_\varepsilon)}{4}\frac{d(x)^2}{t}\right), \tag{3.19}$$

for every $t \in (0, \tau_{1,\varepsilon}]$ and every $x \in \overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}$.

From (3.17), we have

$$f_+(\xi) = f_{1+\varepsilon}(\sqrt{1-2\eta_\varepsilon}\,\xi) > \Psi\left(-\frac{\xi^2}{4}\left(1-\frac{\eta_\varepsilon}{2}\right)\right) \quad \text{if } \xi \geqslant \frac{\xi_\varepsilon}{\sqrt{1-2\eta_\varepsilon}}; \tag{3.20}$$

$$f_-(\xi) = f_{1-\varepsilon}(\sqrt{1+2\eta_\varepsilon}\,\xi) < \Psi\left(-\frac{\xi^2}{4}\left(1+\frac{\eta_\varepsilon}{2}\right)\right) \quad \text{if } \xi \geqslant \frac{\xi_\varepsilon}{\sqrt{1+2\eta_\varepsilon}}. \tag{3.21}$$

Now, consider the two functions $w_\pm = w_\pm(x,t)$ defined by (3.6). It follows from (3.19), (3.20) and (3.21) that there exists $\tau_{2,\varepsilon} \in (0, \tau_{1,\varepsilon}]$ satisfying

$$w_- < u < w_+ \quad \text{in } (\overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}) \times (0, \tau_{2,\varepsilon}]. \tag{3.22}$$

Since $d \in C^2(\overline{\Omega_{\rho_0}})$ and $|\nabla d| = 1$ in $\overline{\Omega_{\rho_0}}$, we have

$$\partial_t w_\pm - \Delta\phi(w_\pm) = -f_\pm' t^{-1}\left\{h_\pm + \sqrt{t}\,\phi'(f_\pm)\Delta d\right\} \quad \text{in } \overline{\Omega_{\rho_0}} \times (0, +\infty).$$

Therefore, it follows from (3.16) that there exists $\tau_{3,\varepsilon} \in (0, \tau_{2,\varepsilon}]$ satisfying

$$\partial_t w_- - \Delta\phi(w_-) < 0 < \partial_t w_+ - \Delta\phi(w_+) \quad \text{in } \Omega_{\rho_0} \times (0, \tau_{3,\varepsilon}].$$

Observe that

$$w_- = u = w_+ = 0 \quad \text{in } \Omega_{\rho_0} \times \{0\},$$

$$w_- = f_-(0) < 1 = f_1(0) = u < f_+(0) = w_+ \quad \text{on } \partial\Omega \times (0, \tau_{3,\varepsilon}],$$

$$w_- < u < w_+ \quad \text{on } \Gamma_{\rho_0} \times (0, \tau_{3,\varepsilon}].$$

Note that the last inequalities above come from (3.22).

Thus, (3.7) holds true with $\tau = \tau_{3,\varepsilon}$, by the comparison principle and (3.22). □

In the next lemma, instead of (3.1)–(3.3), we will work with the following problem:

$$\left(\phi'(f_c) f_c'\right)' + \frac{1}{2}\xi f_c' = 0 \quad \text{in } \mathbb{R}, \tag{3.23}$$

$$f_c(\xi) \to c \quad \text{as } \xi \to -\infty, \qquad f_c(\xi) \to 0 \quad \text{as } \xi \to +\infty, \tag{3.24}$$

$$f_c' < 0 \quad \text{in } \mathbb{R}. \tag{3.25}$$

In Appendix A we will prove that, for every $c > 0$, (3.23)–(3.25) has a unique C^2 solution $f_c = f_c(\xi)$. Note that, if we put $w(s,t) = f_c(t^{-\frac{1}{2}} s)$ for $s \in \mathbb{R}$ and $t > 0$, then w satisfies the one-dimensional initial value problem:

$$\partial_t w = \partial_s^2 \phi(w) \quad \text{in } \mathbb{R} \times (0, +\infty) \quad \text{and} \quad w = c\chi_{(-\infty,0]} \quad \text{on } \mathbb{R} \times \{0\}.$$

Also, let us consider the signed distance function $d^* = d^*(x)$ of $x \in \mathbb{R}^N$ to the boundary $\partial\Omega$ defined by

$$d^*(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \notin \Omega. \end{cases}$$

If $\partial\Omega$ is bounded and of class C^2, there exists a number $\rho_0 > 0$ such that $d^*(x)$ is C^2-smooth on a compact neighborhood $\mathcal{N}$ of the boundary $\partial\Omega$ given by

$$\mathcal{N} = \left\{x \in \mathbb{R}^N \colon -\rho_0 \leqslant d^*(x) \leqslant \rho_0\right\}.$$

For simplicity we have used the same letter $\rho_0 > 0$ as in Lemma 3.1.

Lemma 3.2. *Let $\partial\Omega$ be bounded and of class C^2, set $\rho_1 = \max\{2R, \rho_0\}$ and let $u = u(x,t)$ be the solution of problem* (1.5).

Then, for every $\varepsilon \in (0, 1/4)$, there exist two C^2 functions $f_\pm = f_\pm(\xi) : \mathbb{R} \to \mathbb{R}$ satisfying

$$0 < f_\pm(\xi) \leqslant \alpha e^{-\beta\xi^2} \quad \text{for every } \xi \in [0, +\infty); \tag{3.26}$$

$$f_\pm \to f_1 \quad \text{as } \varepsilon \to 0 \text{ uniformly on } [0, +\infty), \tag{3.27}$$

where α and β are positive constants independent of ε, and there exists a number $\tau = \tau_\varepsilon > 0$ such that the functions $w_\pm$, defined by

$$w_\pm(x,t) = f_\pm\left(t^{-\frac{1}{2}} d^*(x)\right) \quad \text{for } (x,t) \in \mathbb{R}^N \times (0, +\infty), \tag{3.28}$$

satisfy the inequalities:

$$w_- \leqslant u \leqslant w_+ \quad \textit{in}\ \overline{\mathcal{N} \cup \Omega_{\rho_1}} \times (0, \tau]. \tag{3.29}$$

Proof. Let f_c be the solution of problem (3.23)–(3.25); by writing $v_c = v_c(\xi) = \phi(f_c(\xi))$ for $\xi \in \mathbb{R}$, we have:

$$-v_c'(0) = \frac{1}{2}\int_0^\infty f_c(\xi)\,d\xi \quad \text{for } c > 0; \tag{3.30}$$

$$0 < f_{c_1} < f_{c_2} \quad \text{in } \mathbb{R} \text{ if } 0 < c_1 < c_2 < +\infty; \tag{3.31}$$

$$0 > v_{c_1}'(0) > v_{c_2}'(0) \quad \text{if } 0 < c_1 < c_2 < +\infty. \tag{3.32}$$

In Appendix A we will give a proof of (3.30)–(3.32). Then [1, Lemma 4, p. 383] tells us that (3.15) also holds for the solution f_c of this problem.

Let $0 < \varepsilon < \frac{1}{4}$. We can find a sufficiently small $0 < \eta_\varepsilon \ll \varepsilon$ and two C^2 functions $f_\pm = f_\pm(\xi)$ for $\xi \in \mathbb{R}$ satisfying:

$$f_\pm(\xi) = f_{1\pm\varepsilon}\big(\sqrt{1 \mp 2\eta_\varepsilon}\,\xi\big) \quad \text{if } \xi \geqslant \eta_\varepsilon; \tag{3.33}$$

$$f_\pm' < 0 \quad \text{in } \mathbb{R}; \tag{3.34}$$

$$f_-(-\infty) < 1 = f_1(-\infty) < f_+(-\infty) \quad \text{and} \quad f_- < f_1 < f_+ \quad \text{in } \mathbb{R}; \tag{3.35}$$

$$\big(\phi'(f_\pm)f_\pm'\big)' + \frac{1}{2}\xi f_\pm' = h_\pm(\xi) f_\pm' \quad \text{in } \mathbb{R}. \tag{3.36}$$

In Appendix A we will prove (3.35) by choosing $\eta_\varepsilon > 0$ sufficiently small.

Here, we also have (3.16), (3.26), and (3.27). Moreover, it follows from (3.15) that there exists $\xi_\varepsilon > 1$ satisfying (3.17). Proceeding similarly yields (3.18), (3.19), (3.20) and (3.21).

Now, consider the functions $w_\pm$ defined by (3.28). Then we also have (3.22) and, since $d^* \in C^2(\mathcal{N})$ and $|\nabla d^*| = 1$ in $\mathcal{N}$, we obtain that

$$\partial_t w_\pm - \Delta\phi(w_\pm) = -f_\pm' t^{-1}\big\{h_\pm + \sqrt{t}\,\phi'(f_\pm)\Delta d^*\big\} \quad \text{in } \mathcal{N} \times (0, +\infty).$$

Therefore, it follows from (3.16) that there exists $\tau_{3,\varepsilon} \in (0, \tau_{2,\varepsilon}]$ satisfying:

$$\partial_t w_- - \Delta\phi(w_-) < 0 < \partial_t w_+ - \Delta\phi(w_+) \quad \text{in } \mathcal{N} \times (0, \tau_{3,\varepsilon}],$$

$$w_- \leqslant u \leqslant w_+ \quad \text{in } \mathcal{N} \times \{0\},$$

$$w_- < u < w_+ \quad \text{on } \partial\mathcal{N} \times (0, \tau_{3,\varepsilon}].$$

Note that, in the last inequalities, the ones on $\Gamma_{\rho_0} \times (0, \tau_{3,\varepsilon}]$ come from (3.22) and the others on $(\partial\mathcal{N} \setminus \Gamma_{\rho_0}) \times (0, \tau_{3,\varepsilon}]$ come from the former formula of (3.35).

Thus, (3.29) follows, with $\tau = \tau_{3,\varepsilon}$, from the comparison principle and (3.22). $\square$

In the proof of Theorem 1.1, we will also use a geometric lemma from [7] adjusted to our situation.

Lemma 3.3. (*See* [7, Lemma 2.1, p. 376].) *Let $\kappa_j(y_0) < \frac{1}{R}$ for every $j = 1, \dots, N-1$. Then we have*:

$$\lim_{s\to 0^+} s^{-\frac{N-1}{2}} \mathcal{H}^{N-1}\big(\Gamma_s \cap B_R(x_0)\big) = 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}},$$

where $\mathcal{H}^{N-1}$ is the standard $(N-1)$-dimensional Hausdorff measure, and ω_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$.

Proof of Theorem 1.1. We distinguish two cases:

$$\text{(I)}\quad \partial\Omega \text{ is bounded and of class } C^2; \qquad \text{(II)}\quad \partial\Omega \text{ is otherwise.}$$

Let us first show how we obtain case (II) once we have proved case (I). Indeed, we can find two C^2 domains, say Ω_1 and Ω_2, with bounded boundaries, and a ball $B_\delta(y_0)$ with the following properties: Ω_1 and $\mathbb{R}^N \setminus \overline{\Omega_2}$ are bounded; $B_R(x_0) \subset \Omega_1 \subset \Omega \subset \Omega_2$;

$$B_\delta(y_0) \cap \partial\Omega \subset \partial\Omega_1 \cap \partial\Omega_2 \quad \text{and} \quad \overline{B_R(x_0)} \cap \big(\mathbb{R}^N \setminus \Omega_i\big) = \{y_0\} \quad \text{for } i = 1, 2.$$

Let $u_i = u_i(x,t)$ $(i = 1, 2)$ be the two bounded solutions of either problem (1.2)–(1.4) or problem (1.5) where Ω is replaced by Ω_1 or Ω_2, respectively. Since $\Omega_1 \subset \Omega \subset \Omega_2$, it follows from the comparison principle that

$$u_2 \leqslant u \quad \text{in } \Omega \times (0, +\infty) \quad \text{and} \quad u \leqslant u_1 \quad \text{in } \Omega_1 \times (0, +\infty).$$

Therefore, it follows that for every $t > 0$

$$t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u_2(x,t)\,dx \leqslant t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x,t)\,dx \leqslant t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u_1(x,t)\,dx.$$

These two inequalities show that case (I) implies case (II).

Now, let us consider case (I). First, we take care of problem (1.2)–(1.4). Lemma 3.1 implies that for every $t \in (0, \tau]$

$$t^{-\frac{N+1}{4}} \int_{B_R(x_0)} w_-\,dx \leqslant t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u\,dx \leqslant t^{-\frac{N+1}{4}} \int_{B_R(x_0)} w_+\,dx. \tag{3.37}$$

Also, with the aid of the co-area formula, we have:

$$\int_{B_R(x_0)} w_\pm\,dx = t^{\frac{N+1}{4}} \int_0^{2Rt^{-\frac{1}{2}}} f_\pm(\xi)\xi^{\frac{N-1}{2}} \big(t^{\frac{1}{2}}\xi\big)^{-\frac{N-1}{2}} \mathcal{H}^{N-1}\big(\Gamma_{t^{\frac{1}{2}}\xi} \cap B_R(x_0)\big)\,d\xi.$$

Thus, when $\kappa_j(y_0) < \frac{1}{R}$ for every $j = 1, \dots, N-1$, by Lebesgue's dominated convergence theorem, (3.4), and Lemma 3.3, we get

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} w_\pm \, dx = 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}} \int_0^\infty f_\pm(\xi) \xi^{\frac{N-1}{2}} \, d\xi.$$

Moreover, again by Lebesgue's dominated convergence theorem, (3.4), and (3.5), we see that

$$\lim_{\varepsilon\to 0} \int_0^\infty f_\pm(\xi)\xi^{\frac{N-1}{2}} \, d\xi = \int_0^\infty f_1(\xi)\xi^{\frac{N-1}{2}} \, d\xi.$$

Therefore, since $\varepsilon > 0$ is arbitrarily small in (3.37), it follows that (1.10) holds true, where we set

$$c(\phi, N) = 2^{\frac{N-1}{2}} \omega_{N-1} \int_0^\infty f_1(\xi)\xi^{\frac{N-1}{2}} \, d\xi.$$

It remains to consider the case where $\kappa_j(y_0) = \frac{1}{R}$ for some $j \in \{1, \dots, N-1\}$. Choose a sequence of balls $\{B_{R_k}(x_k)\}_{k=1}^\infty$ satisfying:

$$R_k < R, \qquad y_0 \in \partial B_{R_k}(x_k), \quad \text{and} \quad B_{R_k}(x_k) \subset B_R(x_0) \quad \text{for every } k \geqslant 1, \text{ and } \lim_{k\to\infty} R_k = R.$$

Since $\kappa_j(y_0) \leqslant \frac{1}{R} < \frac{1}{R_k}$ for every $j = 1, \dots, N-1$ and every $k \geqslant 1$, we can apply the previous case to each $B_{R_k}(x_k)$ to see that for every $k \geqslant 1$

$$\begin{aligned} \liminf_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x,t) \, dx &\geqslant \liminf_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B_{R_k}(x_k)} u(x,t) \, dx \\ &= c(\phi, N) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R_k} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}}. \end{aligned}$$

Hence, letting $k \to \infty$ yields that

$$\liminf_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x,t) \, dx = +\infty,$$

which completes the proof for problem (1.2)–(1.4).

The proof of (1.10) in the case of problem (1.5) runs similarly with the aid of Lemmas 3.2 and 3.3. □

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Appendix A

Here, for the reader's convenience, we give proofs of several facts used in Section 3, and prove a comparison principle (Theorem A.1) for $\partial_t u = \Delta\phi(u)$ over general domains Ω including the case where $\partial\Omega$ is unbounded.

Proof of (3.12)–(3.14). First of all, (3.12) and (3.13) imply (3.14). It suffices to prove (3.12) and (3.13). Let $c > 0$. By integrating Eq. (3.1) on $[0, \eta]$ for every $\eta > 0$ and integrating by parts, we get

$$v_c'(\eta) - v_c'(0) + \frac{1}{2}\eta f_c(\eta) - \frac{1}{2}\int_0^\eta f_c(\xi)\, d\xi = 0.$$

Then, with the aid of (3.9) and (3.11), letting $\eta \to \infty$ yields (3.12).

Let $0 < c_1 < c_2 < +\infty$. Since $f_{c_1}(0) = c_1 < c_2 = f_{c_2}(0)$, suppose that there exists $\xi_0 > 0$ satisfying

$$f_{c_1}(\xi_0) = f_{c_2}(\xi_0) \quad \text{and} \quad f_{c_1}(\xi) < f_{c_2}(\xi) \quad \text{for every } \xi \in [0, \xi_0).$$

Then it follows from the uniqueness of solutions of Cauchy problems for ordinary differential equations that

$$v_{c_2}'(\xi_0) < v_{c_1}'(\xi_0) < 0. \tag{A.1}$$

Thus, we distinguish two cases:

(i) There exists $\xi_1 \in (\xi_0, \infty)$ satisfying

$$f_{c_1}(\xi_1) = f_{c_2}(\xi_1) \quad \text{and} \quad f_{c_1}(\xi) > f_{c_2}(\xi) \quad \text{for every } \xi \in (\xi_0, \xi_1).$$

(ii) For every $\xi \in (\xi_0, \infty)$, $f_{c_1}(\xi) > f_{c_2}(\xi)$.

In case (i), by the uniqueness, we also have

$$v_{c_1}'(\xi_1) < v_{c_2}'(\xi_1) < 0. \tag{A.2}$$

By integrating Eq. (3.1) on $[\xi_0, \xi_1]$ for f_{c_1} and f_{c_2} and integrating by parts, we see that for $j = 1, 2$

$$v_{c_j}'(\xi_1) - v_{c_j}'(\xi_0) + \frac{1}{2}\xi_1 f_{c_j}(\xi_1) - \frac{1}{2}\xi_0 f_{c_j}(\xi_0) - \frac{1}{2}\int_{\xi_0}^{\xi_1} f_{c_j}(\xi)\, d\xi = 0.$$

Then, considering the difference of these two equalities yields

$$v_{c_1}'(\xi_1) - v_{c_2}'(\xi_1) - \big(v_{c_1}'(\xi_0) - v_{c_2}'(\xi_0)\big) - \frac{1}{2}\int_{\xi_0}^{\xi_1} \big(f_{c_1}(\xi) - f_{c_2}(\xi)\big)\, d\xi = 0.$$

This contradicts (A.1), (A.2) and the situation of case (i).

In case (ii), by integrating Eq. (3.1) on $[\xi_0, \infty)$ for f_{c_1} and f_{c_2} and integrating by parts, we see that for $j = 1, 2$

$$-v'_{c_j}(\xi_0) - \frac{1}{2}\xi_0 f_{c_j}(\xi_0) - \frac{1}{2}\int_{\xi_0}^{\infty} f_{c_j}(\xi)\,d\xi = 0.$$

Then, considering the difference of these two equalities yields

$$-\big(v'_{c_1}(\xi_0) - v'_{c_2}(\xi_0)\big) - \frac{1}{2}\int_{\xi_0}^{\infty}\big(f_{c_1}(\xi) - f_{c_2}(\xi)\big)\,d\xi = 0.$$

This contradicts (A.1) and the situation of case (ii). □

Proof of the existence and uniqueness of the solution of problem (3.23)–(3.25). Let $c > 0$ and define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(s) = \phi(c) - \phi(c - s) \quad \text{for } s \in \mathbb{R}. \tag{A.3}$$

Then ψ satisfies the same condition (1.1) as ϕ does. It was shown in [1] that, for every $a > 0$, there exists a unique C^2 solution $g_a = g_a(\xi)$ of the problem:

$$\big(\psi'(g_a)g'_a\big)' + \frac{1}{2}\xi g'_a = 0 \quad \text{in } [0, +\infty), \tag{A.4}$$

$$g_a(0) = a, \qquad g_a(\xi) \to 0 \quad \text{as } \xi \to +\infty, \tag{A.5}$$

$$g'_a < 0 \quad \text{in } [0, +\infty). \tag{A.6}$$

Hence, writing $V_a = V_a(\xi) = \psi(g_a(\xi))$ for $\xi \in [0, +\infty)$ and proceeding similarly yield that

$$-V'_a(0) = \frac{1}{2}\int_0^{\infty} g_a(\xi)\,d\xi \quad \text{for } a > 0;$$

$$0 < g_{a_1} < g_{a_2} \quad \text{on } [0, +\infty) \text{ if } 0 < a_1 < a_2 < +\infty;$$

$$0 > V'_{a_1}(0) > V'_{a_2}(0) \quad \text{if } 0 < a_1 < a_2 < +\infty.$$

For $a \in (0, c)$, define $f_{a,-} = f_{a,-}(\xi)$ by

$$f_{a,-}(\xi) = c - g_a(-\xi) \quad \text{for } \xi \in (-\infty, 0].$$

Then, in view of (A.3)–(A.6), $f_{a,-}$ satisfies the following:

$$\big(\phi'(f_{a,-})f'_{a,-}\big)' + \frac{1}{2}\xi f'_{a,-} = 0 \quad \text{in } (-\infty, 0],$$

$$f_{a,-}(0) = c - a, \qquad f_{a,-}(\xi) \to c \quad \text{as } \xi \to -\infty,$$

$$f'_{a,-} < 0 \quad \text{in } (-\infty, 0].$$

Let $f_{a,+} = f_{a,+}(\xi)$ ($\xi \in [0, +\infty)$) be the unique C^2 solution f_{c-a} of problem (3.1)–(3.3) where c is replaced by $c - a$. Then we have

$$\left(\phi(f_{a,-})\right)'\big|_{\xi=0} = V_a'(0) \quad \text{and} \quad \left(\phi(f_{a,+})\right)'\big|_{\xi=0} = v_{c-a}'(0),$$

where $v_{c-a}(\xi) = \phi(f_{a,+}(\xi))$ for $\xi \in [0, +\infty)$. Observe that both $V_a'(0)$ and $v_{c-a}'(0)$ are continuous as functions of a on the interval $[0, c]$, $V_a'(0)$ is strictly decreasing, $v_{c-a}'(0)$ is strictly increasing, and $\lim_{a\to 0} V_a'(0) = \lim_{a\to c} v_{c-a}'(0) = 0$. Therefore, there exists a unique $a_* \in (0, c)$ satisfying $V_{a_*}'(0) = v_{c-a_*}'(0)$, and hence the unique C^2 solution $f_c = f_c(\xi)$ of problem (3.23)–(3.25) is given by

$$f_c(\xi) = \begin{cases} f_{a_*,+}(\xi) & \text{if } \xi \in [0, +\infty), \\ f_{a_*,-}(\xi) & \text{if } \xi \in (-\infty, 0). \end{cases} \qquad \square$$

Proof of (3.30)–(3.32). The proof of (3.12) also works for (3.30). (3.30) and (3.31) imply (3.32). Thus it suffices to prove (3.31). Let $0 < c_1 < c_2 < +\infty$. Since $\lim_{\xi\to-\infty} f_{c_1}(\xi) = c_1 < c_2 = \lim_{\xi\to-\infty} f_{c_2}(\xi)$, there exists $\xi_* < 0$ satisfying

$$f_{c_1}(\xi) < f_{c_2}(\xi) \quad \text{for every } \xi \leqslant \xi_*.$$

Hence we can begin with supposing that there exists $\xi_0 > \xi_*$ satisfying

$$f_{c_1}(\xi_0) = f_{c_2}(\xi_0) \quad \text{and} \quad f_{c_1}(\xi) < f_{c_2}(\xi) \quad \text{for every } \xi \in [\xi_*, \xi_0).$$

Therefore, the rest of the proof runs along that of (3.13). $\square$

Proof of (3.35). In view of (3.31) and (3.32), by continuity, we can find a sufficiently small $0 < \eta_\varepsilon \ll \varepsilon$ and two C^2 functions $f_\pm = f_\pm(\xi)$ for $\xi \in \mathbb{R}$ satisfying (3.33), (3.34), (3.36) and the following:

$$f_- < f_1 < f_+ \quad \text{on } [0, +\infty); \tag{A.7}$$

$$0 < f_{1-\frac{3}{2}\varepsilon} < \tilde{f}_- < f_{1-\frac{1}{2}\varepsilon} < f_{1+\frac{1}{2}\varepsilon} < \tilde{f}_+ < f_{1+\frac{3}{2}\varepsilon} \quad \text{at } \xi = 0; \tag{A.8}$$

$$0 > v_{1-\frac{3}{2}\varepsilon}' > \left(\phi(\tilde{f}_-)\right)' > v_{1-\frac{1}{2}\varepsilon}' > v_{1+\frac{1}{2}\varepsilon}' > \left(\phi(\tilde{f}_+)\right)' > v_{1+\frac{3}{2}\varepsilon}' \quad \text{at } \xi = 0, \tag{A.9}$$

where we put $\tilde{f}_\pm(\xi) = f_\pm(\xi \pm 2\eta_\varepsilon^2)$ for $\xi \in \mathbb{R}$. Notice that $\tilde{f}_\pm = \tilde{f}_\pm(\xi)$ satisfy

$$\left(\phi'(\tilde{f}_\pm)\tilde{f}_\pm'\right)' + \frac{1}{2}\xi \tilde{f}_\pm' = 0 \quad \text{in } (-\infty, 0]. \tag{A.10}$$

In order to prove (3.35), it suffices to show that

$$f_{1-\frac{3}{2}\varepsilon} < \tilde{f}_- < f_{1-\frac{1}{2}\varepsilon} \quad \text{and} \quad f_{1+\frac{1}{2}\varepsilon} < \tilde{f}_+ < f_{1+\frac{3}{2}\varepsilon} \quad \text{in } (-\infty, 0]. \tag{A.11}$$

Indeed, (3.34) implies that $f_- < \tilde{f}_-$ and $\tilde{f}_+ < f_+$ in $\mathbb{R}$, and hence (A.11) and (3.31) give us

$$f_- < f_1 < f_+ \quad \text{on } (-\infty, 0].$$

Combining this with (A.7) yields that

$$f_- < f_1 < f_+ \quad \text{in } \mathbb{R}. \tag{A.12}$$

Also, since $\lim_{\xi\to-\infty} f_\pm = \lim_{\xi\to-\infty} \tilde{f}_\pm$, (A.11) implies that

$$1-\frac{3}{2}\varepsilon \leqslant f_-(-\infty) \leqslant 1-\frac{1}{2}\varepsilon < 1 = f_1(-\infty) < 1+\frac{1}{2}\varepsilon \leqslant f_+(-\infty) \leqslant 1+\frac{3}{2}\varepsilon. \tag{A.13}$$

Therefore, (A.12) and (A.13) yield (3.35).

Thus, it remains to prove (A.11). (A.11) consists of four inequalities. Since we will see that all the proofs are similar, let us prove the fourth one:

$$\tilde{f}_+ < f_{1+\frac{3}{2}\varepsilon} \quad \text{in } (-\infty, 0]. \tag{A.14}$$

By (A.8), we have $\tilde{f}_+ < f_{1+\frac{3}{2}\varepsilon}$ at $\xi = 0$. Hence, suppose that there exists $\xi_0 < 0$ satisfying

$$\tilde{f}_+(\xi_0) = f_{1+\frac{3}{2}\varepsilon}(\xi_0) \quad \text{and} \quad \tilde{f}_+ < f_{1+\frac{3}{2}\varepsilon} \quad \text{on } (\xi_0, 0]. \tag{A.15}$$

Then, by the uniqueness we also have

$$v'_{1+\frac{3}{2}\varepsilon} > \big(\phi(\tilde{f}_+)\big)' \quad \text{at } \xi = \xi_0. \tag{A.16}$$

By (A.9), we have

$$\big(\phi(\tilde{f}_+)\big)' > v'_{1+\frac{3}{2}\varepsilon} \quad \text{at } \xi = 0. \tag{A.17}$$

Here, integrating Eqs. (A.10) for $\tilde{f}_+$ and (3.23) for $f_{1+\frac{3}{2}\varepsilon}$ on the interval $[\xi_0, 0]$, integrating by parts, considering the difference of the two resultant equalities, and using the fact that $\tilde{f}_+(\xi_0) = f_{1+\frac{3}{2}\varepsilon}(\xi_0)$, yield that

$$v'_{1+\frac{3}{2}\varepsilon}(0) - \big(\phi(\tilde{f}_+)\big)'(0) - \big\{v'_{1+\frac{3}{2}\varepsilon}(\xi_0) - \big(\phi(\tilde{f}_+)\big)'(\xi_0)\big\} - \frac{1}{2}\int_{\xi_0}^{0} \big(f_{1+\frac{3}{2}\varepsilon}(\xi) - \tilde{f}_+(\xi)\big)\,d\xi = 0.$$

On the other hand, by combining (A.15), (A.16), and (A.17), we see that the left-hand side of this equality is negative, which is a contradiction. Therefore, we get (A.14). □

In the next theorem, we prove a comparison principle over general domains including the case where their boundaries are unbounded, by adjusting a proof that Bertsch, Kersner and Peletier gave for the Cauchy problem (see [3, Appendix, pp. 1005–1008]). Observe that, when $\Omega = \mathbb{R}^N$ (and (A.19) is dropped), there is no need to use the approximating sequences $\{D_j\}$ and $\{D_{j,k}\}$ constructed in our proof below, since the sequence of balls $\{B_{R_k}(0)\}$ suffices, as in [3].

Theorem A.1 *(Comparison principle). Let $T>0$ and let Ω be a domain in $\mathbb{R}^N$, with $N \geqslant 2$, where $\partial\Omega$ is not necessarily bounded. Assume that $u, v \in C^{2,1}(\Omega \times (0,T]) \cap L^\infty(\Omega \times (0,T]) \cap C^0(\overline{\Omega} \times (0,T])$ satisfy the following*:

$$\partial_t u - \Delta\phi(u) \leqslant \partial_t v - \Delta\phi(v) \quad \text{in } \Omega \times (0,T], \tag{A.18}$$

$$u \leqslant v \quad \text{on } \partial\Omega \times (0,T], \tag{A.19}$$

$$u(\cdot,t) \to u_0(\cdot) \quad \text{and} \quad v(\cdot,t) \to v_0(\cdot) \quad \text{in } L^1_{loc}(\Omega) \text{ as } t \downarrow 0, \tag{A.20}$$

where $u_0, v_0 \in L^\infty(\Omega)$ satisfy the inequality $u_0 \leqslant v_0$ in Ω.

Then $u \leqslant v$ in $\Omega \times (0,T]$.

Proof. (a) *Approximating the domain Ω*. Let $d = d(x)$ be the distance of x from the closed set $\mathbb{R}^N \setminus \Omega$ and let $U = \{x \in \mathbb{R}^N\colon\ d(x) < 1\}$. From a lemma due to Calderón and Zygmund [14, Lemma 3.6.1, p. 136] (see also [4, Lemma 3.2, p. 185]) it follows that there exist a function $\delta = \delta(x) \in C^\infty(U \cap \Omega)$ and a positive number $M = M(N)$ such that

$$M^{-1} d(x) \leqslant \delta(x) \leqslant M\, d(x) \quad \text{for all } x \in U \cap \Omega. \tag{A.21}$$

Since $\delta \in C^\infty(U \cap \Omega)$, in view of (A.21) and the definition of U, Sard's theorem (see [11,12]) yields that there exists a strictly decreasing sequence of positive numbers $\{\rho_j\}$ with $\lim_{j\to\infty} \rho_j = 0$ and $\rho_1 < M^{-1}$ such that every level set

$$\gamma_j = \bigl\{x \in U \cap \Omega\colon\ \delta(x) = \rho_j\bigr\} \tag{A.22}$$

is a union of smooth hypersurfaces in $\mathbb{R}^N$. For each $j \in \mathbb{N}$, denote by D_j the set satisfying $\partial D_j = \gamma_j$ and $\overline{D_j} \subset \Omega$ (D_j is in general a union of smooth domains). Moreover, in view of (A.21), we may have

$$\overline{D_j} \subset D_{j+1} \quad \text{for } j \in \mathbb{N} \quad \text{and} \quad \Omega = \bigcup_{j=1}^\infty D_j. \tag{A.23}$$

Without loss of generality, we may also assume that the origin belongs to all the D_j's.

The intersection $D_j \cap B_R(0)$ of D_j with the ball $B_R(0)$ may not be a finite union of Lipschitz domains; however, again by Sard's theorem, the restriction to γ_j of the C^∞-smooth map $x \mapsto |x|^2$ is regular at almost any of its values, and hence there exists a strictly increasing and diverging sequence $\{R_k\}$ of positive numbers such that each $\partial B_{R_k}(0)$ is transversal to all the γ_j's; thus, for each pair of j and k, $D_j \cap B_{R_k}(0)$ is a finite union of Lipschitz domains with piecewise C^∞-smooth boundaries. Therefore, by using a partition of unity, we can modify the boundary of $D_j \cap B_{R_k}(0)$ near the compact submanifold $\gamma_j \cap \partial B_{R_k}(0)$ to get a family $\{D_{j,k}\}$ of finite unions of smooth domains, each one approximating $D_j \cap B_{R_k}(0)$, and satisfying the relations

$$\begin{gathered} D_{j-1} \cap B_{R_k}(0) \subset D_{j,k} \subset D_{j+1} \cap B_{R_k}(0) \bigl(\subset B_{R_k}(0)\bigr) \quad \text{and} \\ \partial D_{j,k} \cap \overline{D_{j-1}} = \partial B_{R_k}(0) \cap \overline{D_{j-1}}, \end{gathered} \tag{A.24}$$

for every $j \geqslant 2$ and $k \in \mathbb{N}$.

(b) *Constructing test functions.* Set

$$A = A(x,t) = \begin{cases} \frac{\phi(u)-\phi(v)}{u-v} & \text{if } u(x,t) \neq v(x,t),\ x \in \Omega,\ \text{and } t > 0, \\ \delta_1 & \text{otherwise.} \end{cases}$$

Then $\delta_1 \leqslant A \leqslant \delta_2$ on $\mathbb{R}^{N+1}$ and we can approximate A by a sequence $\{A_n\}$ of regularizations satisfying

$$A_n \in C^\infty\big(\mathbb{R}^{N+1}\big) \quad \text{and} \quad \delta_1 \leqslant A_n \leqslant \delta_2 \quad \text{in } \mathbb{R}^{N+1} \text{ for each } n \in \mathbb{N}, \tag{A.25}$$

$$A - A_n \to 0 \quad \text{in } L^2_{loc}\big(\mathbb{R}^{N+1}\big) \text{ as } n \to \infty. \tag{A.26}$$

Let $0 < \tau < s < T$ and choose $\chi \in C_0^\infty(\mathbb{R}^N)$, with support $\operatorname{supp}\chi$ contained in Ω, such that $0 \leqslant \chi \leqslant 1$ in $\mathbb{R}^N$. In view of (A.21), there exist $j_0, k_0 \in \mathbb{N}$ such that

$$\operatorname{supp}\chi \subset D_{j,k} \quad \text{for every pair of } j \geqslant j_0 \text{ and } k \geqslant k_0. \tag{A.27}$$

Now, choose an integer $k \geqslant k_0$ and then a number $\varepsilon > 0$. Since $u, v \in C^0(\overline{\Omega} \times (0,T])$, it follows from (A.19) that there exists $\mu > 0$ satisfying

$$\phi(u) \leqslant \phi(v) + \varepsilon \quad \text{in } \overline{\Omega_\mu \cap B_{R_{k+1}}(0)} \times [\tau, T], \tag{A.28}$$

where Ω_μ is given by (1.11). Hence, by (A.21) and (A.24), we see that there exists $j_1 \geqslant j_0$ such that

$$\phi(u) \leqslant \phi(v) + \varepsilon \quad \text{on } (\partial D_{j,k} \setminus D_{j-1}) \times [\tau, T] \text{ for every } j \geqslant j_1. \tag{A.29}$$

For each $j \geqslant j_1$ and $n \in \mathbb{N}$, let $w_{n,j} \in C^\infty(\overline{D_{j,k}} \times [0,s)) \cap C^0(\overline{D_{j,k}} \times [0,s])$ be the unique bounded solution of the problem:

$$\partial_t w_{n,j} + A_n \Delta w_{n,j} = \delta_2 w_{n,j} \quad \text{in } D_{j,k} \times [0,s), \tag{A.30}$$

$$w_{n,j} = 0 \quad \text{on } \partial D_{j,k} \times [0,s), \tag{A.31}$$

$$w_{n,j}(x,s) = e^{-|x|}\chi(x) \quad \text{for every } x \in D_{j,k}. \tag{A.32}$$

Then, by the parabolic regularity theory (see [6]), we see that

$$w_{n,j} \in C^\infty\big(\big(\overline{D_{j,k}} \times [0,s]\big) \setminus \big(\{0\} \times \{s\}\big)\big) \quad \text{and} \quad \nabla w_{n,j} \in L^\infty\big(D_{j,k} \times [0,s]\big),$$

and, as in [3, Lemma B, p. 1007], we can prove the following lemma.

Lemma A.2. *There exists a constant $c > 0$ depending only on χ such that, for each $j \geqslant j_1$ and $n \in \mathbb{N}$, the solutions $w_{n,j}$ have the following properties*:

(i) $0 \leqslant w_{n,j} \leqslant e^{-|x|}$ *in* $\overline{D_{j,k}} \times [0,s]$,

(ii) $\displaystyle\int_0^s dt \int_{D_{j,k}} A_n (\Delta w_{n,j})^2\, dx \leqslant c$,

(iii) $$\sup_{0\leqslant t\leqslant s}\int\limits_{D_{j,k}}\left|\nabla w_{n,j}(x,t)\right|^2 dx\leqslant c,$$

(iv) $$0\leqslant -\frac{\partial w_{n,j}}{\partial\nu}\leqslant ce^{-R_k}\quad \text{on } \big(\partial D_{j,k}\cap\partial B_{R_k}(0)\big)\times[0,s],$$

where ν denotes the unit outward normal vector to $\partial D_{j,k}$.

Remark A.3. The fact that $D_{j,k}\subset B_{R_k}(0)$ (see (A.24)) guarantees that the same barrier function as in [3, Lemma B, p. 1007] can be used to prove (iv). The proofs of the others are the same.

(c) *Completion of the proof.* For each $j\geqslant j_1$ and $n\in\mathbb{N}$, multiplying (A.18) by $w=w_{n,j}$ and integrating by parts the resultant inequality over $D_{j,k}\times[\tau,s]$ yield that

$$\begin{aligned}
0&\geqslant \int\limits_{D_{j,k}\times[\tau,s]}\big\{\partial_t(u-v)-\Delta\big[\phi(u)-\phi(v)\big]\big\}w\,dx\,dt\\
&=\int\limits_{D_{j,k}}\big[(u-v)(x,t)w(x,t)\big]_\tau^s\,dx-\int\limits_{D_{j,k}\times[\tau,s]}(u-v)\partial_t w\,dx\,dt\\
&\quad-\int\limits_\tau^s dt\int\limits_{\partial D_{j,k}}\frac{\partial}{\partial\nu}\big[\phi(u)-\phi(v)\big]w\,d\sigma+\int\limits_{D_{j,k}\times[\tau,s]}\nabla\big[\phi(u)-\phi(v)\big]\cdot\nabla w\,dx\,dt\\
&=\int\limits_{D_{j,k}}\big\{(u-v)(x,s)e^{-|x|}\chi(x)-(u-v)(x,\tau)w(x,\tau)\big\}\,dx\\
&\quad-\int\limits_{D_{j,k}\times[\tau,s]}(u-v)\partial_t w\,dx\,dt+\int\limits_{D_{j,k}\times[\tau,s]}\nabla\big[\phi(u)-\phi(v)-\varepsilon\big]\cdot\nabla w\,dx\,dt;
\end{aligned}\tag{A.33}$$

here we used (A.31) and (A.32), and we modified the last term a little for later use. The last in (A.33) term equals

$$\begin{aligned}
&\int\limits_\tau^s dt\int\limits_{\partial D_{j,k}\setminus\overline{D_{j-1}}}\big[\phi(u)-\phi(v)-\varepsilon\big]\frac{\partial w}{\partial\nu}\,d\sigma\\
&\quad+\int\limits_\tau^s dt\int\limits_{\partial D_{j,k}\cap\overline{D_{j-1}}}\big[\phi(u)-\phi(v)-\varepsilon\big]\frac{\partial w}{\partial\nu}\,d\sigma-\int\limits_{D_{j,k}\times[\tau,s]}\big[\phi(u)-\phi(v)-\varepsilon\big]\Delta w\,dx.
\end{aligned}$$

Since $\frac{\partial w}{\partial\nu}\leqslant 0$ on $\partial D_{j,k}\times[0,s]$, it follows from (A.29) that the first term above is nonnegative; also, in the third term, we write:

$$\phi(u)-\phi(v)-\varepsilon=\big\{A_n+(A-A_n)\big\}(u-v)-\varepsilon.$$

Therefore, it follows from (A.33) and (A.30) that

$$0 \geqslant \int\limits_{D_{j,k}} (u-v)(x,s)e^{-|x|}\chi(x)\,dx - \int\limits_{D_{j,k}} (u-v)(x,\tau)w(x,\tau)\,dx$$
$$+ \int\limits_{\tau}^{s} dt \int\limits_{\partial D_{j,k}\cap\overline{D_{j-1}}} \left[\phi(u)-\phi(v)-\varepsilon\right]\frac{\partial w}{\partial\nu}\,d\sigma - \delta_2 \int\limits_{D_{j,k}\times[\tau,s]} (u-v)w\,dx\,dt$$
$$- \int\limits_{D_{j,k}\times[\tau,s]} (u-v)(A-A_n)\Delta w\,dx\,dt + \varepsilon \int\limits_{D_{j,k}\times[\tau,s]} \Delta w\,dx\,dt. \tag{A.34}$$

Since u and v are bounded, there exists a constant $K > 0$ such that

$$\max\bigl\{|u-v|, \bigl|\phi(u)-\phi(v)-\varepsilon\bigr|\bigr\} \leqslant K \quad \text{in } \Omega\times[0,T].$$

Combining (A.24) with (iv) of Lemma A.2 yields that the third term in (A.34) is bounded from below by

$$-cKe^{-R_k}TN\omega_N R_k^{N-1},$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. By using (i) of Lemma A.2, we see that the fourth term in (A.34) is bounded from below by

$$-\delta_2 \int\limits_{\tau}^{s} dt \int\limits_{\Omega} \max\{u-v,0\}e^{-|x|}\,dx.$$

With the aid of (A.25), (ii) of Lemma A.2 yields that the fifth and the sixth terms in (A.34) are bounded from below by

$$-K\frac{\sqrt{c}}{\sqrt{\delta_1}}\left(\int\limits_0^T dt \int\limits_{D_{j,k}} (A-A_n)^2\,dx\right)^{\frac{1}{2}} \quad \text{and} \quad -\varepsilon\frac{\sqrt{c}}{\sqrt{\delta_1}}\sqrt{T|D_{j,k}|},$$

respectively, where $|D_{j,k}|$ denotes the N-dimensional Lebesgue measure of $D_{j,k}$. Consequently, with these bounds and by using (A.27) in the first term in (A.34), from (A.34) we obtain:

$$\int\limits_{\Omega} (u-v)(x,s)e^{-|x|}\chi(x)\,dx \leqslant \int\limits_{D_{j,k}} (u-v)(x,\tau)w_{n,j}(x,\tau)\,dx$$
$$+ cKe^{-R_k}TN\omega_N R_k^{N-1} + \delta_2 \int\limits_{\tau}^{s} dt \int\limits_{\Omega} \max\{u-v,0\}e^{-|x|}\,dx$$
$$+ K\frac{\sqrt{c}}{\sqrt{\delta_1}}\left(\int\limits_0^T dt \int\limits_{D_{j,k}} (A-A_n)^2\,dx\right)^{\frac{1}{2}} + \varepsilon\frac{\sqrt{c}}{\sqrt{\delta_1}}\sqrt{T|D_{j,k}|}.$$

Since $\varepsilon > 0$ is arbitrarily chosen and $D_{j,k} \subset B_{R_k}(0)$, we can remove the last term in the above inequality. Also, letting $n \to \infty$ and $\tau \to 0$ with in mind (A.26) and (A.20), respectively, yield that

$$\int_{\Omega} (u-v)(x,s)e^{-|x|}\chi(x)\,dx \leqslant cKe^{-R_k}TN\omega_N R_k^{N-1} + \delta_2 \int_{\Omega\times[0,s]} \max\{u-v,0\}e^{-|x|}\,dx\,dt.$$

By letting $k \to \infty$, we remove the first term in the right-hand side of this inequality. Then, since $\chi \in C_0^\infty(\mathbb{R}^N)$ is an arbitrary function satisfying that $0 \leqslant \chi \leqslant 1$ in $\mathbb{R}^N$ and its support is contained in Ω, we conclude that for every $s \in [0,T]$

$$\int_{\Omega} \max\bigl\{(u-v)(x,s),0\bigr\}e^{-|x|}\,dx \leqslant \delta_2 \int_0^s dt \int_{\Omega} \max\{u-v,0\}e^{-|x|}\,dx. \tag{A.35}$$

Finally, Gronwall's lemma implies that $u \leqslant v$ in $\Omega \times (0,T]$. $\square$

References

[1] F.V. Atkinson, L.A. Peletier, Similarity solutions of the nonlinear diffusion equation, Arch. Ration. Mech. Anal. 54 (1974) 373–392.
[2] H. Berestycki, L.A. Caffarelli, L. Nirenberg, Monotonicity for elliptic equations in unbounded Lipschitz domains, Comm. Pure Appl. Math. 50 (1997) 1089–1111.
[3] M. Bertsch, R. Kersner, L.A. Peletier, Positivity versus localization in degenerate diffusion equations, Nonlinear Anal. 9 (1985) 987–1008.
[4] A.P. Calderón, A. Zygmund, Local properties of solutions of elliptic partial differential equations, Studia Math. 20 (1961) 171–225.
[5] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.
[6] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monogr., vol. 23, Amer. Math. Soc., 1968.
[7] R. Magnanini, S. Sakaguchi, Interaction between degenerate diffusion and shape of domain, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007) 373–388.
[8] R. Magnanini, S. Sakaguchi, Stationary isothermic surfaces for unbounded domains, Indiana Univ. Math. J. 56 (2007) 2723–2738.
[9] R. Magnanini, S. Sakaguchi, Nonlinear diffusion with a bounded stationary level surface, Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010) 937–952.
[10] R. Magnanini, S. Sakaguchi, Stationary isothermic surfaces and some characterizations of the hyperplane in the N-dimensional Euclidean space, J. Differential Equations 248 (2010) 1112–1119.
[11] A. Sard, The measure of the critical values of differentiable maps, Bull. Amer. Math. Soc. 48 (1942) 883–890.
[12] S. Sternberg, Lectures on Differential Geometry, second ed., Chelsea Publishing Co., New York, 1983.
[13] S.R.S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, Comm. Pure Appl. Math. 20 (1967) 431–455.
[14] W.P. Ziemer, Weakly Differentiable Functions, Grad. Texts in Math., vol. 120, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, 1989.